1. Introduction

We give an introduction to the basic theory of state/signal systems via boundary control. More precisely, we discuss the connection between some basic notions of boundary control state/signal systems on one hand, and classical boundary triplets on the other hand. Boundary triplets and their generalizations have been extensively utilized in the theory of self-adjoint extensions of symmetrical operators in Hilbert spaces, see e.g. [Gorbachuk and Gorbachuk, 1991], [Derkach and Malamud, 1995], [Behrndt and Langer, 2007], and the references therein.

The notions related to standard input/state/output boundary control systems are discussed in Section 2, where we also introduce the boundary control state/signal system. In Section 3 we briefly discuss the concept of conservativity in the state/signal framework and in Section 4 we illustrate the abstract concepts using the example of a finite-length conservative LC-transmission line with distributed inductance and capacitance.

We conclude this chapter in Section 5, where we recall the definition of a boundary triplet for a symmetric operator and compare this object to a boundary control state/signal system. In particular, we show that every boundary triplet can be transformed into a conservative boundary control state/signal system in impedance form, but that the converse is not true. We make a few final remarks about common generalizations of boundary triplets, which leads over to Chapter ??.
where we treat more general passive state/signal systems, not only conservative systems or systems of boundary-control type. There we show how conservative state/signal systems are related to boundary relations.

2. Boundary control systems

In this section we introduce boundary control state/signal systems by first describing their predecessors, input/state/output systems of boundary-control type.

In boundary control one often investigates systems that can be abstractly written in the form

$$
\Sigma_{i/s/o} : \begin{cases}
\dot{x}(t) = Lx(t), \\
u(t) = \Gamma_0 x(t), & t \in \mathbb{R}^+, \ x(0) = x_0 \text{ given,} \\
y(t) = \Gamma_1 x(t),
\end{cases}
$$

where \( \mathbb{R}^+ = [0, \infty) \) and \( \dot{x} = \frac{dx}{dt} \). Here the initial state \( x_0 \) and the current state \( x(t) \) belong to the Hilbert state space \( \mathcal{X} \), the input \( u(t) \) belongs to the Hilbert input space \( \mathcal{U} \), and the output \( y(t) \) belongs to the Hilbert output space \( \mathcal{Y} \). The main operator \( L \) is an unbounded operator in \( \mathcal{X} \) with domain \( \text{dom} (L) \), and the boundary control operator \( \Gamma_0 \) is an unbounded operator \( \mathcal{X} \to \mathcal{U} \) with the same domain as \( L \). The observation operator \( \Gamma_1 : \mathcal{X} \to \mathcal{Y} \) may be bounded or unbounded, and it is defined at least on \( \text{dom} (L) \). All of these operators are linear. We denote the system (2.1) with these properties by \( \Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \).

In order for (2.1) to generate a dynamical system with good properties at least the properties listed in the following definition need to be assumed; see e.g. [Salamon, 1987], [Staffans, 2005], or [Malinen and Staffans, 2006] for details.

**Definition 1.** Assume that \( \Sigma_{i/s/o} = (L, \Gamma_0, \Gamma_1; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is as described above. Then \( \Sigma_{i/s/o} \) is a boundary control input/state/output (i/s/o) node if \( \Sigma_{i/s/o} \) satisfies the following conditions:

1. The input operator \( \Gamma_0 \) is surjective and strictly unbounded in the sense that \( \ker (\Gamma_0) \) is dense in \( \mathcal{X} \).
2. The restriction \( A := L|_{\ker(\Gamma_0)} \) of \( L \) to \( \ker (\Gamma_0) \) generates a \( C_0 \)-semigroup \( t \mapsto \mathfrak{A}^t, \ t \in \mathbb{R}^+ \).

A boundary control state/signal system is analogous to a boundary control i/s/o system, but we no longer specify which part of the “boundary signal” \( w(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \) is the input, and which part is the output. Instead we combine the input and output spaces into one signal space \( \mathcal{W} := \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix} = \mathcal{U} \times \mathcal{Y} \), and denote \( \Gamma := \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix} \). Then \( \Gamma : \text{dom} (L) \to \mathcal{W} \),
and (2.1) can be rewritten in the form

$$\begin{align*}
\Sigma : \quad \begin{cases} 
\dot{x}(t) = Lx(t), \\
w(t) = \Gamma x(t),
\end{cases} \quad t \in \mathbb{R}^+, \ x(0) = x_0 \text{ given}.
\end{align*}$$

