Stability in possibilistic linear programming with continuous fuzzy number parameters

Mario Fedrizzi
fedrizzi@cs.unitn.it

Robert Fullér
rfuller@abo.fi

Abstract

We prove that possibilistic linear programming problems (introduced by Buckley in [2]) are well-posed, i.e. small changes of the membership function of the parameters may cause only a small deviation in the possibilistic distribution of the objective function.

1 Introduction

We consider certain possibilistic linear programming problems, which have been introduced in [2]. In contrast to classical linear programming (where a small error of measurement may produce a large variation in the objective function), we show that the possibility distribution of the objective function of a possibilistic linear program with continuous fuzzy number parameters is stable under small perturbations of the parameters. First, in this section, we will briefly review possibilistic linear programming and set up notations.

A fuzzy number is a fuzzy set \( \tilde{a}, \tilde{a} : \mathbb{R} \to [0,1] = I \), which is normal, continuous, fuzzy convex and compactly supported. The fuzzy numbers will represent the continuous possibility distributions for fuzzy parameters. A possibility linear programming is

\[
\max_{/\min} Z = x_1 \tilde{c}_1 + \cdots + x_n \tilde{c}_n,
\]

subject to \( x_1 \tilde{a}_{i1} + \cdots + x_n \tilde{a}_{in} \ast \tilde{b}_i, \ 1 \leq i \leq m, \ x \geq 0 \).

where \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j \) are fuzzy numbers, \( x = (x_1, \ldots, x_n) \) is a vector of (nonfuzzy) decision variables, and \( \ast \) denotes \( <, \leq, =, \geq \) or \( > \) for each \( i \).

We will assume that all fuzzy numbers \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j \) are non-interactive [11].

Following Buckley [2], we define \( \text{Poss}[Z = z] \), the possibility distribution of the objective function \( Z \). We first specify the possibility that \( x \) satisfies the \( i \)-th constraints. Let

\[
\Pi(a_i, b_i) = \min\{\tilde{a}_{i1}(a_{i1}), \ldots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)\},
\]

where \( a_i = (a_{i1}, \ldots, a_{in}) \), which is the joint distribution of \( \tilde{a}_{ij}, j = 1, \ldots, n \), and \( \tilde{b}_i \). Then

\[
\text{Poss}[x \in F_i] = \sup_{a_i, b_i} \{ \Pi(a_i, b_i) \mid a_{i1}x_1 + \cdots + a_{in}x_n \ast b_i \},
\]

which is the Possibility that \( x \) is feasible with respect to the \( i \)-th constraint. Therefore, for \( x \geq 0 \),

\[
\text{Poss}[x \in \mathcal{F}] = \min_{1 \leq i \leq m} \text{Poss}[x \in \mathcal{F}_i],
\]

which is the Possibility that \( x \) is feasible. We next construct \( \text{Poss}[Z = z|x] \) which is the conditional possibility that \( Z \) equals \( z \) given \( x \). The joint distribution of the \( \tilde{c}_j \) is

\[
\Pi(c) = \min\{\tilde{c}_1(c_1), \ldots, \tilde{c}_n(c_n)\}
\]

where \( c = (c_1, \ldots, c_n) \). Therefore,

\[
\text{Poss}[Z = z|x] = \sup_c \{\Pi(c)|c_1x_1 + \cdots + c_n x_n = z\}.
\]

Finally, applying Bellman and Zadeh’s method of fuzzy decision making [1], the Possibility distribution of the objective function is defined as follows

\[
\text{Poss}[Z = z] = \sup_{x \geq 0} \min_{x \in \mathcal{F}} \{\text{Poss}[Z = z|x], \text{Poss}[x \in \mathcal{F}]\}
\]

It should be noted that Buckley [3] showed that the solution to an appropriate linear program gives the correct \( z \) values in \( \text{Poss}[Z = z] = \alpha \) for each \( \alpha \in I \).

Let \( \tilde{a} \) be a fuzzy number. Then for any \( \theta \geq 0 \) we define \( \omega(\tilde{a}, \theta) \), the modulus of continuity of \( \tilde{a} \) as

\[
\omega(\tilde{a}, \theta) = \max_{|u-v| \leq \theta} |\tilde{a}(u) - \tilde{a}(v)|.
\]

The following statements hold [8]:

\[
\text{If } 0 \leq \theta \leq \theta' \text{ then } \omega(\tilde{a}, \theta) \leq \omega(\tilde{a}, \theta') \quad (2)
\]

\[
\text{If } \alpha > 0, \beta > 0, \text{ then } \omega(\tilde{a}, \alpha + \beta) \leq \omega(\tilde{a}, \alpha) + \omega(\tilde{a}, \beta). \quad (3)
\]

\[
\lim_{\theta \to 0} \omega(\tilde{a}, \theta) = 0 \quad (4)
\]

