LINEAR STATIONARY
INPUT/STATE/OUTPUT AND
STATE/SIGNAL SYSTEMS

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Preface (Nov 15, 2013)

The theory presented in this book arose as a product of a continued collaboration between the two authors during the years 2003–2017. The basis for this collaboration was our common interest in passive linear input/state/output systems theory. At the time when this project started O. Staffans was preparing a joint article [Ball and Staffans, 2006] with Prof. J. Ball which, in particular, explored the connections between conservative input/state/output systems theory on one hand and some results in the behavioral theory introduced by J. Willems in the late 80’s on the other hand. After extensive discussions on the above approach we understood that this opens up a new direction in the study of passive linear stationary systems. We called the new class of systems that arose in this way passive state/signal systems. Our first article [Arov and Staffans, 2005] on state/signal system was completed and submitted for publication in the fall of 2003, and it was followed by many others in the years to follow. The bulk of the work was done during D. Arov’s regular visits to Åbo Akademi during August–October 2003–2010 and to the Aalto University during August–October 2011–2017, with an average length of almost 3 months. These visits were financed by the Academy of Finland, the Magnus Ehrnrooth Foundation, and the Finnish Society of Sciences and Letters.

In the fall of 2009 it was decided that the theory was sufficiently mature to be presented in terms of a book, and the writing of this book begun on August 30, 2009. By the end of November, 2009, a preliminary list of contents was ready. Two significant factors in this decision were the research grant from the Academy of Finland which relieved O. Staffans from teaching duties during the Academic year 2009–10, and the leave of absence for D. Arov for extensive periods of time from the South-Ukrainian Pedagogical University based on a joint exchange agreement with Åbo Akademi.

The book that we originally planned to write was supposed to be devoted to autonomous systems in discrete time. In 2011 we realized that it would be more important to instead write a book on linear stationary systems in continuous time, and in 2013 it was clear that it was not feasible to write only one book on systems in continuous times. The continuous time theory contains a number of mathematical difficulties that must first be sorted out, and this is done in the present volume. The next volume will be devoted primarily to the theory of passive state/signal systems in continuous time.

We thank Academy of Finland, the Magnus Ehrnrooth Foundation, and the Finnish Society of Sciences and Letters for their financial support, without which this work could not have been carried out. We also thank Åbo Akademi and the Aalto University for excellent working facilities, and the South-Ukrainian Pedagogical University for giving D. Arov ample time to devote to research.
Above all we are grateful to our wives Natalyja and Satu-Marjatta for their constant support, understanding, and patience while this work was carried out.
List of Notations (Jan 02, 2016)

Basic Sets and Symbols

\( \mathbb{C} \): The complex plane.

\( \mathbb{C}_+^\omega, \overline{\mathbb{C}}_+^\omega \): 
\( \mathbb{C}_+^\omega := \{ z \in \mathbb{C} \mid \Re z > \omega \} \) and 
\( \overline{\mathbb{C}}_+^\omega := \{ z \in \mathbb{C} \mid \Re z \geq \omega \} \).

\( \mathbb{C}_-^\omega, \overline{\mathbb{C}}_-^\omega \): 
\( \mathbb{C}_-^\omega := \{ z \in \mathbb{C} \mid \Re z < \omega \} \) and 
\( \overline{\mathbb{C}}_-^\omega := \{ z \in \mathbb{C} \mid \Re z \leq \omega \} \).

\( \mathbb{C}^+, \overline{\mathbb{C}}^+ \): 
\( \mathbb{C}^+ := \mathbb{C}_0^+ \) and 
\( \overline{\mathbb{C}}^+ := \overline{\mathbb{C}}_0^+ \).

\( \mathbb{C}^-, \overline{\mathbb{C}}^- \): 
\( \mathbb{C}^- := \mathbb{C}_0^- \) and 
\( \overline{\mathbb{C}}^- := \overline{\mathbb{C}}_0^- \).

\( \mathbb{D}_r^+ \), \( \overline{\mathbb{D}}_r^+ \): 
\( \mathbb{D}_r^+ := \{ z \in \mathbb{C} \mid |z| > r \} \) and 
\( \overline{\mathbb{D}}_r^+ := \{ z \in \mathbb{C} \mid |z| \geq r \} \).

\( \mathbb{D}_r^- \), \( \overline{\mathbb{D}}_r^- \): 
\( \mathbb{D}_r^- := \{ z \in \mathbb{C} \mid |z| < r \} \) and 
\( \overline{\mathbb{D}}_r^- := \{ z \in \mathbb{C} \mid |z| \leq r \} \).