As before, the initial state $x_0$ and the current state $x(t)$ belong to the Hilbert state space $\mathcal{X}$. The (interaction) signal $w(t)$ belongs to the signal space $\mathcal{W}$, which we take to be an arbitrary Krein space (the reason for this will be explained below). We thus no longer assume that $\mathcal{W}$ is of the form $\mathcal{W} = [U^\dagger Y]$, where $U$ and $Y$ are the input and output spaces of a boundary control i/s/o node. The main operator $L$ is still an unbounded operator $\mathcal{X} \to \mathcal{X}$ with domain $\text{dom}(L)$, and the boundary operator $\Gamma$ is an unbounded operator $\mathcal{X} \to \mathcal{W}$ with the same domain as $L$. We denote this system by $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$.

Note that (2.2) can be written in the graph form:

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \ x(0) = x_0,$$

where the generating subspace $V$ is the graph of $[\frac{\text{d}}{\text{d}t}]$:

$$V := \left\{ [\begin{bmatrix} Lx \\ \Gamma x \end{bmatrix}] \mid x \in \text{dom}(L) \right\}.$$  

The unbounded operator $[\frac{\text{d}}{\text{d}t}]$ is assumed to be closed, and this is equivalent to assuming that $V$ is a closed subspace of the node space $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$. The generating subspace is the key to generalizing the state/signal theory beyond boundary control, as we shall see in Chapter ???. We define the dynamics of a state/signal system using the generating subspace $V$.

**Definition 2.** Let $V$ be a closed subspace of $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$.

1. The pair $[x \ w]$ is a **classical trajectory** generated by $V$ on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $w \in C(\mathbb{R}^+; \mathcal{W})$, and $[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}] \in V$ for all $t > 0$.

2. The pair $[x \ w]$ is a **generalized trajectory** generated by $V$ on $\mathbb{R}^+$ if $x \in C(\mathbb{R}^+; \mathcal{X})$, $w \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W})$, and there exists a sequence of classical trajectories $[x_n \ w_n]$ such that $x_n \to x$ uniformly on all bounded intervals $[0, T]$ and $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W})$.

Note that $[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}] \in V$ for all $t > 0$ in item (1) of Definition 2 if and only if $[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}] \in V$ for all $t \in \mathbb{R}^+$ when we interpret $\dot{x}(0)$ as the right-sided derivative of $x$ at zero. We are now ready to define a boundary control s/s system.

**Definition 3.** A **boundary control state/signal (s/s) node** is a quadruple $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ such that:
(1) The space $\mathcal{X}$ is a Hilbert space and $\mathcal{W}$ is a Krein space.
(2) The operator $[L]: \mathcal{X} \to [\mathcal{X}, \mathcal{W}]$ is closed and densely defined.
(3) The range of $\Gamma$ is dense in $\mathcal{W}$.

By the boundary control state/signal system induced by a boundary control s/s node $(L, \Gamma; \mathcal{X}, \mathcal{W})$ we mean this node together with the sets of classical and generalized trajectories generated by $V$ in (2.3) on $\mathbb{R}^+$. We denote both the node and the system by $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$.

In Definition 4 below we will equip the node space $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$ with an indefinite inner product which makes it a Krein space.

3. CONSERVATIVE STATE/SIGNAL SYSTEMS IN BOUNDARY CONTROL

In this chapter we shall focus our attention on s/s systems $\Sigma$ whose classical trajectories on $\mathbb{R}^+$ satisfy the power equality

$$\frac{d}{dt} \|x(t)\|_{\mathcal{X}}^2 = [w(t), w(t)]_{\mathcal{W}}, \quad t \in \mathbb{R}^+. \tag{3.1}$$

Here $\|x(t)\|_{\mathcal{X}}^2$ stands for (two times) the internal energy stored in the state $x$ at time $t$ and $[w(t), w(t)]_{\mathcal{W}}$ represents (two times) the power (energy flow per time unit) entering the system through the signal $w(t)$ at time $t$. This explains why we need to take $\mathcal{W}$ to be a Krein space: we must allow the inner product $[\cdot, \cdot]_{\mathcal{W}}$ in $\mathcal{W}$ to be indefinite: if the inner product in $\mathcal{W}$ is non-negative, then no energy can leave the system via the (interaction) signal, and if the inner product in $\mathcal{W}$ is non-positive, then no energy can enter the system via the signal.