An \( \alpha \)-level set of a fuzzy numer \( \tilde{a} \) is a non-fuzzy set denoted by \( [\tilde{a}]^{\alpha} \) and is defined by

\[
[\tilde{a}]^{\alpha} = \begin{cases} 
\{t \in \mathbb{R} | \tilde{a}(t) \geq \alpha\} & \text{if } \alpha > 0 \\
\text{cl}(\text{supp} \tilde{a}) & \text{if } \alpha = 0
\end{cases}
\]

where \( \text{cl}(\text{supp} \tilde{a}) \) denotes the closure of the support of \( \tilde{a} \).

If \( \tilde{a} \) and \( \tilde{b} \) are fuzzy numbers and \( \lambda \in \mathbb{R} \) then \( \tilde{a} + \lambda \tilde{b}, \tilde{a} - \lambda \tilde{b}, \lambda \tilde{a} \) are defined by Zadeh’s extension principle [12] in the usual way.

Let \( \tilde{a} \) and \( \tilde{b} \) be fuzzy numbers and

\[
[\tilde{a}]^{\alpha} = [a_1(\alpha), a_2(\alpha)], \quad [\tilde{b}]^{\alpha} = [b_1(\alpha), b_2(\alpha)].
\]

It is well known that

\[
[\tilde{a} + \tilde{b}]^{\alpha} = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]. \quad (5)
\]

It is easy to see that

\[
\text{Poss}[\tilde{a} \ast \tilde{b}] = \sup_{x+y} \min \{\tilde{a}(x), \tilde{b}(y)\} = \sup_{t \in 0} (\tilde{a} - \tilde{b})(t) \quad (6)
\]

where \( \ast \) stands for \( <, \leq, =, \geq \) or \( > \). We metricize the set of fuzzy numbers by the metric [9]

\[
D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max_{i=1,2} [a_i(\alpha) - b_i(\alpha)],
\]

2
2 Auxiliary propositions

Lemma 2.1 [9]. Let \( \tilde{a}, \tilde{b}, \tilde{c} \) and \( \tilde{d} \) be fuzzy numbers. Then

\[
D(\lambda \tilde{a}, \lambda \tilde{b}) = |\lambda|D(\tilde{a}, \tilde{b}), \quad \text{for } \lambda \in \mathbb{R},
\]

\[
D(\tilde{a} + \tilde{c}, \tilde{b} + \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d})
\]

\[
D(\tilde{a} - \tilde{c}, \tilde{b} - \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d})
\]

Lemma 2.2 Let \( \lambda \neq 0, \mu \neq 0 \) be real numbers and let \( \tilde{a} \) and \( \tilde{b} \) be fuzzy numbers. Then

\[
\omega(\lambda \tilde{a}, \theta) = \omega(\tilde{a}, \theta/|\lambda|), \quad \text{for } \theta \geq 0.
\]

Proof. From the equation \((\lambda \tilde{a})(t) = \tilde{a}(t/\lambda)\) for \( t \in \mathbb{R} \) we have

\[
\omega(\lambda \tilde{a}, \theta) = \max_{|u-v| \leq \theta} |(\lambda \tilde{a})(u) - (\lambda \tilde{a})(v)| = \max_{|u-v| \leq \theta} |\tilde{a}(u/\lambda) - \tilde{a}(v/\lambda)| = \max_{|u-v/\lambda| \leq \theta/|\lambda|} |\omega(u/\lambda) - \omega(v/\lambda)| = \omega(\tilde{a}, \theta/|\lambda|),
\]

which proves (7). As to (8), let \( \theta > 0 \) be arbitrary and \( u, t \in \mathbb{R} \) such that \( |u - t| \leq \theta \). Then with the notations \( \tilde{c} := \lambda \tilde{a}, \tilde{d} := \mu \tilde{b} \) we need to show that

\[
|(\tilde{c} + \tilde{d})(u) - (\tilde{c} + \tilde{d})(t)| \leq \omega(\theta/|\lambda|).
\]

We assume without loss of generality that \( t < u \). From (5) it follows that there are real numbers \( t_1, t_2, u_1, u_2 \) with the properties

\[
t = t_1 + t_2, \quad u = u_1 + u_2, \quad t_1 \leq u_1, t_2 \leq u_2
\]

\[
\tilde{c}(t_1) = \tilde{d}(t_2) = (\tilde{c} + \tilde{d})(t),
\]

\[
\tilde{c}(u_1) = \tilde{d}(u_2) = (\tilde{c} + \tilde{d})(u).
\]

