On computation of the compositional rule of inference under triangular norms

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Abstract

This paper is devoted to the derivation of exact calculation formulas for the compositional rule of inference under Archimedean t-norms, when both the observation and the relation parts are given by Hellendoorn’s $\phi$-function [6].

Keywords: Triangular norm, compositional rule of inference, Lagrange’s multipliers method,

1 Introduction

The inference process from imprecise or vague premises is becoming more and more important for knowledge-based systems, especially for fuzzy expert systems [9,12,13,15]. In approximate reasoning there are several kinds of inference rules, which deal with the problem of deduction of conclusions in an imprecise setting. An important problem is the (approximate) computation of the membership function of the conclusion in these schemes [1, 4, 6, 7, 8, 10]. This paper deals with the derivation of exact calculation formulas for the compositional rule of inference, which has the global scheme [14]

Observation: $x$ has property $P$
Relation: $x$ and $y$ have relation $W$
Conclusion: $y$ has property $Q$

where the membership function of the conclusion $Q$ is defined by a sup-t-norm composition of $P$ and $W$

$$Q(y) = \sup\text{t-norm } (P(x), W(x, y))$$

In [6] Hellendoorn showed the closure property of the compositional rule of inference under sup-min composition and presented exact calculation formulas for the membership function of the conclusion when both the observation and relation parts are given by $S$-, $\pi$-, or $\phi$-function. Our results are connected with those presented in [6] and we generalize them. We shall determine the exact membership function of the conclusion, when both the observation and the part of the relation (rule) are given by concave $\phi$-function [6]; and the t-norm is Archimedean with a strictly convex additive generator function. The efficiency of our method stems from the fact that the distributions, involved in the relation and observation, are represented by a parametrized $\phi$-function. The deduction process then consists of some simple computations performed on the parameters.

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2 Definitions

A fuzzy interval \( \tilde{a} \) is a fuzzy set of real numbers \( \mathbb{R} \) with a continuous, compactly supported, unimodal and normalized membership function. It is known \([2]\), that any fuzzy interval \( \tilde{a} \) can be described by the following membership function:

\[
\tilde{a}(x) = \begin{cases} 
L \left( \frac{a - t}{\alpha} \right) & \text{if } t \in [a - \alpha, a] \\
1 & \text{if } t \in [a, b] \\
R \left( \frac{t - b}{\beta} \right) & \text{if } t \in [b, b + \beta] \\
0 & \text{otherwise}
\end{cases}
\]

where \([a, b]\) is the peak of \( \tilde{a} \); \( a \) and \( b \) are the lower and upper modal values; \( L \) and \( R \) are shape functions: \([0, 1] \rightarrow [0, 1] \), with \( L(0) = R(0) = 1 \) and \( L(1) = R(1) = 0 \) which are non-increasing, continuous mappings. These fuzzy intervals are called of type L-R and denoted by \( \tilde{a} = (a, b, \alpha, \beta)_{LR} \). The support of \( \tilde{a} \) is \([a - \alpha, b + \beta]\).

A function \( T : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is said to be a triangular norm \([11]\) (t-norm for short) iff \( T \) is symmetric, associative, non-decreasing in each argument, and \( T(x, 1) = x \) for all \( x \in [0, 1] \). Recall that a t-norm \( T \) is Archimedean if its additive generator \( f \) is continuous and \( T(x, x) < x \) for all \( x \in (0, 1) \). Every Archimedean t-norm \( T \) is representable by a continuous and decreasing function \( f : [0, 1] \rightarrow [0, \infty] \) with \( f(1) = 0 \) and

\[
T(x, y) = f^{-1}(f(x) + f(y))
\]

where \( f^{-1} \) is the pseudo-inverse of \( f \), defined by

\[
f^{-1}(y) = \begin{cases} 
f^{-1}(y) & \text{if } y \in [0, f(0)] \\
0 & \text{otherwise}
\end{cases}
\]

The function \( f \) is the additive generator of \( T \). If \( T \) is a t-norm and \( \tilde{a}_1, \tilde{a}_2 \) are fuzzy sets of the real line (i.e. fuzzy quantities) then their \( T \)-sum \( \tilde{A} := \tilde{a}_1 + \tilde{a}_2 \) is defined by

\[
\tilde{A}(z) = \sup_{x_1 + x_2 = z} T(\tilde{a}_1(x_1), \tilde{a}_2(x_2)), \quad z \in \mathbb{R}
\]

which can be written in the form

\[
\tilde{A}(z) = f^{-1}(f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2)))
\]

supposing that \( f \) is the additive generator of \( T \).