\( \mathbb{D}^+ \), \( \overline{\mathbb{D}}^+ \): 
\( \mathbb{D}^+ := \mathbb{D}_1^+ \) and 
\( \overline{\mathbb{D}}^+ := \overline{\mathbb{D}}_1^+ \).

\( \mathbb{D}^- \), \( \overline{\mathbb{D}}^- \): 
\( \mathbb{D}^- := \mathbb{D}_1^- \) and 
\( \overline{\mathbb{D}}^- := \overline{\mathbb{D}}_1^- \).

\( \mathbb{R} \): 
\( \mathbb{R} := (-\infty, \infty) \).

\( \mathbb{R}^+ \), \( \mathbb{R}^+_1 \): 
\( \mathbb{R}^+ := (0, \infty) \) and 
\( \mathbb{R}^+_1 := [0, \infty) \).

\( \mathbb{R}^- \), \( \mathbb{R}^-_1 \): 
\( \mathbb{R}^- := (-\infty, 0) \) and 
\( \mathbb{R}^-_1 := (-\infty, 0] \).

\( T \): The unit circle in the complex plane.

\( N \): 
\( N \) is the set of natural numbers, i.e., \( N := \{1, 2, 3, \ldots \} \).

\( Z \): 
\( Z \) is the set of all integers, i.e., \( Z := \{ \pm 1, \pm 2, \pm 3, \ldots \} \).

\( Z^+, Z^- \): 
\( Z^+ := \{0, 1, 2, \ldots \} \) and 
\( Z^- := \{-1, -2, -3, \ldots \} \).

\( j \): 
\( j := \sqrt{-1} \).

\( 0 \): The number zero, or the zero vector in a vector space, or the zero operator, or the zero-dimensional vector space \( \{0\} \).

\( 1 \): The number one and also the identity operator.

Operators and Related Symbols

\( A, B, C, D \): In connection with an i/s/o system, \( A \) is usually the main operator, \( B \) the control operator, \( C \) the observation operator and \( D \) a feedthrough operator.

\( \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \): Often \( \mathfrak{A} \) is the semigroup, \( \mathfrak{B} \) is the input map, \( \mathfrak{C} \) is the output map, and \( \mathfrak{D} \) is the input/output map of an well-posed i/s/o linear system. See Definition 8.1.12.
\(\hat{A}, \hat{B}, \hat{C}, \hat{D}:\) Often \(\hat{A}\) is the s/s resolvent, \(\hat{B}\) the i/s resolvent, \(\hat{C}\) is the s/o resolvent, and \(\hat{D}\) is the i/o resolvent of a i/s/o node. See Definitions 3.2.8, 4.2.15, 5.1.4, 5.2.11 and 5.3.22.

\(B(U; \mathcal{Y}), B(U):\) The set of bounded linear operators from \(U\) into \(\mathcal{Y}\) respectively from \(U\) into itself.

\(L(U; \mathcal{Y}), L(U):\) The set of linear operators from \(U\) into \(\mathcal{Y}\) respectively from \(U\) into itself.

\(\mathcal{ML}(U; \mathcal{Y}), \mathcal{ML}(U):\) The set of multi-valued linear operators from \(U\) into \(\mathcal{Y}\) respectively from \(U\) into itself.

\(\tau^t:\) The bilateral shift operator on \(\mathcal{R}: \tau^tu(s) := u(s + t), \; t, s \in \mathcal{R}\) (this is a left-shift when \(t > 0\) and a right-shift when \(t < 0\)).

\(\tau^t_+:\) The left shift operator on \(\mathcal{R}^+: \tau^+_u(s) := u(s + t), \; s \in \mathcal{R}^+.\) Here \(t \in \mathcal{R}^+\).

\(\tau^t_-:\) The right shift operator on \(\mathcal{R}^+: \tau^-_u(s) := u(s - t), \; s \in \mathcal{R}^+.\) Here \(t \in \mathcal{R}^+\).

\(\tau^t_+:\) The left shift operator on \(\mathcal{R}^-: \tau^+_u(s) := u(s - t), \; s \in \mathcal{R}^-\). Here \(t \in \mathcal{R}^+\).

\(\tau^t_-:\) The right shift operator on \(\mathcal{R}^-: \tau^-_u(s) := u(s + t), \; s \in \mathcal{R}^-\). Here \(t \in \mathcal{R}^+\).

\(\iota_I:\) The embedding operator \(L^p_{\text{loc}}(I) \hookrightarrow L^p_{\text{loc}}(\mathcal{R}): (\iota_Iu)(t) := u(t), \; t \in I\) and \((\iota_Iu)(t) := 0, \; t \notin I\). Here \(I \subset \mathcal{R}\).