Since from (7) we have

\[
|\tilde{c}(u_1) - \tilde{c}(t_1)| \leq \omega(\tilde{a}, |u_1 - t_1|/|\lambda|)
\]

and

\[
|\tilde{d}(u_2) - \tilde{d}(t_2)| \leq \omega(\tilde{b}, |u_2 - t_2|/|\mu|),
\]

it follows by (8), (9), (10), (11) that

\[
|(\tilde{c} + \tilde{d})(u) - (\tilde{c} + \tilde{d})(t)| \leq min\{\omega(\tilde{a}, |u_1 - t_1|/|\lambda|), \omega(\tilde{b}, |u_2 - t_2|/|\mu|)\} \leq \omega(\theta/|\lambda|).
\]
Lemma 2.3 Let \( \tilde{a} \) be a fuzzy number and \([\tilde{a}]^\alpha = [a_1(\alpha), a_2(\alpha)]\). Then \( a_1 : [0, 1] \to \mathbb{R} \) is strictly increasing and

\[
\forall t \in \text{cl}(\text{supp} \tilde{a}), \quad \tilde{a}(a_1) = \alpha,
\]

for \( \alpha \in [0, 1] \),

\[
a_1(\tilde{a}(t)) \leq t \leq a_1(\tilde{a}(t) + 0),
\]

for \( a_1(0) \leq t < a_1(1) \), where

\[
a_1(\tilde{a}(t) + 0) = \lim_{\epsilon \to +0} a_1(\tilde{a}(t) + \epsilon). \tag{12}
\]

Lemma 2.4 Let \( \tilde{a} \) and \( \tilde{b} \) be fuzzy numbers. Then

(i) \( D(\tilde{a}, \tilde{b}) \geq |a_1(\alpha + 0) - b_1(\alpha + 0)| \), for \( 0 \leq \alpha < 1 \),

(ii) \( \tilde{a}(a_1(\alpha + 0)) = \alpha \), for \( 0 \leq \alpha < 1 \),

(iii) \( a_1(\alpha) \leq a_1(\alpha + 0) < a_1(\beta) \), for \( 0 \leq \alpha < \beta \leq 1 \).

Proof. (i) From the definition of the metric \( D \) we have

\[
|a_1(\alpha + 0) - b_1(\alpha + 0)| = \lim_{\epsilon \to +0} |a_1(\alpha + \epsilon) - b_1(\alpha + \epsilon)| = \lim_{\epsilon \to +0} |a_1(\alpha + \epsilon) - b_1(\alpha + \epsilon)| \leq \sup_{\gamma \in I} |a_1(\gamma) - b_1(\gamma)| \leq D(\tilde{a}, \tilde{b}).
\]

(ii) Since \( \tilde{a}(a_1(\alpha + \epsilon)) = \alpha + \epsilon \), for \( \epsilon \leq 1 - \alpha \), we have

\[
\tilde{a}(a_1(\alpha + 0)) = \lim_{\epsilon \to +0} A(a_1(\alpha + \epsilon)) = \lim_{\epsilon \to +0} (\alpha + \epsilon) = \alpha.
\]

(iii) From strictly monotony of \( a_1 \) it follows that \( a_1(\alpha + \epsilon) < a_1(\beta) \), for \( \epsilon < \beta - \alpha \). Therefore,

\[
a_1(\alpha) \leq a_1(\alpha + 0) = \lim_{\epsilon \to +0} a_1(\alpha + \epsilon) < a_1(\beta),
\]

which completes the proof.

Lemma 2.5 Let \( \delta \geq 0 \) and let \( \tilde{a}, \tilde{b} \) be fuzzy numbers. If \( D(\tilde{a}, \tilde{b}) \leq \delta \), then