Since \( f \) is continuous and decreasing, \( f^{-1} \) is also continuous and non-increasing, we have

\[
\tilde{A}(z) = f^{-1}\left( \inf_{x_1 + x_2 = z} (f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2))) \right).
\]

3 Auxiliary lemma

**Lemma 3.1** Let \( T \) be an Archimedean t-norm with additive generator \( f \) and let \( \tilde{a}_i = (a_i, b_i, \alpha, \beta)_{LR} \), \( i = 1, 2 \) be fuzzy intervals of type L-R. If \( L \) and \( R \) are twice differentiable, concave functions, and \( f \) is a twice
As mentioned above, the investigated membership function is
differentiable, strictly convex function, then the membership function of their T-sum \( \tilde{A} = \tilde{a}_1 + \tilde{a}_2 \) is

\[
\tilde{A}_2(z) = \begin{cases} 
1 & \text{if } A \leq z \leq B \\
 f[-1] \left( 2 \times f \left( \frac{A - z}{2\alpha} \right) \right) & \text{if } A - 2\alpha \leq z \leq A \\
 f[-1] \left( 2 \times f \left( \frac{z - B}{2\beta} \right) \right) & \text{if } B \leq z \leq B + 2\beta \\
0 & \text{otherwise}
\end{cases}
\]

where \( A = a_1 + a_2 \) and \( B = b_1 + b_2 \).

**Proof.** As mentioned above, the investigated membership function is

\[
\tilde{A}(z) = f[-1] \left( \inf_{x_1 + x_2 = z} \left( f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2)) \right) \right).
\]

(1)

It is easy to see that the support of \( \tilde{A} \) is included in the interval \([A - 2\alpha, B + 2\beta]\). From the decomposition rule of fuzzy intervals into two separate parts [3] it follows that the peak of \( \tilde{A} \) is \([A, B]\). Moreover, if we consider the right hand side of \( \tilde{A} \) (i.e. \( z \leq x \leq B + 2\beta \)) then only the right hand sides of terms \( \tilde{a}_1 \) and \( \tilde{a}_2 \) become relevant in (1) (i.e. \( b_i \leq x \leq b_i + \beta, \ i = 1, 2 \)). The same holds for the left hand side of \( \tilde{A} \) (i.e. \( A - 2\alpha \leq z \leq B \)). Let us now consider the right hand side of \( \tilde{A} \), so let \( B \leq z \leq B + 2\beta \). (A similar method can be used for the computation of \( \tilde{A}(z) \), if \( z \in [A - 2\alpha, A] \).) The constraints

\[
x_1 + x_2 = z, \quad b_i \leq x_i \leq b_i + \beta, \ i = 1, 2,
\]

determine a compact and convex domain \( \mathcal{K} \subset \mathbb{R}^2 \) which is the intersection of the rectangle,

\[
\mathcal{B} := \{(x_1, x_2) \in \mathbb{R}^2 | b_i \leq x_i \leq b_i + \beta, \ i = 1, 2, \}
\]

and the hyperplane

\[
\mathcal{P} := \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 = z \}.
\]

In order to determine \( \tilde{A}(z) \), we have to solve the following mathematical programming problem:

\[
\phi(x_1, x_2) := f(\tilde{a}_1(x_1)) + f(\tilde{a}_2(x_2)); \quad \text{subject to } (x_1, x_2) \in \mathcal{K}.
\]

Since \( \mathcal{K} \) is compact and \( \phi \) is continuous, the infimum can be changed to minimum.

(1) (for the complete proof see: Fuzzy Sets and Systems, 51(1992) 267-275)

It should be noted that Lemma 1 can be extended to the case of \( n \) fuzzy intervals [5].