The equality (3.1) says that the system has no internal energy sources or sinks. However, the equality is not enough to make the system $\Sigma$ conservative: we need an additional hypermaximality condition. We give the full definition of a conservative boundary control s/s system in Definition 5 below.

After integration over the interval $[s, t] \subset \mathbb{R}^+$, one can rewrite (3.1) in the equivalent form

$$\|x(t)\|_{\mathcal{X}}^2 - \|x(s)\|_{\mathcal{X}}^2 = \int_s^t [w(v), w(v)]_{\mathcal{W}} dv, \quad s, t \in \mathbb{R}^+, \quad s \leq t. \tag{3.2}$$

By the continuity of the inner product this inequality remains valid for generalized trajectories as well.

Carrying out the differentiation in (3.1), we get a third equivalent condition in terms of classical trajectories, namely

$$-(\dot{x}(t), x(t))_{\mathcal{X}} - (x(t), \dot{x}(t))_{\mathcal{X}} + [w(t), w(t)]_{\mathcal{W}} = 0, \quad t \in \mathbb{R}^+. \tag{3.3}$$

Using item (1) of Definition 2, we see that (3.3) always holds if

$$-(z, x)_{\mathcal{X}} - (x, z)_{\mathcal{X}} + [w, w]_{\mathcal{W}} = 0, \quad \begin{bmatrix} z \\ w \end{bmatrix} \in V. \tag{3.4}$$

It is now natural to make the following definition:
Definition 4. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space. The corresponding node space is the product space $\mathfrak{N} = \mathcal{X} \times \mathcal{X} \times \mathcal{W}$ equipped with the indefinite inner product induced by the quadratic form in (3.4):

$$
(z_1, x_2, w_1, x_1, z_2, w_2)_{\mathfrak{N}} = -(z_1, x_2)_{\mathcal{X}} - (x_1, z_2)_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}}.
$$

Note that the quadratic form in (3.4) is strictly indefinite, i.e., it takes both positive and negative values whenever $\mathcal{X} \neq \{0\}$. Furthermore, the inner product in (3.5) makes the node space $\mathfrak{N}$ a Kreĭn space.

The equality (3.4) says that $V$ is a neutral subspace of $\mathfrak{N}$ with respect to the inner product (3.5), i.e., that $[v, v]_{\mathfrak{N}} = 0$ for all $v \in V$. The condition that a subspace $V$ is a neutral subspace of $\mathfrak{N}$ can equivalently be written $V \subseteq V^{[\perp]}$, where

$$
V^{[\perp]} := \{ k \in \mathfrak{N} \mid [k, k']_{\mathfrak{N}} = 0 \text{ for all } k' \in V \}.
$$

If instead $V^{[\perp]} \subset V$, then $V$ is called co-neutral, and if $V^{[\perp]} = V$, then $V$ is called Lagrangian or hypermaximal neutral.

Definition 5. A boundary control s/s system $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ is conservative if its generating subspace $V$ in (2.3) is a Lagrangian subspace of the node space $\mathfrak{N}$, i.e., if $V = V^{[\perp]}$.

Since every orthogonal companion is closed, necessarily every Lagrangian subspace is closed. Moreover, in [Kurula et al., 2010, Thm 4.3] it was proved that if $V$ in (2.3) is Lagrangian then $\ker (\Gamma)$ is dense in $\mathcal{X}$ and $\text{ran}(\Gamma)$ is dense in $\mathcal{W}$. Since $\ker(\Gamma) \subseteq \text{dom}(\Gamma) = \text{dom}(L)$, the operator $[\frac{1}{L}]$ is closed and automatically densely defined. Thus the conditions in Definition 3 are satisfied for every Lagrangian subspace $V$ of the type (2.3). See also [Derkach et al., 2006, Cor. 2.4].