\[
\sup_{t \in I} |\tilde{a}(t) - \tilde{b}(t)| \leq \max\{\omega(\tilde{a}, \delta), \omega(\tilde{b}, \delta)\}. \tag{13}
\]
Proof. Let \( t \in \mathbb{R} \) be arbitrarily fixed. It will be sufficient to show that
\[
|\hat{a}(t) - \hat{b}(t)| \leq \max \{\omega(\hat{a}, \delta), \omega(\hat{b}, \delta)\}.
\]
If \( t \in \text{supp} \hat{a} \cup \text{supp} \hat{b} \) then we obtain (13) trivially. Suppose that \( t \in \text{supp} \hat{a} \cup \text{supp} \hat{b} \). With no loss of
generality we will assume \( 0 \leq \hat{b}(t) < \hat{a}(t) \). Then either of the following must occur:
\[
\begin{align*}
(a) & \quad t \in (b_1(0), b_1(1)), \\
(b) & \quad t \leq b_1(0), \\
(c) & \quad t \in (b_2(1), b_2(0)) \\
(d) & \quad t \geq b_2(0).
\end{align*}
\]
In this case of (a) from Lemma 2.4 (with \( \alpha = \hat{b}(t), \beta = \hat{a}(t) \)) and Lemma 2.3(iii) it follows that
\[
\hat{a}(a_1(\hat{b}(t) + 0)) = \hat{b}(t), \quad t \geq a_1(\hat{a}(t)) \geq a_1(\hat{b}(t) + 0)
\]
and
\[
D(\hat{a}, \hat{b}) \geq |a_1(\hat{b}(t) + 0) - a_1(\hat{b}(t) + 0)|.
\]
Therefore from continuity of \( \hat{a} \) we get
\[
|\hat{a}(t) - \hat{b}(t)| = |\hat{a}(t) - \hat{a}(a_1(\hat{b}(t) + 0))| = \omega(\hat{a}, |t - a_1(\hat{b}(t) + 0)|) = \omega(\hat{a}, t - a_1(\hat{b}(t) + 0)) \leq \omega(\hat{a}, b_1(\hat{b}(t) + 0) - a_1(\hat{b}(t) + 0)) \leq \omega(\hat{a}, \delta).
\]
In this case of (b) we have \( \hat{b}(t) = 0 \); therefore from Lemma 2.3(i) it follows that
\[
|\hat{a}(t) - \hat{b}(t)| = |\hat{a}(t) - 0| = |\hat{a}(t) - \hat{a}(a_1(0))| \leq \omega(\hat{a}, |t - a_1(0)|) \leq \omega(\hat{a}, b_1(0) - a_1(0)) \leq \omega(\hat{a}, \delta).
\]
A similar reasoning yields in the cases of (c) and (d); instead of properties \( a_1 \) we use the properties of \( a_2 \).

3 Stability theorem

An important question [4, 7, 13] is the influence of the perturbations of the fuzzy parameters to the
Possibility distribution of the objective function. We will assume that there is a collection of fuzzy
parameters \( \hat{a}_{ij}^\delta \), \( \hat{b}_i^\delta \), \( \hat{c}_j^\delta \) available with the property
\[
\max_{i,j} D(\hat{a}_{ij}, \hat{a}_{ij}^\delta) \leq \delta, \quad \max_i D(\hat{b}_i, \hat{b}_i^\delta) \leq \delta, \quad \max_j D(\hat{c}_j, \hat{c}_j^\delta) \leq \delta. \tag{14}
\]
Then we have to solve the following problem:
\[
\max / \min \ Z^\delta = x_1 \hat{c}_1^\delta + \cdots + x_n \hat{c}_n^\delta \tag{15}
\]
subject to \( x_1 \hat{a}_{1i}^\delta + \cdots + x_n \hat{a}_{ni}^\delta \leq b_i^\delta \), \( 1 \leq i \leq m \), \( x \geq 0 \).

Let us denote by Possx \( x \in \mathcal{F}_x^\delta \) the Possibility that \( x \) is feasible with respect to the \( i \)-th constraint in (15).
Then the Possibility distribution of the objective function \( Z^\delta \) in (15) is defined as follows:
\[
\text{Poss}[Z^\delta = z] = \sup_{x \geq 0} (\min \{\text{Poss}[Z^\delta = z \mid x], \text{Poss}[x \in \mathcal{F}_x^\delta] \}).
\]
The next theorem shows a stability property (with respect to perturbations (14) of the Possibility distribution of the objective function of the Possibilistic linear programming problems (1) and (15).
Theorem 3.1 Let \( \delta \geq 0 \) be a real number and let \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{a}_i^\delta, \tilde{c}_j, \tilde{c}_j^\delta \) be continuous fuzzy numbers. If (14) hold, then

\[
\sup_{z \in \mathbb{R}} \left| \text{Poss}[Z^\delta = z] - \text{Poss}[Z = z] \right| \leq \omega(\delta)
\]

(16)

where

\[
\omega(\delta) = \max_{i,j} \{ \omega(\tilde{a}_{ij}, \delta), \omega(\tilde{a}_i^\delta, \delta), \omega(\tilde{b}_i, \delta), \omega(\tilde{c}_j, \delta), \omega(\tilde{c}_j^\delta, \delta) \}.
\]

Proof. It is sufficient to show that

\[
| \text{Poss}[Z = z \mid x] - \text{Poss}[Z^\delta = z \mid x] | \leq \omega(\delta), \ z \in \mathbb{R},
\]

(17)

\[
| \text{Poss}[x \in F] - \text{Poss}[x \in F^\delta] | \leq \omega(\delta),
\]

(18)

for each \( x \in \mathbb{R} \) and \( 1 \leq i \leq m \), because (16) follows from (17) and (18). We shall prove only (18), because the proof of (17) is carried out analogously.