### 4 Computation of the compositional rule of inference

Following [6] we use the \( \phi \)-function for the representation of linguistic notions

\[
\phi(x; a, b, c, d) = \begin{cases} 
1 & \text{if } b \leq x \leq c \\
\phi_1 \left( \frac{x - a}{b - c} \right) & \text{if } a \leq x \leq b, \ a < b, \\
\phi_2 \left( \frac{x - c}{d - c} \right) & \text{if } c \leq x \leq d, \ c < d, \\
0 & \text{otherwise}
\end{cases}
\]
where $\phi_1 : [0, 1] \to [0, 1]$ is continuous, monoton increasing function and $\phi_1(0) = 0$, $\phi_1(1) = 1$; 
$\phi_2 : [0, 1] \to [0, 1]$ is continuous, monoton decreasing function and $\phi_2(0) = 1$, $\phi_2(1) = 0$ So $\phi$ is a function which is 0 left of $a$, increases to 1 in $(a, b)$, is 1 in $[b, c]$, decreases to 0 in $(c, d)$ and is 0 right of $d$ (for the sake of simplicity, we do not consider the cases $a = b$ or $c = d$). It should be noted that $\phi$ can be considered as the membership function of the fuzzy interval $\tilde{a} = (b, c, b - a, d - c)_{LR}$, with $R(x) = \phi_2(x)$ and $L(x) = \phi_1(1 - x)$.

We consider the compositional rule of inference,

\[
\begin{align*}
\text{Observation:} & \quad x \quad \text{has property } P \\
\text{Relation:} & \quad x \quad \text{and } y \quad \text{have relation } W \\
\text{Conclusion:} & \quad y \quad \text{has property } Q
\end{align*}
\]

where, the membership functions of $P$ and $W$ are defined by means of a particular $\phi$-function, and the membership function of the conclusion $Q$ is defined by sup-$T$ composition of $P$ and $W$

\[Q(y) = (P \circ_T W)(y) = \sup_x T(P(x), W(x, y)).\]

The following theorem presents an efficient method for the exact computation of the membership function of the conclusion.

**Theorem 4.1** Let $T$ be an Archimedean t-norm with additive generator $f$ and let $P(x) = \phi(x; a, b, c, d)$ and $W(x, y) = \phi(y - x; a + u, b + u, c + v, d + v)$. If $\phi_1$ and $\phi_2$ are twice differentiable, concave functions, and $f$ is a twice differentiable, strictly convex function, then

\[
Q(y) = \begin{cases} 
1 & \text{if } 2b + u \leq y \leq 2c + v \\
2f(\phi_1(\frac{y - 2a - u}{2(b - a)})) & \text{if } 2a + u \leq y \leq 2b + u \\
2f(\phi_2(\frac{y - 2c - v}{2(d - c)})) & \text{if } 2c + v \leq y \leq 2d + v \\
0 & \text{otherwise}
\end{cases}
\]


**Remark 4.1** It should be noted that we have calculated the membership function of $Q$ under the assumption that the left and right spreads of $P$ do not differ from the left and right spreads of $W$ (the lengths of their tops can be different). To determine the exact membership function of $Q$ in the general case: $P(x) = \phi(x; a_1, a_2, a_3, a_4)$ and $W(x, y) = \phi(y - x; b_1, b_2, b_3, b_4)$ seems to be very difficult and it is possible, that there does not exist an explicit formula for it.

## 5 Applications

Using Theorem 1 we compute the exact membership function of the conclusion $Q$ in the case of Yager’s, Dombi’s and Hamacher’s parametrized t-norm. Let us consider the following scheme

\[
\begin{align*}
P(x) &= \phi(x; a, b, c, d) \\
W(y, x) &= \phi(y - x; a + u, b + u, c + v, d + v) \\
Q(y) &= (P \circ_T W)(y)
\end{align*}
\]
Denoting
\[ \sigma := \frac{(y - 2a - u)}{2(b - a)}, \quad \theta := \frac{y - 2c - v}{2(d - c)}, \]
we get the following formulas for the membership function of the conclusion \( Q \).