Remark 6. In the boundary control case the neutrality condition $V \subseteq V^{[\perp]}$ means that

$$
(Lx, x)_{\mathcal{X}} + (x, Lx)_{\mathcal{X}} = [\Gamma x, \Gamma x]_{\mathcal{W}}, \quad x \in \text{dom}(L).
$$

However, if $V$ is only neutral, then $V$ might for instance be the degenerate trivial system $\{0\}$. This case is excluded by the hypermaximality condition $V \supset V^{[\perp]}$, which in the case of boundary control means that

$$
(z_1, x)_{\mathcal{X}} + (x, Lx)_{\mathcal{X}} = [w_1, \Gamma x]_{\mathcal{W}}, \quad x \in \text{dom}(L) \implies \left[ \begin{array}{c} z_1 \\ x_1 \\ w_1 \end{array} \right] \in V.
$$

Letting $\mathcal{X}$ be a Hilbert space, $\mathcal{W}$ be a Kreĭn space, and $[\frac{1}{L}] : \mathcal{X} \to [\mathcal{W}]$, we thus have that $\Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W})$ is a conservative boundary control s/s system if and only if the conditions (3.7) and (3.8) are satisfied.
4. An example: the transmission line

An ideal transmission line of length $\ell$ can be modeled by the following equations, where $\xi \in [0, \ell]$ and $t \in \mathbb{R}^+$:

$$
\frac{\partial}{\partial t} \begin{bmatrix} i(\xi, t) \\ v(\xi, t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\mathcal{L}(\xi)} \frac{\partial}{\partial \xi} \\ \frac{1}{\mathcal{C}(\xi)} \frac{\partial}{\partial \xi} & 0 \end{bmatrix} \begin{bmatrix} i(\xi, t) \\ v(\xi, t) \end{bmatrix},
$$

(4.1)

$$
w(t) = \begin{bmatrix} i(0, t) \\ v(0, t) \end{bmatrix} = \begin{bmatrix} i(\xi, 0) \\ v(\xi, 0) \end{bmatrix} = \begin{bmatrix} i_0(\xi) \\ v_0(\xi) \end{bmatrix}.
$$

Here $i(\xi, t)$ and $v(\xi, t)$ are the current and voltage, respectively, at the point $\xi \in [0, \ell]$ at time $t \in \mathbb{R}^+$. The functions $\mathcal{L}(\cdot) > 0$ and $\mathcal{C}(\cdot) > 0$ represent the distributed inductance and capacitance, respectively, of the line. For simplicity we assume that $\mathcal{C}(\cdot)$ and $\mathcal{L}(\cdot)$ are continuous on $[0, \ell]$, which implies that $\mathcal{C}$ and $\mathcal{L}$ are both bounded and bounded away from zero. The transmission line is illustrated in Figure 1.

![Figure 1. An ideal $\mathcal{LC}$-transmission line of length $\ell$ with distributed inductance $\mathcal{L}$ and capacitance $\mathcal{C}$. Here $i(\xi, t)$ and $v(\xi, t)$ denote the current and the voltage, respectively, at the point $\xi \in [0, \ell]$ at time $t \in \mathbb{R}^+$.

The natural state at time $t$ of this transmission line is the current-voltage vector $x(t) = \begin{bmatrix} i(\cdot, t) \\ v(\cdot, t) \end{bmatrix}$, $t \in \mathbb{R}^+$, and the initial state is $x(0) = \begin{bmatrix} i(0, 0) \\ v(0, 0) \end{bmatrix} = \begin{bmatrix} i_0(\cdot) \\ v_0(\cdot) \end{bmatrix} =: x_0$. We take the state space $\mathcal{X}$ to be $L^2([0, \ell]; \mathbb{C}^2)$ with inner product $(\cdot, \cdot)_\mathcal{X}$ defined by

$$
(4.2) \quad \left( \begin{bmatrix} i_1(\xi) \\ v_1(\xi) \end{bmatrix}, \begin{bmatrix} i_2(\xi) \\ v_2(\xi) \end{bmatrix} \right)_\mathcal{X} = \int_0^\ell \left( \mathcal{L}(\xi) i_1(\xi) \overline{i_2(\xi)} + \mathcal{C}(\xi) v_1(\xi) \overline{v_2(\xi)} \right) d\xi.
$$

In our setting the corresponding quadratic form $(x(t), x(t))_\mathcal{X}$ is equivalent to the standard inner product on $L^2([0, \ell]; \mathbb{C}^2)$ and its value is two times the energy stored in the state $x(t)$ of the transmission line at time $t$.