\(\iota_+, \iota_-:\) \(\iota_+ := \iota_{[0, \infty)}\) and \(\iota_- := \iota_{(-\infty, 0]}\).

\(\rho_I:\) The restriction operator \(L^p_{\text{loc}}(\mathcal{R}) \rightarrow L^p_{\text{loc}}(I): (\rho_Iu)(t) := u(t), \; t \in I\). Here \(I \subset \mathcal{R}\). \(\rho_I\rho_I = 1_{L^p_{\text{loc}}(I)}\) and \(\iota_I\rho_I = \pi_I\).

\(\rho_+, \rho_-:\) \(\rho_+ := \rho_{[0, \infty)}\) and \(\rho_- := \rho_{(-\infty, 0]}\).

\(\pi_I:\) The projection operator in \(L^p_{\text{loc}}(\mathcal{R})\) with range \(L^p_{\text{loc}}(I)\) and kernel \(L^p_{\text{loc}}(\mathcal{R} \setminus I): (\pi_Iu)(s) := u(s)\) if \(s \in I\) and \((\pi_Iu)(s) := 0\) if \(s \notin I\). Here \(I \subset \mathcal{R}\). \(\rho_I\pi_I = \rho_I\) and \(\pi_I\pi_I = \iota_I\).

\(\pi_+, \pi_-:\) \(\pi_+ := \pi_{[0, \infty)}\) and \(\pi_- := \pi_{(-\infty, 0]}\).

\(\text{w-lim:}\) The weak limit in a \(B\)-space. Thus \(\text{w-lim}_{n \to \infty} x_n = x\) in \(\mathcal{X}\) iff \(\text{lim}_{n \to \infty} x^*x_n = x^*x\) for all \(x^* \in \mathcal{X}^*\).

\(\langle x, x^* \rangle:\) In a \(B\)-space setting \(x^*x := \langle x, x^* \rangle\) is the continuous linear functional \(x^*\) evaluated at \(x\). In a Hilbert space setting this is the inner product of \(x\) and \(x^*\).

\(E^\perp:\) \(E^\perp = \{x^* \in \mathcal{X}^* \mid \langle x, x^* \rangle = 0\ \text{for all } x \in E\}\). This is the annihilator of \(E \subset \mathcal{X}\).

\(\perp F:\) \(\perp F = \{x \in \mathcal{X} \mid \langle x, x^* \rangle = 0\ \text{for all } x^* \in F\}\). This is the pseudo-annihilator of \(F \subset \mathcal{X}^*\). In the reflexive case \(\perp F = F^\perp\), and in the non-reflexive case \(\perp F = F^\perp \cap \mathcal{X}\).

\(A^*:\) The (anti-linear) dual of the operator \(A\).

\(A \geq 0:\) \(A\) is (self-adjoint and) positive definite.

\(A \succ 0:\) \(A \geq \epsilon\) for some \(\epsilon > 0\), hence \(A\) is invertible.

\(\text{dom}(A):\) The domain of the (unbounded) operator \(A\).
rng (A): The range of the operator A.
ker (A): The null space (kernel) of the operator A.
mul (A): The multi-valued part of the operator A.
dim (X): The dimension of the space X.
ρ (A): The resolvent set of the operator A (see Definitions 3.1.1 and 5.2.1). The resolvent set is always open.
σ (A): The spectrum of the operator A (see Definitions 3.1.1 and 5.2.1). The spectrum is always closed.
ρi/s/o (S): The i/s/o resolvent set of S (see Definitions 5.1.2 and 5.2.8).
ρ (Σ): The resolvent set of the i/s/o or s/s system Σ (see Definitions 5.1.2, 5.2.8, and 5.3.1).
ω (A): The growth bound of the semigroup A. See (4.1.9).
TI, TIC: TI stands for the set of all shift-invariant, and TIC stands for the set of all shift-invariant and causal operators. See Definition 8.1.37 for details.

Vector Spaces

H-space: A topological vector space $X$ which is isomorphic to a Hilbert space, i.e., the topology in $X$ is induced by a norm induced by a Hilbert space inner product. See Definitions ?? and A.1.6.

B-space: A topological vector space $X$ which is isomorphic to a Banach space, i.e., the topology in $X$ is induced by a Banach space norm. See Definitions 1.1.2.

F-space: A topological vector space $X$ which is isomorphic to a Fréchet space, i.e., the topology in $X$ is induced by a Fréchet space metric. See Definitions 1.1.2.

