Riccati equations and optimal control for
infinite-dimensional linear systems

Kalle M. Mikkola*
Helsinki University of Technology
Institute of Mathematics
P.O. Box 1100; FIN-02015 HUT, Finland
kalle.mikkola@iki.fi
http://www.math.hut.fi/~kmikkola

Olof Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
Olof.Staffans@abo.fi
http://www.abo.fi/~staffans/

Abstract

We generalize the standard theory on algebraic
Riccati equations and optimization to infinite-
dimensional well-posed linear systems, thus
completing the work of George Weiss, Olof
Staffans and others. We show that the optimal
control is given by the stabilizing solution of an
integral Riccati equation. If, e.g., the input op-
erator is not maximally unbounded, then this in-
tegral Riccati equation is equivalent to an alge-
braic Riccati equation.

Our theory covers all quadratic (possibly in-
definite) cost functions, but the optimal state
feedback need not be well-posed unless the cost
function is uniformly positive or the system is
sufficiently regular. If one allows controls that
do not stabilize the state, just the output, then
the definition of the stabilizing solution becomes
more complicated. We treat this and some
other phenomena that are met also in the finite-
dimensional setting but more important in the
infinite-dimensional one.

A linear time-invariant system is typically
governed by

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

(for \( t \geq 0 \), \( x(0) = x_0 \), where the generat-
ors \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(H \times U, H \times Y) \) are matrices, or
more generally, linear operators between Hilbert
spaces \((U, H, Y)\) of arbitrary dimensions.

Given \( x_0 \) and \( u \), the state \( x \) and output \( y \) equal

\[
\begin{aligned}
x(t) &= \mathcal{A}^t x_0 + \mathcal{B}^t u, \\
y &= \mathcal{C} x_0 + \mathcal{D} u, \\
\mathcal{A}^t &= e^{At}, \\
\mathcal{B}^t u &= \int_0^t \mathcal{A}^{t-s} Bu(s) \, ds,
\end{aligned}
\]

\((\mathcal{C} x_0)(t) = C \mathcal{A}^t x_0, \quad (\mathcal{D} u)(t) = C \mathcal{B}^t u + Du(t)\).

We study Well-Posed Linear Systems
(WPLSs) (“Salamon–Weiss class”), i.e.,
time-invariant systems of form (1), with
\( \mathcal{A}^t, \mathcal{B}^t, \mathcal{C}^t, \mathcal{D} \) linear, bounded, compatible with
each other and continuous on \( H \times \mathcal{L}^2_{\text{loc}} \). It fol-
low that \( \mathcal{A}^t \) is a \( C_0 \)-semigroup and \( A, B, C \) exist
to satisfy \( \dot{x} = Ax + Bu \) (and \( y = Cx \) when \( u = 0 \)),
but \( A, B, C \) may be unbounded. Such systems
are equivalent to Lax–Phillips scattering sys-
tems and to the operator-based models of Béla
Sz.-Nagy and Ciprian Foiaș [S04]. If also \( D \)
e exists, then the WPLS is called regular. [M04a]

**Theorem 1** If B is bounded \((B \in \mathcal{B}(U, H))\), then
the following are equivalent:

(i) For each initial state \( x_0 \in H \), there exists a
unique control \( u : \mathbb{R}_+ \to U \) that minimizes

\[
\mathcal{J}(x_0, u) := \int_0^\infty \left( \| x(t) \|_H^2 + \| u(t) \|_U^2 \right) \, dt.
\]

*Corresponding author
(ii) The algebraic Riccati equation (ARE)
\[ \mathcal{P}B^* \mathcal{P} = A^* \mathcal{P} + \mathcal{P}A + I \]
has an exponentially stabilizing solution.

(iii) For each \( x_0 \in H \), there exists \( u \in L^2 \) such that \( J(x_0, u) < \infty \).

Any solution \( \mathcal{P} \) of (ii) is unique, and the (state-feedback) control \( u(t) = Kx(t) \) strictly minimizes the cost \( J(x_0, \cdot) \) for any \( x_0 \in H \). Moreover, the minimal cost equals \( \langle x_0, \mathcal{P}x_0 \rangle_H \).

In fact, a solution of (3) is exponentially stabilizing if \( B \) is nonnegative.