The operator $L$ is given by

$$
L := \begin{bmatrix} 0 & -\frac{1}{\mathcal{L}(\xi)} \frac{\partial}{\partial \xi} \\ \frac{1}{\mathcal{C}(\xi)} \frac{\partial}{\partial \xi} & 0 \end{bmatrix}, \quad \text{dom} (L) := W^{1,2}([0, \ell]; \mathbb{C}^2),
$$
where $W^{1,2}([0, \ell]; \mathbb{C}^2)$ is the Sobolev space of absolutely continuous functions in $L^2([0, \ell]; \mathbb{C}^2)$ which have a distribution derivative in $L^2([0, \ell]; \mathbb{C}^2)$. The signal space $\mathcal{W}$ is $\mathbb{C}^4$ equipped with the indefinite inner product

$$(4.3) \quad \left[ \begin{array}{c} \dot{i}_{01} \\ \dot{v}_{01} \\ \dot{i}_{02} \\ \dot{v}_{02} \\ i_{v1} \\ v_{v1} \\ i_{v2} \\ v_{v2} \end{array} \right] \mathcal{W} = \left( \begin{array}{c} \dot{i}_{01} \\ \dot{v}_{01} \\ \dot{i}_{02} \\ \dot{v}_{02} \\ i_{v1} \\ v_{v1} \\ i_{v2} \\ v_{v2} \end{array} \right) \mathcal{W} \quad \text{and} \quad J_\mathcal{W} = \left[ \begin{array}{cc} 0 & 1 \\ \overline{1} & 0 \end{array} \right].$$

The boundary operator $\Gamma$ has the same domain as $L$, and it is given by

$$\Gamma \left[ \begin{array}{c} i(\cdot) \\ v(\cdot) \end{array} \right] = \left[ \begin{array}{c} i(0) \\ v(0) \\ -i(t) \\ v(t) \end{array} \right].$$

The operator $[\frac{\partial}{\partial t}]$ is closed as an operator from $\mathcal{X}$ to $[\frac{\partial}{\partial \xi}]$ with domain $\text{dom}(\frac{\partial}{\partial t}) = \text{dom}(L) = W^{1,2}([0, \ell]; \mathbb{C}^2)$. With these definitions, the transmission line can be modeled as an example of a boundary control system in the sense of Definition 3, as we now show.

We next derive the appropriate Lagrangian identity. Combining equations (4.1) and (4.2), we make the following computations for $t > 0$:

$$\frac{d}{dt} \|x(t)\|^2_{\mathcal{X}} = 2\text{Re} (x(t), \dot{x}(t))_{\mathcal{X}}$$

$$= 2\text{Re} \int_0^\ell \left( \mathcal{L}(\xi)i(\xi, t) \frac{\partial}{\partial t} \dot{i}(\xi, t) + \mathcal{C}(\xi)v(\xi, t) \frac{\partial}{\partial t} \dot{v}(\xi, t) \right) d\xi$$

$$= -2 \int_0^\ell \text{Re} \left( i(\xi, t) \frac{\partial}{\partial \xi} \dot{v}(\xi, t) + \frac{\partial}{\partial \xi} i(\xi, t)v(\xi, t) \right) d\xi$$

$$= -2 \int_0^\ell \text{Re} \frac{\partial}{\partial \xi} \left( i(\xi, t)v(\xi, t) \right) d\xi$$

$$= -2\text{Re} \left[ i(\xi, t)v(\xi, t) \right]_{\xi=0}^\ell$$

$$= 2\text{Re} i(0, t)v(0, t) - 2\text{Re} i(\ell, t)v(\ell, t)$$

$$= \left( \begin{array}{c} i_{v0} \\ v_{v0} \\ i_{v1} \\ v_{v1} \end{array} \right) \mathcal{W} = [\Gamma x(t), \Gamma x(t)]_{\mathcal{W}},$$

where we have used that $(''$ denotes spatial derivative)

$$2\text{Re} (i\bar{v}' + \bar{i}'v) = i\bar{v}' + \bar{i}'v + i'\bar{v}' = 2\text{Re} (i\bar{v}')'$$

in the fourth equality. Thus, $[w(t), w(t)]_{\mathcal{W}} = [\Gamma x(t), \Gamma x(t)]_{\mathcal{W}}$ is two times the power entering the transmission line through the terminals at the ends $\xi = 0$ and $\xi = \ell$ of the line at time $t \geq 0$.

These computations tell us that the generating subspace $V$ is a neutral subspace of the node space $\mathcal{R}$, i.e., that (3.7) holds. It is not difficult to show that this subspace is not only neutral, but even Lagrangian, so
that (3.8) also holds; see Example 11 below for the proof idea. Thus, the transmission line gives rise to a conservative boundary control s/s system.