$U$: Frequently the input space of a i/s/o system.
$X$: Frequently the state space of a i/s/o or s/s system.
$Y$: Frequently the output space of a i/s/o system.
$W$: Frequently the signal space of a s/s system.

$X_\bullet$, $X_\circ$: $X_\bullet$ is the interpolation space and $X_\circ$ is the extrapolation space induced by a closed operator A in $X$ with dense domain. See Definitions 5.1.17 and 5.1.22.

$A_\bullet$, $A_\circ$: $A_\bullet$ is the part of A in $X_\bullet$ and $A_\circ$ is the extension of A to a closed operator in $X_\circ$.

$A_\bullet$, $A_\circ$: $A_\bullet$ is restriction of the $C_0$ semigroup $A$ in $X$ to a $C_0$ semigroup in $X_\bullet$ and $A_\circ$ is the extension of $A\mathfrak{A}$ to a $C_0$ semigroup in $X_\circ$.

$\oplus$: $X = X_1 \oplus X_2$ means that the topological vector space $X$ is the direct sum of $X_1$ and $X_2$, i.e., both $X_1$ and $X_2$ are closed subspaces of $X$, and every $x \in X$ has a unique representation of the form $x = x_1 + x_2$ where $x_1 \in X_1$ and $x_2 \in X_2$. 

$\dot{+}$: $X = X_1 \dot{+} X_2$ means that the topological vector space $X$ is the sum of $X_1$ and $X_2$, i.e., both $X_1$ and $X_2$ are closed subspaces of $X$, and every $x \in X$ has a unique representation of the form $x = x_1 + x_2$ where $x_1 \in X_1$ and $x_2 \in X_2$. 

$\mathfrak{A}$, $\mathfrak{A}_0$: $\mathfrak{A}_0$ is restriction of the $C_0$ semigroup $\mathfrak{A}$ in $X$ to a $C_0$ semigroup in $X_\bullet$ and $\mathfrak{A}_0$ is the extension of $A\mathfrak{A}$ to a $C_0$ semigroup in $X_\circ$. 

⊕: $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ means that the Hilbert space $\mathcal{X}$ is the orthogonal direct sum of the Hilbert spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e., $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ and $\mathcal{X}_1 \perp \mathcal{X}_2$.

$[U \times Y]$: The cross product of the two $B$-spaces $U$ and $Y$. Thus, $[U \times Y] = [U] + [Y]$. Also denoted by $U \times Y$.

$U \times Y$: The cross product of the two $B$-spaces $U$ and $Y$. Also denoted by $[U \times Y]$.

Special Functions

$\chi_I$: The characteristic function of the set $I$.

$e^{\omega}$: $e^{\omega}(t) = e^{\omega t}$ for $\omega, t \in \mathbb{R}$.

$log$: The natural logarithm.

Function Spaces

$V(I; Z)$: Functions of type $V (= L^p, C, BC, etc.)$ on the interval $I \subset \mathbb{R}$ with range in $Z$.

$V_{\text{loc}}(I; Z)$: Functions which are locally of type $V$, i.e., they are defined on $I \subset \mathbb{R}$ with range in $Z$ and they belong to $V(K; Z)$ for every compact subinterval $K \subset I$.

$V_\omega(I; Z)$: Functions in $V(I; Z)$ with compact support.

$V_{\omega, \text{loc}}(I; Z)$: Functions in $V_{\text{loc}}(I; Z)$ whose support is bounded to the left.

$V_\omega(I; Z)$: The set of functions $u$ for which $(t \mapsto e^{-\omega t}u(t)) \in V(I; Z)$. See also the special cases listed below.

$V_{\omega, \text{loc}}(I; Z)$: Functions in $V_{\omega}(I; Z)$ whose support is bounded to the left.

$V_\omega(I; Z)$: The set of functions $u \in V_{\text{loc}}(I; Z)$ which satisfy $\rho I \cap \mathbb{R}^- - u \in V_\omega(I \cap \mathbb{R}^+; Z)$.

$V_\omega(I; Z)$: The closure of $V_\omega(I; Z)$ in $V(I; Z)$. Functions in $V_\omega(I; Z)$ “vanish at infinity”. See also the special cases listed below.

$BC$: The space of bounded continuous functions with the sup-norm.

$BC_\omega$: Functions in $BC$ that tend to zero at $\pm \infty$.

$BC_{\omega}$: Functions $u$ for which $(t \mapsto e^{-\omega t}u(t)) \in BC$.

$BC_{\omega, \text{loc}}$: Continuous functions whose restrictions to $\mathbb{R}^-$ belong to $BC_\omega$.