As above, for any initial state \( x_0 \in H \), we want to minimize the cost \( J(x_0, \cdot) \) over
\[ \mathcal{U}_{\text{exp}}(x_0) := \{ u \in L^2(\mathbb{R}_+; U) \mid x \in L^2(\mathbb{R}_+; H) \} \]
the set of exponentially stabilizing controls, as in Theorem 1. To cover more general cost functions, we allow for any \( J \in \mathcal{B}(Y) \) in

\[ J(x_0, u) := \int_0^\infty \langle y(t), J y(t) \rangle_y \, dt. \]  \hspace{1cm} (4)

(Take \( C := \begin{bmatrix} 0 \\ I \end{bmatrix} \), \( D := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), \( J := \begin{bmatrix} I & 0 \end{bmatrix} \) get (2.).

Now we can generalize the above result for arbitrary cost functions; for simplicity, we require the indicator (or signature operator) \( S := D^* J D \) to be uniformly positive:

**Theorem 2** If \( D^* J D \geq \varepsilon I \) for some \( \varepsilon > 0 \) and \( B \) is bounded, then the following are equivalent:

(i) There is a unique minimizing control over \( \mathcal{U}_{\text{exp}}(x_0) \) for each initial state \( x_0 \in H \).

(ii) The ARE

\[ \begin{align*}
K^* SK &= A^* \mathcal{P} + \mathcal{P}A + C^* J C, \\
S &= D^* J D, \\
K &= -S^{-1}(B^* \mathcal{P} + D^* J C),
\end{align*} \]  \hspace{1cm} (5)

has an exponentially stabilizing solution.

Any solution \( \mathcal{P} \) of (ii) is unique, and the state feedback \( u(t) = Kx(t) \) (a.e.) minimizes (4). The minimal cost equals \( \langle x_0, \mathcal{P}x_0 \rangle_H \).

\[
\begin{array}{c}
\text{Figure 1: State feedback connection}
\end{array}
\]

(Here \( \mathcal{K}_1 := \lim_{s \to +\infty} Ks(s-A)^{-1} \). We have \( \mathcal{K} = K \) if \( K \) is bounded (e.g., if \( C \) is).) The equations are given on Dom(\( A \)), \( U \) and Dom(\( A \)).

Under an external disturbance \( u_\varepsilon : \mathbb{R}_+ \to U \) (i.e., \( u = \mathcal{K}x + u_\varepsilon \)), we get \( J(x_0, u) = \langle x_0, \mathcal{P}x_0 \rangle + \langle u_\varepsilon, y_\varepsilon \rangle \).

Next we give up the boundedness of \( B \) but require that the system is regular (i.e., that the transfer function has a weak limit at \( +\infty \)):

**Theorem 3** For any regular WPLS, the following are equivalent:

(i) There is an optimal regular state-feedback operator \( K \in \mathcal{B}(\text{Dom}(A), U) \);

(ii) The ARE

\[ \begin{align*}
K^* SK &= A^* \mathcal{P} + \mathcal{P}A + C^* J C, \\
S &= D^* J D + \lim_{s \to +\infty} \overline{B^* \mathcal{P}}(s-A)^{-1} B, \\
SK &= -(\overline{B^* \mathcal{P}} + D^* J C),
\end{align*} \]  \hspace{1cm} (6)

has an exponentially stabilizing solution.

Any solution \( \mathcal{P} \) of (ii) is unique, and the state feedback \( u(t) = \mathcal{K}x(t) \) (a.e.) is optimal. For this control, the cost is given by \( J(x_0, u) = \langle x_0, \mathcal{P}x_0 \rangle_H \).

(Here \( \overline{B^*} := \lim_{s \to +\infty} B^* s(s-A)^{-1} \).) By \( u \) being optimal we mean that \( \frac{d}{du} J(x_0, u) \) is Fréchet derivative). The control \( u \) is minimizing iff \( S \geq 0 \). (The indefinite case gives, e.g., the maximin control of a \( H^\infty \) problem.) The optimal control is unique iff the indicator \( S \) is one-to-one.
If, e.g., the input operator $B$ is not maximally unbounded ($\|(s-A)^{-1}B\| \leq Ms^{-1/2-\varepsilon}$ for all $s > M$), then $S = D^* JD$ and any optimal state-feedback is regular.