(i) Yager's t-norm with \( p > 1 \). Here
\[ T(x, y) = 1 - \min\left\{ 1, \sqrt[p]{(1 - x)^p + (1 - y)^p} \right\}. \]
with generator \( f(t) = (1 - t)^p \), and
\[ Q(y) = \begin{cases} 1 - 2^{1/p}(1 - \phi_1(\sigma)) & \text{if } 0 < \sigma < \phi_1^{-1}(2^{-1/p}), \\ 1 & \text{if } 2b + u \leq y \leq 2c + v, \\ 1 - 2^{1/p}(1 - \phi_2(\theta)) & \text{if } 0 < \theta < \phi_2^{-1}(2^{-1/p}), \end{cases} \]
(ii) Hamacher's t-norm with \( p \leq 2 \). Here
\[ T(x, y) = \frac{xy}{p + (1 - p)(x + y - xy)} \]
with generator
\[ f(t) = \ln \frac{p + (1 - p)t}{t}, \]
and
\[ Q(y) = \begin{cases} p/(\tau_1^2 - 1 + p) & \text{if } 0 < \sigma < 1, \\ 1 & \text{if } 2b + u \leq y \leq 2c + v, \\ p/(\tau_2^2 - 1 + p) & \text{if } 0 < \theta < 1, \end{cases} \]
where
\[ \tau_1 = \frac{p + (1 - p)\phi_1(\sigma)}{\phi_1(\sigma)}, \quad \tau_2 = \frac{p + (1 - p)\phi_2(\sigma)}{\phi_2(\sigma)} \]
(iii) Dombi's t-norm with \( p > 1 \). Here
\[ T(x, y) = \frac{1}{1 + \sqrt[p]{(1/x - 1)^p + (1/y - 1)^p}} \]
with additive generator
\[ f(t) = \left( \frac{1}{t} - 1 \right)^p, \]
and
\[ Q(y) = \begin{cases} 1/(1 + 2^{1/p}\tau_3) & \text{if } 0 < \sigma < 1, \\ 1 & \text{if } 2b + u \leq y \leq 2c + v, \\ 1/(1 + 2^{1/p}\tau_4) & \text{if } 0 < \theta < 1, \end{cases} \]
where
\[ \tau_3 = \frac{1}{\phi_1(\sigma)} - 1, \quad \tau_4 = \frac{1}{\phi_2(\sigma)} - 1 \]

**Example 1** We illustrate Theorem 1 by the following example:
\[
\begin{array}{ll}
x \text{ is close to } 3 & \phi(x; 1, 3, 4, 7) \\
x \text{ and } y \text{ are approximately equal} & \phi(y - x; -2, 0, 0, 3) \\
y \text{ is more or less close to } [3, 4] & Q(y)
\end{array}
\]

where

\[
Q(y) = \begin{cases} 
1 & \text{if } 2b + u \leq y \leq 2c + v \\
0 & \text{otherwise}
\end{cases}
\]

\[
f^{-1}\left(2f\phi_1\left(y - \frac{2a - u}{2(b - a)}\right)\right) & \text{if } 2a + u \leq y \leq 2b + u \\
f^{-1}\left(2f\phi_2\left(y - \frac{2c - v}{2(d - c)}\right)\right) & \text{if } 2c + v \leq y \leq 2d + v
\]

We have used the membership function \(\phi(y - x; -2, 0, 0, 3)\) to describe "\(x \text{ and } y \text{ are approximately equal}\". This means that the membership degree is one, iff \(x \text{ and } y\) are equal in the classical sense. If \(y - x > 2\) or \(x - y > 3\), then the degree of membership is 0. The conclusion \(Q\) has been called "\(y \text{ is more or less close to } [3, 4]\)”, because \(P(t) = Q(T) = 1\), when \(t \in [3, 4]\) and \(P(t) < Q(t)\) otherwise.