$BC_{\omega, \omega}$: Functions $u$ for which $(t \mapsto e^{-\omega t}u(t)) \in BC_\omega$.

$BC_{\omega, \omega, \text{loc}}$: Continuous functions whose restrictions to $\mathbb{R}^-$ belong to $BC_{\omega, \omega}$.

$BUC$: Bounded uniformly continuous functions with the sup-norm.

$BUC^n$: Functions which together with their $n$ first derivatives belong to $BUC$.

$C$: Continuous functions. The same space as $BC_{\text{loc}}$.

$C^n$: $n$ times continuously differentiable functions. The same space as $BC^n_{\text{loc}}$.

$L^p$, $1 \leq p < \infty$: Strongly measurable functions with norm $\left\{ \int \|u(t)\|^p \, dt \right\}^{1/p}$.
\( L^\infty \): Strongly measurable functions with norm \( \text{ess sup} \|u(t)\| \).

\( L^p_{\text{loc}} \): Functions which belong locally to \( L^p \).

\( L^p_c \): Functions in \( L^p \) with compact support.

\( L^p_{c,\text{loc}} \): Functions in \( L^p_{\text{loc}} \) whose support is bounded to the left.

\( L^p_\omega \): Functions \( u \) for which \( (t \mapsto e^{-\omega t}u(t)) \in L^p \).

\( L^p_{\omega,\text{loc}}(\mathbb{R}; \mathcal{Z}) \): Functions \( u \in L^p_{\text{loc}}(\mathbb{R}; \mathcal{Z}) \) which satisfy \( \rho - u \in L^p_\omega(\mathbb{R}^-; \mathcal{Z}) \).
In this chapter we lay the foundations for the theory of linear stationary s/s (state/signal) systems in continuous time. Our main interest lies in the class of well-posed s/s systems. However, we shall also encounter systems which are not well-posed. For this reason we start by discussing a very general case, i.e., we start by introducing a class of s/s systems which includes all the specific subclasses that we shall consider later in this book. For this class of systems we introduce, among others, the notions of classical, generalized, and mild time domain trajectories, and notions related to existence and uniqueness of trajectories, controllability, observability, minimality, compressions, dilations, and intertwinements. These properties will be studied in more detail in later chapters.

In the basic setting both the state space $\mathcal{X}$ and the signal space $\mathcal{W}$ of the s/s system will be a vector space with a Hilbert space topology (i.e., the topology is induced by some Hilbert space norm). The reason for this choice is that later $\mathcal{W}$ will be taken to be a Krein space, but to begin with the exact Krein space inner product in $\mathcal{W}$ is not important, only the fact that every Krein space is a vector space with a Hilbert space topology. The exact Hilbert space inner product in the state space $\mathcal{X}$ is also not important until the last two chapters of this book.
1.1.1. The s/s system and its trajectories. In this book we study linear stationary s/s (state/signal) systems in continuous time, which we in the sequel simply refer to as s/s systems. The equations defining the time domain dynamics of such a system can be written in different ways. The one that we use the most is the form

\[
\begin{bmatrix}
\dot{x}(t) \\
x(t) \\
w(t)
\end{bmatrix} \in V, \quad t \in I,
\]

where \( I \subset \mathbb{R} \) is an arbitrary interval (finite or infinite, open or closed or semi-closed). Here the values of the state \( x(t) \) and its time-derivative \( \dot{x}(t) \) lie in the state space \( \mathcal{X} \), the values of the signal \( w(t) \) lie in a signal space \( \mathcal{W} \), and \( V \) is a given subspace of \( \mathbb{R}^{\mathcal{X} \times \mathcal{W}} \). We take both \( \mathcal{X} \) and \( \mathcal{W} \) to be vector spaces with Hilbert space topologies, i.e., the topologies in \( \mathcal{X} \) and \( \mathcal{W} \) are induced by some Hilbert space norm (see Remark 1.1.4 below for an explanation of this choice).

1.1.1. REMARK. Throughout this monograph, if \( x \) is a continuously differentiable function on some nontrivial interval \( I \) and \( t \in I \), then we let \( \dot{x}(t) \) denote

(i) the two-sided derivative of \( x \) at \( t \) if \( t \) is an interior point of \( I \),
(ii) the right-sided derivative of \( x \) at \( t \) if \( t \) is the left end-point of \( I \),
(iii) the left-sided derivative of \( x \) at \( t \) if \( t \) is the right end-point of \( I \).

