Multiobjective linguistic optimization

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Abstract

Generalizing our earlier results on optimization with linguistic variables [3, 6, 7] we introduce a novel statement of fuzzy multiobjective mathematical programming problems and provide a method for finding a fair solution to these problems. Suppose we are given a multiobjective mathematical programming problem in which the functional relationship between the decision variables and the objective functions is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part consists of a linguistic value of the objective functions. We suggest the use of Tsukamoto’s fuzzy reasoning method to determine the crisp functional relationship between the decision variables and objective functions. We model the anding of the objective functions by a t-norm and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy multiobjective problem.

1 Introduction

Fuzzy multiobjective optimization problems can be stated and solved in many different ways (for good surveys see [5, 12, 16, 17, 20]). Usually the authors consider optimization problems of the form

$$\max / \min \{ f_1(x), \ldots, f_K(x) \}; \text{ subject to } x \in X,$$

where $f_k$, $k = 1, \ldots, K$, or/and $X$ are defined by fuzzy terms. Then they are searching for a crisp $x^*$ which (in a certain) sense maximizes the $f_k$’s under the (fuzzy) constraints $X$. For example, fuzzy multiobjective linear programming (FMOLP) problems can be stated as

$$\max / \min \{ \tilde{c}_1x, \ldots, \tilde{c}_Kx \}; \text{ subject to } \tilde{A}x \leq \tilde{b},$$

(1)

where $x \in \mathbb{R}^n$ is the vector of crisp decision variables, $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{b} = (\tilde{b}_i)$ and $\tilde{c}_j = (\tilde{c}_{ij})$ are fuzzy quantities, the inequality relation $\leq$ is given by a certain fuzzy relation and the (implicit) $X$ is a fuzzy set describing the concept "$x$ satisfies $Ax \leq b". In many important cases (e.g. in strategy formation processes) the values of the objective functions are not known for all $x \in \mathbb{R}^n$, and we are able to describe the causal link between $x$ and the $f_k$’s linguistically using fuzzy if-then rules. In this paper we consider constrained fuzzy optimization problems of the form

$$\max / \min \{ f_1(x), \ldots, f_K(x) \}; \text{ subject to } \{ R_1(x), \ldots, R_m(x) \mid x \in X \}$$

(2)

where where $x_1, \ldots, x_n$ are linguistic variables, $X \subseteq \mathbb{R}^n$ is a (crisp or fuzzy) set of constraints on the domain of $x_1, \ldots, x_n$, and

$$R_i(x) : \text{ if } x_1 \text{ is } A_{i1} \text{ and } \ldots \text{ and } x_n \text{ is } A_{in} \text{ then } f_1(x) \text{ is } C_{i1} \text{ and } \ldots \text{ and } f_K(x) \text{ is } C_{iK},$$

constitutes the only knowledge available about the values of \( f_k(x) \), and \( A_{ij} \) and \( C_{ik} \) are fuzzy numbers. In our earlier work [3] we interpreted FMOLP problems (1) with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes, where the antecedents of the scheme correspond to the constraints of the FMOLP problem and the facts of the scheme are the objectives of the FMOLP problem. Generalizing the fuzzy reasoning approach introduced in [6, 7] we determine the crisp value of the \( f_k \)'s at \( y \in X \) by Tsukamoto’s fuzzy reasoning method [18], and obtain an optimal solution to (2) by solving the resulting (usually nonlinear) optimization problem

\[
\max / \min \text{t-norm} (f_1(y), \ldots, f_K(y)); \text{ subject to } y \in X. \tag{3}
\]

We will illustrate the proposed method by a simple example.

2 Multiobjective optimization under fuzzy if-then rules

Consider the FMOP problem (2) with continuous \( A_{ij} \) representing the linguistic values of \( x_i \), and with strictly monotone and continuous \( C_{ik}, i = 1, \ldots, m \) representing the linguistic values of \( f_k, k = 1, \ldots, K \). To find a fair solution to the fuzzy optimization problem (2) we first determine the crisp value of the \( k \)-th objective function \( f_k \) at \( y \in \mathbb{R}^n \) from the fuzzy rule base \( \mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_m\} \) using Tsukamoto’s fuzzy reasoning method as

\[
f_k(y_1, \ldots, y_n) := \frac{\alpha_1 C^{-1}_{1k}(\alpha_1) + \cdots + \alpha_m C^{-1}_{mk}(\alpha_m)}{\alpha_1 + \cdots + \alpha_m}
\]

where

\[
\alpha_i = \text{t-norm}(A_{i1}(y_1), \ldots, A_{in}(y_n))
\]

denotes the firing level of the \( i \)-th rule, \( \mathcal{R}_i, i = 1, \ldots, m \). To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function).

In this manner the constrained optimization problem (2) turns into the crisp (usually nonlinear) mathematical programming problem (3).

Example 2.1 Consider the optimization problem

\[
\max \{f_1(x), f_2(x)\} \tag{4}
\]

subject to \( 0 \leq x_1, x_2 \leq 1, \ x_1 + x_2 = 3/4, \)

where

\[
\mathcal{R}_1(x) : \text{if } x_1 \text{ is small and } x_2 \text{ is small then } f_1(x_1, x_2) \text{ is small and } f_2(x_1, x_2) \text{ is big,} \\
\mathcal{R}_2(x) : \text{if } x_1 \text{ is small and } x_2 \text{ is big then } f_1(x_1, x_2) \text{ is big and } f_2(x_1, x_2) \text{ is small,}
\]

and the universe of discourse for the linguistic values of \( f_1 \) and \( f_2 \) is also the unit interval \([0,1]\) (see Figure 1). We will compute the firing levels of the rules by the product t-norm. Let the membership functions in the rule base \( \mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2\} \) be defined by

\[
\text{small}(y) = 1 - y, \ \text{big}(y) = y.
\]

Let \( y_1, y_2 \in [0, 1] \) be an input to the fuzzy system. Then the firing levels of the rules are

\[
\alpha_1 = (1 - y_1)(1 - y_2), \quad \alpha_2 = (1 - y_1)y_2,
\]
It is clear that if \( y_1 = 1 \) then no rule applies because \( \alpha_1 = \alpha_2 = 0 \). So we can exclude the value \( y_1 = 1 \) from the set of feasible solutions.