References


6 Follow ups

The results of this paper have been improved and/or generalized in the following works.

in journals

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There are many works that have proposed modifications to the classical CRI in an attempt to enhance the efficiency in its inferencing (see, for example, the works of Fullér and coauthors [48]-[50] and Moser and Navara [51]-[53]). (page 295)

A26-c25 B. Jayaram, On the Law of Importation \((x \land y) \rightarrow z \equiv (x \rightarrow (y \rightarrow z))\) in Fuzzy Logic, IEEE Transactions on Fuzzy Systems, vol.16, no.1, pp.130-144, Feb. 2008
http://dx.doi.org/10.1109/TFUZZ.2007.895969

http://dx.doi.org/10.1016/j.ins.2007.03.023

A26-c23 Hong DH, T-sum of bell-shaped fuzzy intervals, FUZZY SETS AND SYSTEMS, 158 (7): 739-746 APR 1 2007
http://dx.doi.org/10.1016/j.fss.2006.10.021

A26-c22 Hong DH On types of fuzzy numbers under addition KYBERNETIKA, 40 (4): 469-476 2004


A26-c20 Hong DH, On shape-preserving additions of fuzzy intervals, JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS, 267 (1): 369-376, MAR 1 2002
In particular if \( \alpha_1 = \alpha_2, \beta_1 = \beta_2 \) then \( \tilde{A}^* = \tilde{A}_1 \oplus_T \tilde{A}_2 \), which generalizes the results by Fullér and Zimmermann [A24, A26].

Now, in the case of different spreads, we are naturally asked about how to determine the exact membership function. Fullér and Zimmermann [A26] mentioned in their remark 1 that it seems to be very difficult and complicated to determine the exact membership function of \( T \)-sum of LR-fuzzy numbers. (page 241)

We also consider the compositional rule of inference under triangular norms stated by Fullér and Zimmermann [A26]. (page 252)
The aim of this paper is to provide a close upper bound of the membership function for the compositional rule of inference under Archimedean t-norm, where both the observation and the relation parts are given by Hellendoorn’s φ-function (1980). In particular, if the left and right spreads of the observation part is the same as those of the relation part, then this upper bound is the exact membership function, which generalizes the earlier results by Fullér and Zimmermann (1992) in that the assumption of twice differentiability is deleted. (page 25)

Fullér and Zimmermann [A26] wrote a paper which deals with the derivation of exact calculation formulas for the compositional rule of inference, which has the global scheme [7]

Observation: \( x \) has property \( P \)

Relation: \( x \) and \( y \) have relation \( W \)

Conclusion: \( y \) has property \( Q \)

where the membership function of the conclusion \( Q \) is defined by sup-t-norm composition of \( P \) and \( W \):

\[
Q(y) = \sup_x \text{t-norm}(P(x), W(x, y)).
\]

(page 26)

In [6] Hellendoorn showed the closure property of the compositional rule of inference under sup-min-norm composition and presented exact calculation formulas for the membership function of the conclusion when both the observation and relation parts are given by \( S-, \pi-, \) or \( \phi- \)functions. Fullér and Zimmermann’s results are connected with those presented in [6] and they generalize them as follows:

**Theorem 1.1** [A26] Let \( T \) be an Archimedean t-norm with additive generator \( f \) and let \( P(x) = \phi(x; a, b, c, d) \) and \( W(x, y) = \phi(y - x; a + u, b + u, c + v, d + v) \). If \( \phi_1 \) and \( \phi_2 \) are twice differentiable, concave functions, and \( f \) is a twice differentiable, strictly convex function, then

\[
Q(y) = \begin{cases}
1 & \text{if } 2b + u \leq y \leq 2c + v \\
\frac{f^{-1}}{2f} \left( \frac{\phi_1 \left( \frac{y - 2a - u}{2(b - a)} \right)}{\phi_2 \left( \frac{y - 2c - v}{2(d - c)} \right)} \right) & \text{if } 2a + u \leq y \leq 2b + u \\
\frac{f^{-1}}{2f} \left( \frac{\phi_2 \left( \frac{y - 2c - v}{2(d - c)} \right)}{\phi_2 \left( \frac{y - 2c - v}{2(d - c)} \right)} \right) & \text{if } 2c + v \leq y \leq 2d + v \\
0 & \text{otherwise.}
\end{cases}
\]

(page 26)

From Theorem 2.4 we know that if the left and right spreads of \( P \) are equal to the left and right spreads of \( W \), respectively, then the exact membership function of the conclusion \( Q \) can be determined without the condition of differentiability of \( \phi_1, \phi_2 \) and \( f \) in Fullér and Zimmermann’s theorem [A26]. (page 29)

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