Since we in this book make heavy use of the class of topological vector spaces which are isomorphic to a Hilbert or Banach space we introduce the following special names for these classes of spaces:

1.1.2. DEFINITION.

(i) By a \( H \)-space \( \mathcal{X} \) we mean a topological vector space which is isomorphic to a Hilbert space, i.e., the topology in \( \mathcal{X} \) is induced by a norm that arises from some Hilbert space inner product in \( \mathcal{X} \).
(ii) By a \( B \)-space \( \mathcal{X} \) we mean a topological vector space which is isomorphic to a Banach space, i.e., the topology in \( \mathcal{X} \) is induced by a Banach space norm in \( \mathcal{X} \).
(iii) By and admissible norm for the \( H \)-space or \( B \)-space \( \mathcal{X} \) we mean any norm in \( \mathcal{X} \) which is compatible with the topology of \( \mathcal{X} \).

Section A.1 for a short review of \( H \)-spaces.

1.1.3. DEFINITION.

(i) By a s/s (state/signal) node we mean a triple \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \), where \( \mathcal{X} \) and \( \mathcal{W} \) are \( H \)-spaces and \( V \) is a subspace of the product space space \( \mathcal{R} = \mathbb{R}^{\mathcal{X} \times \mathcal{W}} \). The spaces \( \mathcal{X} \) and \( \mathcal{W} \) are called the state space respectively signal space of \( \Sigma \), the product space \( \mathcal{R} = \mathbb{R}^{\mathcal{X} \times \mathcal{W}} \) is called the node space of \( \Sigma \), and the subspace \( V \) of \( \mathcal{R} \) is called the generating subspace of \( \Sigma \).
(ii) By a closed s/s node we mean a s/s node \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) with a closed generating subspace \( V \).

---

1In this book we shall always tacitly assume that the interval \( I \) on which a trajectory is defined is nontrivial (it contains more than one point).
(iii) $\Sigma$ is finite-dimensional if both its state space $X$ and its signal space $W$ are finite-dimensional.

1.1.4. Remark. The reason why we do not require $X$ and $W$ to be Hilbert spaces (or Banach spaces) in the above definition is that we do not want to fix the inner products in $X$ and $W$ at this point. Later we shall take $X$ to be a Hilbert space and $W$ to be a Krein space, but to begin with the exact inner products in $X$ and $W$ are not important, only the fact that the topology of every $H$-space is induced by a Hilbert space norm. Neither this Hilbert space norm nor the corresponding Hilbert space inner product in $W$ is unique. In the first part of the book we shall use the assumption that $X$ and $W$ are $H$-spaces only in order to guarantee that every closed subspace of $X$ and $W$ has a direct complement\footnote{Throughout this book, whenever we say “direct sum” we mean the “topological direct sum”, and whenever we mention the “complement” of a subspace we mean the “topological complement”, i.e., both the subspace itself and its complement are required to be closed.}, and that it is possible to apply the closed graph theorem to show that every closed operator with closed domain in $X$ or $W$ is continuous. See also Remark 1.1.7 below.

In the study of generalized trajectories of $\Sigma$ we shall allow the signal $w$ to belong to $L^1$ over $I$ if $I$ is finite, and locally to $L^1$ over $I$ if $I$ is infinite. In the case where a function takes its values in a $H$-space or $B$-space the corresponding $L^1$-space is defined as follows. (For later reference we formulate this definition so that it can also be applied when $L^1$ is replaced by $L^p$ for some $p \in [1, \infty]$.)

1.1.5. Notation. Let $I$ be an interval, let $Z$ be a $B$-space, and let $p \in [1, \infty)$.

(i) The space $C(I; Z)$ is the space of all continuous $Z$-valued functions $z$ defined on the interval $I$. A sequence $z_n \in C(I; Z)$ tends to a function $z$ in $C(I; Z)$ as $n \to \infty$ if the restriction of $z_n$ to every closed finite subinterval $I'$ of $I$ tend to the restriction of $z$ to $I'$ in $C(I'; X)$ as $n \to \infty$. If $I$ is closed and finite, then $C(I; Z)$ is a $B$-space, and each admissible norm $\|\cdot\|_Z$ in $Z$ induces the admissible norm $\|z\|_{C(I; Z)} = \sup_{s \in I} \|z(s)\|_Z$ in $C(I; Z)$.

(ii) The space $L^p(I; Z)$ is the space of all $Z$-valued measurable function $z$ defined on the interval $I$ satisfying $(\int_I \|z(s)\|_Z^p \, ds)^{1/p} < \infty$ for some admissible norm $\|\cdot\|_Z$ in $Z$. This is a $B$-space (it is an $H$-space if $Z$ is an $H$-space and $p = 2$), and each admissible norm $\|\cdot\|_Z$ in $Z$ induces the admissible norm $\|z\|_{L^p(I; Z)} = (\int_I \|z(s)\|_Z^p \, ds)^{1/p}$ in $L^p(I; Z)$.