However, in general a regular WPLS may have an irregular optimal control, and not all WPLSs are regular. Thus, to cover all WPLSs, we must use the IRE instead of the ARE:

**Theorem 4** There is an optimal state feedback iff the following integral Riccati equation (IRE) has an exponentially stabilizing solution:

\[
\mathcal{K}^{-1} S \mathcal{K} = \mathcal{A}^* \mathcal{P} \mathcal{A}^* - \mathcal{P} + \mathcal{C}^* \mathcal{C}, \quad (7a)
\]
\[
\mathcal{X}^* S \mathcal{X} = \mathcal{D}^* \mathcal{J} \mathcal{D} + \mathcal{B}^* \mathcal{P} \mathcal{B}^* \quad \text{for all } t \in [0, T], \quad \text{or equivalently (integrate it over } [0, t]), (7b)
\]
\[
\mathcal{X}^* S \mathcal{X} = -(\mathcal{P} \mathcal{C} \mathcal{C}^* \mathcal{P} + \mathcal{B}^* \mathcal{P} \mathcal{B}^*). \quad (7c)
\]

(Here $\mathcal{C} := \chi(0, t) \mathcal{C}, \mathcal{D} := \mathcal{X}(0, t) \mathcal{D} \mathcal{X}(0, t) \, \chi(0, t)$ etc., where $\chi(0, t)$ is the characteristic function of $[0, t]$, equivalently, the natural projection $L^2_{\text{loc}} \to L^2([0, t]; U)$ (or its adjoint, the embedding.).)

If $B$ is bounded, $C = \begin{bmatrix} \bar{C} \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ \bar{D} \end{bmatrix}$ (hence $\mathcal{I}(x_0, u) = ||u||_2^2 + ||\tilde{C} x||_2^2$), then, by (5), we get $S = I$, $K = -B^* \mathcal{P}$, hence then the ARE reduces to $\mathcal{P} B \mathcal{B}^* \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + \bar{C} \mathcal{C}$, or equivalently (integrate it over $[0, t]$), to (7a), which now becomes

\[
\mathcal{P} x_0 = \mathcal{A}^* \mathcal{P} \mathcal{A} x_0 + \int_0^t \mathcal{A}^* (\bar{C}^* \bar{C} - \mathcal{P} \mathcal{B} B^* \mathcal{P}) \mathcal{A} x \, ds
\]

(8)

(for all $x_0 \in H$), familiar from classical results.

The optimal control equals $u = \mathcal{K}^{-1} \mathcal{K} x_0$ (i.e., $u = \mathcal{K} x_0 + (I - \mathcal{K}) u_+|u| u_+$), with cost $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P} x_0 \rangle + \langle u_+, S u_+ \rangle$.

However, the optimal state feedback may be ill-posed (i.e., $(\mathcal{K}^{-1})^{-1} : u_+ \to u$ or $\mathcal{K}^{-1}$ need not be well-defined on $L^2_{\text{loc}}$). Nevertheless, if there is a unique optimal control $u^*_0$ for each initial state $x_0 \in H$, then the map $\mathcal{K}^* : x_0 \mapsto u^*_0$ and $\mathcal{P}$ form the exponentially stabilizing solution of the $\mathcal{P}^* \mathcal{I}$-IRE, where the left-hand-side of the IRE is replaced by $\mathcal{K}^* \mathcal{P} \mathcal{A}_{\text{opt}} \mathcal{A}^* \mathcal{K}^* \mathcal{P} \mathcal{B}^* \mathcal{P} \mathcal{B}^* \mathcal{P}$.

In fact, the $\mathcal{P}^* \mathcal{I}$-ARE (as well as the IRE) is exactly the discrete-time ARE for the discretized system $[\mathcal{G}_0^*, \mathcal{G}^*]$; this leads to alternative proofs. We also give frequency-domain variants of the IRE and the $\mathcal{P}^* \mathcal{I}$-IRE in [M04a].