**Remark 7.** Set \( U := \mathbb{C}^2 \), \( R := iL|_{\ker \Gamma} \), and

\[
\Gamma_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} i(0) \\ -i(\ell) \end{bmatrix} \quad \text{and} \quad \Gamma_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} v(0) \\ v(\ell) \end{bmatrix}.
\]

Then \( R \) is a closed, densely defined and symmetric operator in the Hilbert space \( X \), and the triple \((\Gamma_0, -i \Gamma_1; U)\) is a boundary triplet for \( R^* = -iL \) in the standard sense; see below. The boundary triplet and its connection to boundary-control state/signal systems is the topic of the last section of this chapter.

Recall that \([w(t), w(t)]_W\) is two times the power entering the transmission line through the terminals at the ends \( \xi = 0 \) and \( \xi = \ell \) of the line at time \( t \geq 0 \). The decomposition in (4.4) of \( \Gamma \) into an input map \( \Gamma_0 \) and an output map \( \Gamma_1 \) corresponds to choosing the current entering the system at \( \xi = 0 \) and \( \xi = \ell \) as input and the voltages at both ends as output, cf. (2.1). We refer to this as an impedance decomposition of the external signal \( w \).

Several other choices of input and output would have been possible, such as for example

\[
\begin{align*}
\tilde{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \frac{1}{\sqrt{2}} (\Gamma_1 + \Gamma_0) \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v(0) + i(0) \\ v(\ell) - i(\ell) \end{bmatrix} \\
\tilde{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \frac{1}{\sqrt{2}} (\Gamma_1 - \Gamma_0) \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v(0) - i(0) \\ v(\ell) + i(\ell) \end{bmatrix},
\end{align*}
\]

or

\[
\begin{align*}
\hat{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} &:= \begin{bmatrix} i(0) \\ v(0) \end{bmatrix} \quad \text{and} \quad \hat{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} := \begin{bmatrix} -i(\ell) \\ v(\ell) \end{bmatrix}.
\end{align*}
\]

In (4.5) we have

\[
\| \tilde{\Gamma}_0 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \|_{\mathbb{C}^2}^2 - \| \tilde{\Gamma}_1 \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \|_{\mathbb{C}^2}^2 = \left[ \Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix}, \Gamma \begin{bmatrix} i(\cdot) \\ v(\cdot) \end{bmatrix} \right]_W,
\]

where \([\cdot, \cdot]_W\) still denotes the inner product (4.3). This decomposition is an example of a scattering decomposition. In (4.6) we choose voltage and current at \( \xi = 0 \) as input and the voltage and current at \( \xi = \ell \) as output, and this in an example of a transmission decomposition.

**Remark 8.** Making a different choice of input and output signals results in a different map from the input to the output, i.e., a different input/state/output representation, with possibly widely different properties. However, the physical system, i.e., the LC-transmission line with length \( \ell \), is still the same. This “input/output-free” paradigm is inherent in the state/signal philosophy.
5. The connection to boundary triplets

Boundary triplets originate from the extension theory of symmetrical operators on Hilbert spaces. The following definition is adapted from [Gorbachuk and Gorbachuk, 1991, pp. 154–155], using the more recent terminology and notations from [Derkach et al., 2006, Def. 5.1].

**Definition 9.** Let \( R \) be a closed densely defined symmetric operator on the Hilbert space \( X \) with equal (finite or infinite) defect numbers \( n_\pm := \dim \ker (R \mp i) \). Let \( \mathcal{U} \) be another Hilbert space, the “external Hilbert space”, and let \( \Gamma_j, j = 0, 1 \), be linear operators mapping \( \text{dom}(R^*) \) into \( \mathcal{U} \).

The triplet \((\Gamma_0, \Gamma_1; \mathcal{U})\) is called a **boundary triplet** for the operator \( R^* \) if the following two conditions hold:

1. For all \( x_1, x_2 \in \text{dom}(R^*) \) we have
   \[
   (R^*x_1, x_2)_X - (x_1, R^*x_2)_X = (\Gamma_0 x_1, \Gamma_1 x_2)_\mathcal{U} - (\Gamma_1 x_1, \Gamma_0 x_2)_\mathcal{U}.
   \]

2. The range of the combined operator \( \Gamma := \begin{bmatrix} \Gamma_0 & i \Gamma_1 \end{bmatrix} \) is \([\mathcal{U} \, \mathcal{U}]\).