(iii) If $I$ is finite then $L^p_{\text{loc}}(I; Z) = L^p(I; Z)$, and if $I$ is infinite then $L^p_{\text{loc}}(I; Z)$ is the space of all $Z$-valued functions $z$ on $I$ with the property that the restriction of $z$ to every finite subinterval $I'$ of $I$ belongs to $L^p(I'; Z)$. A sequence $z_n \in L^p_{\text{loc}}(I; Z)$ tends to a function $z$ in $L^p_{\text{loc}}(I; Z)$ as $n \to \infty$ if the restriction of $z_n$ to every finite subinterval $I'$ of $I$ tend to the restriction of $z$ to $I'$ in $L^p(I'; X)$ as $n \to \infty$.

1.1.6. Definition. Let $\Sigma = (V; X, W)$ be a s/s node.

(i) By a classical trajectory of $\Sigma$ on the interval $I$ we mean a pair of functions $[\overset{\wedge}{x}] \in \begin{bmatrix} C^1(I; X) \\ C(I; W) \end{bmatrix}$ satisfying (1.1.1) for all $t \in I$ (see also Remark 1.1.1).
(ii) A pair of functions \([\frac{z}{w}]\) is a **generalized trajectory** of \(\Sigma\) on the closed finite interval \(I\) if there exists a sequence of classical trajectories \([\frac{z}{w_n}]\) of \(\Sigma\) on \(I\) which converges to \([\frac{z}{w}]\) in \(C(I;X)\) as \(n \to \infty\).

(iii) A pair of functions \([\frac{z}{w}]\) is a **generalized trajectory** of \(\Sigma\) on a general interval \(I\) if the restriction of \([\frac{z}{w}]\) to every closed finite subinterval \(I'\) of \(I\) is a generalized trajectory of \(\Sigma\) on \(I'\).

(iv) By the s/s (state/signal) system we mean the above s/s node \(\Sigma = (V;\mathcal{X},\mathcal{W})\) together with its sets of classical and generalized trajectories. We use the same notation \(\Sigma = (V;\mathcal{X},\mathcal{W})\) for the s/s system as for the s/s node defining this system.

(v) \(\mathcal{X}, \mathcal{W}, \mathbb{R},\) and \(V\) are also called the **state space**, the **signal space**, the **node space**, and the **generating subspace** of the s/s system \(\Sigma\).

The reason for the name “generating subspace” for \(V\) is that \(V\) uniquely determines the sets of classical and generalized trajectories of the s/s system \(\Sigma\) on any interval \(I\).

1.1.7. **Remark.** In the above definition we require the signal part \(w\) of a generalized trajectory \([\frac{z}{w}]\) of \(\Sigma\) on the interval \(I\) to belong locally to \(L^1\), and the signal component \(w_n\) of a sequence of classical approximations \([\frac{z}{w_n}]\) on a finite interval \(I\) is supposed to converge to \(w\) in \(L^1\) over \(I\). Almost all the results in this monograph remain true if we throughout replace \(L^1\) by \(L^p\) for some \(p \in [1,\infty)\). This will be important in Chapters 8–11 where we shall use the value \(p = 2\) instead of \(p = 1\).

The three most important intervals \(I\) in Definition 1.1.6 are \(I = \mathbb{R}^+, I = \mathbb{R}^-, \) and \(I = \mathbb{R}\). We give the following special names to trajectories defined on these intervals.

1.1.8. **Definition.** By a **future**, **past**, or **two-sided** (classical or generalized) trajectory of a s/s system \(\Sigma = (V;\mathcal{X},\mathcal{W})\) we mean a trajectory defined on \(\mathbb{R}^+, \mathbb{R}^-,\) or \(\mathbb{R}\), respectively.

Many basic results that we prove will be valid for any s/s node, but often we shall require a s/s node to have some of the following “regularity” properties:

1.1.9. **Definition.** A s/s node \(\Sigma = (V;\mathcal{X},\mathcal{W})\) and the corresponding s/s system \(\Sigma\) are **regular** if the generating subspace \(V\) of \(\Sigma\) has the following properties (i), (ii) and (iii):

(i) \(V\) is closed;

(ii) \(V \cap \left[\begin{array}{c} 0 \\ 0 \end{array}\right] = \{0\};

(iii) The subspace

\[
\mathcal{X}_0 := \left\{ x \in \mathcal{X} \mid \left[\begin{array}{c} z \\ w \end{array}\right] \in V \text{ for some } z \in \mathcal{X} \text{ and } w \in \mathcal{W} \right\}
\]

is dense in \(\mathcal{X}\).