The optimal state-feedback is well-posed iff a certain stable spectral factorization problem has a solution. If $\mathcal{J}(0, -)$ is uniformly positive (e.g., as in (2)) and $\mathcal{U}(x_0) \neq 0$ for all $x_0$, then this is the case. This led to the generalization to WPLS of numerous classical results on minimization, state-feedback and dynamic stabilization and coprime factorizations in [M04a].

Similar results also hold for other domains of optimization (admissible controls). E.g., for

\[
\mathcal{U}(x_0) := \{ u \in L^2(\mathbb{R}_+: U) \mid y \in L^2(\mathbb{R}_+: Y) \},
\]

the set of output-stabilizing controls, we must replace “exponentially stabilizing solution” by the (unique) “solution satisfying $u, y \in L^2$ and $\mathcal{G}^* u + \mathcal{G}^* \mathcal{A}^* \mathcal{G}^* x_0, \mathcal{P} \mathcal{A}^* \mathcal{G}^* x_0 \to 0$, as $t \to +\infty$". Fortunately, for the cost function $\mathcal{J} = ||u||_2^2 + ||y||_2^2$ (or with $\mathcal{G}^* u + \mathcal{G}^* \mathcal{A}^* \mathcal{G}^* x_0, \mathcal{P} \mathcal{A}^* \mathcal{G}^* x_0 \to 0$, as $t \to +\infty$". Fortunately, for the cost function $\mathcal{J} = ||u||_2^2 + ||y||_2^2$ (or with $\mathcal{G}^* u + \mathcal{G}^* \mathcal{A}^* \mathcal{G}^* x_0, \mathcal{P} \mathcal{A}^* \mathcal{G}^* x_0 \to 0$, as $t \to +\infty$"

Details, corresponding LQR and $H^\infty$ applications, discrete-time results, more detailed historical remarks etc. are given in [M04a] and [M02]. For bounded $B, C$, many of the above results are well known; see, e.g., [CZ94]. If we neglect the well-posedness of the state feedback and the ARE, most of the results have been established earlier under various assumptions. The well-posedness of the state feedback has been known for Pritchard–Salamon systems, and for several parabolic systems it has been established by, e.g., Irena Lasiecka, Roberto Triggiani and others; see [LT00].

For stable problems, the necessity of the ARE (6) was established by Olof Staffans [S97]; its first and third equations were independently found by Martin Weiss and George Weiss [WW97].

**Reciprocal AREs**

To overcome the difficulties due to unbounded generators, Ruth Curtain and Mark Opmeer have developed the reciprocal RE theory for the LQR problem assuming that $0 \notin \sigma(A)$ [OC04] [C03]. There the bounded operators $A_- := A^{-1}$, $B_- := A^{-1} B$, $C_- := -CA^{-1}$ are used in place of $A, B, C$ — the surprising fact is that the Riccati operator $\mathcal{P}$ remains the same. E.g., the
first equation in the ARE (6), when multiplied by $A^*$ to the left and by $A_-$ the right, becomes

$$K_+S K_- = P A_+ + A_- P + C_- J C_. \quad (9)$$

Here $S = D_+^* J D_-$, $SK_- = -B_+^* P - D_+^* J C_-$ (and $D_- := D + C(0 - A)^{-1}B \in \mathcal{B}(U, Y)$ or the value of the characteristic function at zero). This was generalized to arbitrary WPLSs and cost functions in [M03] (with $(A - \alpha)^{-1}$ in place of $A_-$ and different formulas for $S$ and $K_-$), leading to the first generalization of Theorem 1 and similar results to arbitrary WPLSs; see [M04b]. However the results of [M03] and [M04b] established the well-posedness of the optimal state-feedback pair only for the LQR cost function and similar ones, hence they were later partially shadowed by the IRE methods of [M04a], which provided well-posedness for any uniformly positive cost function. Nevertheless, for several purposes the reciprocal and resolvent AREs provide technically much simpler tools, which have been applied succesfully to the generalization of several results from systems with bounded $B, C$ to general WPLSs; see, e.g. [BMS04]. Moreover, the ARE (9) is sometimes more applicable than the IRE or the ARE (6).

**References**


[OC04] Ruth F. Curtain and Mark R. Opmeer. New Riccati equations for well-posed linear systems. To appear, 2004. (See also their articles on the subject on this same conference CD.)