The individual rule outputs are

\[
\begin{align*}
z_{11} &= 1 - (1 - y_1)(1 - y_2), \\
z_{21} &= (1 - y_1)y_2, \\
z_{12} &= (1 - y_1)(1 - y_2), \\
z_{22} &= 1 - (1 - y_1)y_2,
\end{align*}
\]

and, therefore, the overall system outputs are

\[
\begin{align*}
f_1(y) &= \frac{(1 - y_1)(1 - y_2)(1 - (1 - y_1)(1 - y_2)) + (1 - y_1)y_2(1 - y_1)y_2}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} \\
&= y_1 + y_2 - 2y_1y_2, \\
f_2(y) &= \frac{(1 - y_1)(1 - y_2)(1 - y_1)(1 - y_2) + (1 - y_1)y_2(1 - (1 - y_1)y_2)}{(1 - y_1)(1 - y_2) + (1 - y_1)y_2} \\
&= 1 - (1 - y_1 + y_2 - 2y_1y_2).
\end{align*}
\]

Modeling the anding of the objective functions by the minimum t-norm our original fuzzy problem (4) turns into the following crisp nonlinear mathematical programming problem

\[
\begin{align*}
\min \{ y_1 + y_2 - 2y_1y_2, 1 - (y_1 + y_2 - 2y_1y_2) \} \rightarrow \max \\
\text{subject to } y_1 + y_2 = 3/4, \ 0 \leq y_1 < 1, \ 0 \leq y_2 \leq 1.
\end{align*}
\]

which has the optimal solutions \((1/2, 1/4)\) and \((1/4, 1/2)\), and its optimal value is \((1/2, 1/2)\).

Remark 2.1 We can introduce trade-offs among the objective functions by using an OWA-operator in (5). However, as Yager has pointed out in [19], constrained OWA-aggregations are not easy to solve, because they usually lead to a mixed integer programming problem of very high dimension.

The rules represent our knowledge-base for the fuzzy optimization problem. The fuzzy partitions for linguistic variables will not usually satisfy \( \varepsilon \)-completeness, normality and convexity. In many cases we have only a few (and contradictory) rules. Therefore, we can not make any preselection procedure to remove the rules which do not play any role in the optimization problem. All rules should be considered when we derive the crisp values of the objective function. We have chosen Tsukamoto’s fuzzy reasoning scheme, because the individual rule outputs are crisp numbers, and therefore, the functional relationship between the input vector \( y = (y_1, \ldots, y_n)^T \) and the system outputs \((f_1(y), \ldots, f_K(y))\) can be relatively easily identified (the only thing we have to do is to perform inversion operations).

3 Refining the fuzzy rule base

We can refine the fuzzy rule base by introducing new linguistic variables modeling the linguistic dependencies between the variables and the objectives [1, 2, 4, 11, 15]. Namely, suppose we have the following piece of knowledge

\[
\text{if } \ldots \ 'x_j \text{ is big}' \ \text{and} \ \ldots \ 'f_k(x) \text{ is big}' \ \text{and} \ \ldots
\]
and we have seen from examples that if \( x_j \) is increasing then the value of \( f_k \) is also increasing. Then this new piece of knowledge can be included in the rule base as an additional rule of the form \([9, 10]\)

\[
\text{if \ldots \text{'the more } x_j \text{ is big' and \ldots then \ldots \text{'the more } f_k(x) \text{ is big' and \ldots}}
\]

where 'the more \( x_j \) is big' is interpreted as '\( x_j \) is very big' and 'the more \( f_k(x) \) is big' is defined as

'\( f_k(x) \) is very big', where 'very big' is a new linguistic value of the linguistic variables \( x_j \) and \( f_k \). In other words, \( x_j \) supports the \( k \)-th objective function if \( x_j \) is big.

It can also happen that if \( x_j \) is getting bigger then the value of \( f_k \) is becoming bigger but after a certain threshold the trend changes and \( f_k \) starts to decrease. In this case we can refine the rule base by adding the following two rules (see Fig.2)

\[
\begin{align*}
\text{if \ldots '} x_j \text{ is medium big' and \ldots then \ldots '} f_k(x) \text{ is very big' and \ldots} \\
\text{if \ldots '} x_j \text{ is very big' and \ldots then \ldots '} f_k(x) \text{ is big' and \ldots}
\end{align*}
\]

where '\( x_j \) is medium big' is a new value for the linguistic variable \( x_j \) and its membership function is defined according to the observed (finite) patterns of \( f_k(x) \)

Figure 2: Refinement of the fuzzy rule base.

4 Summary

We have introduced a novel statement of multiobjective fuzzy mathematical programming problems and provided a method for finding a fair solution to these problems. We addressed multiobjective mathematical programming problems in which the functional relationship between the decision variables and the objective functions are not completely known. Our knowledge-base is assumed to consist only of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part consists of a linguistic value of the objective functions. We suggested the use of Tsukamoto’s fuzzy reasoning method to determine the crisp functional relationship between the decision variables and objective functions, and to solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem. Further refinements of the fuzzy rule base (by introducing interdependencies directly among the objective functions) will be the subject of our future research.

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References