Here condition (1) is the **Lagrangian identity** and condition (2) can be interpreted as a regularity condition or a (hyper)maximality condition.

By a direct-sum decomposition \( W = \mathcal{U} + \mathcal{V} \) of a Krein space we mean that \( \mathcal{U} \) and \( \mathcal{V} \) are closed subspaces of \( W \), such that \( \mathcal{U} + \mathcal{V} = W \) and \( \mathcal{U} \cap \mathcal{V} = \{0\} \). This decomposition is **Lagrangian** if \( \mathcal{U} \) and \( \mathcal{V} \) are both Lagrangian subspaces: \( \mathcal{U} = \mathcal{U}_1 \) and \( \mathcal{V} = \mathcal{V}_1 \). For every Hilbert space \( \mathcal{U} \), the direct-sum decomposition

\[
\mathcal{W} = \begin{bmatrix} \mathcal{U} & \{0\} \end{bmatrix} \oplus \begin{bmatrix} \{0\} & \mathcal{U} \end{bmatrix}
\]

of \( \mathcal{W} = \mathcal{U}^2 \) is Lagrangian if \( \mathcal{W} \) has the inner product

\[
[[u_1, y_2], [v_1, w_2]]_{\mathcal{W}} = (u_1, y_2)_\mathcal{U} + (y_1, w_2)_\mathcal{U}.
\]

For instance, the impedance decomposition in the transmission line example, where we take the currents as input and voltages as outputs, is a Lagrangian decomposition.

For a proof of the following result, see [Malinen and Staffans, 2007, Sec. 5]:

**Theorem 10.** Let \( R \) be a closed and densely defined symmetric operator on \( X \) with equal defect numbers, and let \((\Gamma_0, \Gamma_1; \mathcal{U})\) be a boundary triplet for \( R^* \). Take \( \mathcal{W} := \begin{bmatrix} \mathcal{U} & \{0\} \end{bmatrix} \) with the indefinite inner product (5.2) and define \( \Gamma := \begin{bmatrix} \Gamma_0 & i \Gamma_1 \\ \{0\} & \{0\} \end{bmatrix} \) with \( \text{dom}(\Gamma) = \text{dom}(R^*) \).

Then \( \Sigma = (iR^*, \Gamma; X, \mathcal{W}) \) is a boundary control s/s system in the sense of Definition 3. The system is moreover conservative: \( V = V_1 \), where \( V \) is given by (2.3).

Consider the conservative boundary control s/s system \( \Sigma \) in Theorem 10. The **input/state/output representation**

\[
\Sigma_{i/s/o} = \left( iR^*, \begin{bmatrix} \Gamma_0 & i \Gamma_1 \\ \{0\} & \{0\} \end{bmatrix}, X, \begin{bmatrix} \mathcal{U} & \{0\} \\ \{0\} & \mathcal{U} \end{bmatrix} \right)
\]
corresponding to the Lagrangian decomposition (5.1) is an example of an impedance representation of Σ. We investigate these concepts in more detail in Section ?? of Chapter ??.

The converse of Theorem 10 is not true: there do exist conservative boundary control s/s systems which are not induced by any boundary triplet of the type in Definition 9. These examples are of two types:

1. The signal space \( W \) need not have a Lagrangian decomposition. A necessary and sufficient condition for the existence of a Lagrangian decomposition is that \( \text{ind}_+ W = \text{ind}_- W (\leq \infty) \); see Example 11 below. In the case of a boundary triplet we always have at least the Lagrangian decomposition (5.1).

2. Even if the signal space \( W \) has a Lagrangian decomposition the main operator \( L \) need not be closed, and we can thus not have \( L = iR^\ast \). Moreover, the operator \( \Gamma := \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix} \) need not be surjective. See [Malinen and Staffans, 2007] for an example.

More precisely, let \( \Sigma = (L, \Gamma; \mathcal{X}, \mathcal{W}) \) be a conservative boundary control s/s system. According to [Kurula et al., 2010, Prop. 4.5], \( L \) is closed if and only if the range of \( \Gamma \) is closed. Combining this with the condition that \( \Gamma \) has dense range, we obtain that \( L \) is closed if and only if \( \Gamma \) is surjective. The same conclusion can be made based on Prop. 2.3 and Cor. 2.4 of [Derkach et al., 2006].