Let \( x \in \mathbb{R} \) and \( i \in \{1, \ldots, m\} \) arbitrarily fixed. From (6) it follows that

\[
\text{Poss}[x \in F_i] = \sup_{t \geq 0} \left( \sum_{j=1}^{n} \tilde{a}_{ij} x_j - B_i(t) \right),
\]

\[
\text{Poss}[x \in F_i^\delta] = \sup_{t \geq 0} \left( \sum_{j=1}^{n} \tilde{a}_i^\delta x_j - B_i(t) \right).
\]

Applying Lemma 2.1 and (4) and (14) we have

\[
D \left( \sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i, \sum_{j=1}^{n} \tilde{a}_i^\delta x_j - \tilde{b}_i^\delta \right) \leq \delta (|x|_1 + 1),
\]

where \( |x|_1 = |x_1| + \ldots + |x_n| \).

By Lemma 2.2 we get

\[
\max \left\{ \omega \left( \sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i, \delta \right), \omega \left( \sum_{j=1}^{n} \tilde{a}_i^\delta x_j - \tilde{b}_i^\delta, \delta \right) \right\} \leq \omega \left( \frac{\delta}{|x|_1 + 1} \right).
\]

Finally, applying Lemma 2.5 we have

\[
| \text{Poss}[x \in F_i] - \text{Poss}[x \in F_i^\delta] | =
\]

\[
| \sup_{t \geq 0} \left( \sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i(t) \right) - \sup_{t \geq 0} \left( \sum_{j=1}^{n} \tilde{a}_i^\delta x_j - \tilde{b}_i^\delta(t) \right) | \leq \delta (|x|_1 + 1),
\]

\[
\text{sup}_{t \geq 0} \left| \sum_{j=1}^{n} \tilde{a}_{ij} x_j - \tilde{b}_i(t) \left( \sum_{j=1}^{n} \tilde{a}_i^\delta x_j - \tilde{b}_i^\delta(t) \right) \right| \leq \delta (|x|_1 + 1).
\]
\[
\sup_{t \in \mathbb{R}} \left| \left( \sum_{j=1}^{n} \tilde{a}_{ij}x_j - \tilde{b}_i(t) \right) - \left( \sum_{j=1}^{n} \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta(t) \right) \right| \leq \omega\left( \frac{\delta(|x|_1 + 1)}{|x|_1 + 1} \right) = \omega(\delta),
\]
which proves the theorem.

**Remark 3.1** From (4) and (16) it follows that
\[
\sup_{z \in \mathbb{R}} | \text{Poss}[Z^\delta = z] - \text{Poss}[Z = z] | \to 0 \text{ as } \delta \to 0
\]
which means the stability of the possibility distribution of the objective function with respect to perturbations (14).

**Remark 3.2** As an immediate consequence of this theorem we obtain the following result (see [5]): If the fuzzy numbers in (1) and (15) satisfy the Lipschitz condition with constant \( L > 0 \), then
\[
\sup_{z \in \mathbb{R}} | \text{Poss}[Z^\delta = z] - \text{Poss}[Z = z] | \leq L \delta
\]
Furthermore, similar estimations can be obtained in the case of symmetrical trapezoidal fuzzy number parameters (see [11]) and in the case of symmetrical triangular fuzzy number parameters (see [6, 10]).

**Remark 3.3** It is easy to see that in the case of non-continuous fuzzy parameters the possibility distribution of the objective function may be unstable under small changes of the parameters.

**References**


4 Follow ups

The results of this paper have been improved and generalized later in the following works:

http://dx.doi.org/10.1016/j.ins.2010.11.021


http://dx.doi.org/10.1080/03081070108960695

http://dx.doi.org/10.1007/s11766-000-0010-y


Further investigations have been performed for stability of fuzzy linear systems and fuzzy linear programming problems in [A27], [A34], [A31], [8], [A35].

http://dx.doi.org/10.1016/0165-0114(94)90333-6

The approach deals with imprecise parameters treated as fuzzy variables with possibility distributions assigned to them in the form of fuzzy numbers. A fuzzy number is a fuzzy set which is normal, continuous, fuzzy convex and compactly supported [A27].
in proceedings and in edited volumes


The criterion of stability based on uniform metric is given in [A27, 6].


in books