We now give an example of a conservative boundary control s/s system that is not induced by a boundary triplet. In a scattering setting this system has no input and a one-dimensional output, and the \( C_0 \)-semigroup describing the system dynamics is the left shift in \( L^2(\mathbb{R}^+; \mathbb{C}) \).

Example 11. Choose \( \mathcal{X} := L^2(\mathbb{R}^+; \mathbb{C}) \) with its standard Hilbert-space inner product, set \( \mathcal{W} := -\mathbb{C} \), and define

\[
V := \left\{ \begin{bmatrix} \frac{dx}{dt} \\ x \end{bmatrix} \mid x \in W^{1,2}(\mathbb{R}^+; \mathbb{C}) \right\} \subset \mathcal{X} \times \mathcal{X} \times \mathcal{W}.
\]

It is clear that \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V \) implies that \( z = 0 \), and we will now show that \( V = V^{[-1]} \), i.e., that \( (V; \mathcal{X}, \mathcal{W}) \) is a conservative boundary control s/s system. Note that the signal space \( \mathcal{W} \) has no Lagrangian decompositions.
We first prove that $V^\perp \subset V$. By definition $\begin{bmatrix} \tilde{z} \\ \tilde{x} \\ \tilde{w} \end{bmatrix} \in \mathcal{R} = L^2(\mathbb{R}^+; \mathbb{C}) \times L^2(\mathbb{R}^+; \mathbb{C}) \times \mathbb{C}$ and for all $x \in W^{1,2}(\mathbb{R}^+; \mathbb{C})$:
\begin{equation}
\begin{bmatrix} \tilde{z} \\ \tilde{x} \\ \tilde{w} \end{bmatrix} \in \mathcal{V}^\perp \quad \text{if and only if} \quad \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} 
= -\tilde{w} x(0) - \int_0^\infty \left( \tilde{x}(\xi) \frac{dx}{d\xi}(\xi) + \tilde{z}(\xi) x(\xi) \right) \, d\xi = 0.
\end{equation}
In particular, if we let $x$ vary over the set of test functions in $C^\infty$ with support contained in $(0, \infty)$, then $x(0) = 0$, and we find that $\frac{dx}{d\xi} = \tilde{z}$ in the distribution sense. Since both $\tilde{x}$ and $\tilde{z}$ belong to $L^2(\mathbb{R}^+; \mathbb{C})$, this implies that $\tilde{x} \in W^{1,2}(\mathbb{R}^+; \mathbb{C})$. This makes it possible to integrate by parts in (5.3), using that $\tilde{z}(\xi) = \frac{dx}{d\xi}(\xi)$, in order to get that
$$\tilde{w} x(0) = \tilde{x}(0) x(0), \quad x \in W^{1,2}(\mathbb{R}^+; \mathbb{C}).$$
Thus $\tilde{w} = \tilde{x}(0)$, and this proves that $V^\perp \subset V$.

In order to show that $V \subset V^\perp$, we choose $\tilde{x} \in W^{1,2}(\mathbb{R}^+; \mathbb{C})$ arbitrarily, and we set $\tilde{z} := \frac{dx}{d\xi}$ and $\tilde{w} := \tilde{x}(0)$. Then (5.3) holds for all $x, \tilde{x} \in W^{1,2}(\mathbb{R}^+; \mathbb{C})$, i.e., $V \subset V^\perp$. We are done proving that $V = V^\perp$, and therefore, that $(V; \mathcal{X}, \mathcal{W})$ is a conservative boundary control s/s system whose signal space $\mathcal{W} = -\mathcal{C}$ has no Lagrangian decompositions.

The i/s/o case where $\Gamma = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix} : \mathcal{X} \to \mathcal{U}^2$ has dense but non-closed range has been treated using generalized boundary triplets in [Derkach and Malamud, 1995] and using quasi boundary triplets in [Behrndt and Langer, 2007]. Interconnection of conservative boundary control i/s/o systems with surjective $\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$ was worked out in detail in [Kurula et al., 2010].

A considerably more general notion than that of a boundary triplet is that of a boundary relation which was extensively studied in e.g. [Derkach et al., 2006]. The topic of Chapter ??, which is more detailed than he present one, is to show how boundary relations are connected to general (non-boundary control) s/s systems. There the main point is to show that the notion of a boundary relation is connected to the notion of a conservative state/signal system in the same way as the boundary triplet is related to the boundary control s/s system: the former arises as a particular i/s/o impedance representation of the latter.

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