If \(\Sigma\) has properties (i) and (ii) above, then \(\Sigma\) is called **semi-regular**.

1.1.10. **Lemma.** A s/s node \(\Sigma = (V;\mathcal{X},\mathcal{W})\) has property (ii) in Definition 1.1.9 if and only if the generating subspace \(V\) has the graph representation

1.1.3

\[
V = \text{gph}(F) = \left\{ \left[\begin{array}{c} z \\ w \end{array}\right] \in \mathcal{X} \bigg| x \in \text{dom}(F) \text{ and } z = F \left( \left[\begin{array}{c} x \\ w \end{array}\right] \right) \right\}
\]
where $F$ is an operator which is determined uniquely by $V$. The domain of this operator is given by

$$\text{dom}(F) = \begin{bmatrix} 0 & 1x & 0 \\ 0 & 0 & 1w \end{bmatrix} V$$

(1.1.4)

$$= \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} X \\ W \end{bmatrix} \mid \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ for some } z \in X \right\}.$$  

The operator $F$ is closed if and only if $V$ is closed. For all $\begin{bmatrix} x \\ w \end{bmatrix} \in \text{dom}(F)$ the value of $F \begin{bmatrix} x \\ w \end{bmatrix}$ is the unique vector $z \in Z$ for which $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$, and the subspace $X_0$ defined in (1.1.2) is given by

$$X_0 := \left\{ x \in X \mid \begin{bmatrix} x \\ w \end{bmatrix} \in \text{dom}(F) \text{ for some } w \in W \right\}.$$  

Conversely, suppose that $F$ is an operator $[X, W] \rightarrow X$, and define $V$ by (1.1.3). Then $\Sigma = (V; X, W)$ is a s/s node which has property (ii) in Definition 1.1.9.

Proof. Clearly condition (ii) in Definition 1.1.9 is equivalent to the existence of a unique operator $F: [X, W] \rightarrow X$ such that (1.1.3) holds. This operator is closed if and only if $V$ is closed. Formula (1.1.5) follows directly from (1.1.2) and (1.1.4).

The converse statement is trivial. □

The reason why the set $X_0$ in (1.1.2) is important is that this space is related to the set of all possible initial states of trajectories of $\Sigma$. More precisely, if $\begin{bmatrix} x \\ w \end{bmatrix}$ is a classical future trajectory of $\Sigma$, then necessarily $x(0) \in X_0$. Since generalized trajectories are obtained from classical trajectories by approximation, it is also true that if $\begin{bmatrix} x \\ w \end{bmatrix}$ is a generalized future trajectory of $\Sigma$, then necessarily $x(0) \in \overline{X_0}$. Thus, if $X_0$ is not dense in $X$, then the set of possible initial states of all generalized future trajectories of $\Sigma$ is a proper subset of $X$. See also Remark 1.1.13 below.

In addition to the subspace $X_0$ of $X$, there are two additional subspaces which play important roles in the theory.

1.1.11. Lemma. Let $\Sigma = (V; X, W)$ be a s/s system which has property (ii) in Definition 1.1.9. Define

$$V_0 = \begin{bmatrix} 1x & 0 & 0 \\ 0 & 0 & 1w \end{bmatrix} \left( V \cap \left[ \begin{bmatrix} X \\ \{0 \} \end{bmatrix} \right] \right),$$

(1.1.6)

$$W_0 := \begin{bmatrix} 0 & 1w \end{bmatrix} V_0 = \begin{bmatrix} 0 & 0 & 1w \end{bmatrix} \left( V \cap \left[ \begin{bmatrix} X \\ \{0 \} \end{bmatrix} \right] \right).$$

(1.1.7)

Then there exists a closed operator $B_0: W \rightarrow X$ with domain $\text{dom}(B_0) = W_0$ such that

$$V_0 = \text{gph}(B_0) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in \begin{bmatrix} X \\ W_0 \end{bmatrix} \mid z = B_0w \right\}.$$  

(1.1.8)

If $\Sigma$ is semi-regular, then the operator $B_0$ is closed, and in this case $B_0$ is continuous if and only if $W_0$ is closed.

Proof. Let $F$ be the operator defined in Lemma 1.1.10 and defined $B_0 = F \begin{bmatrix} 0 \\ w \end{bmatrix}$ with domain $\text{dom}(B_0) = \left\{ w \in W \mid \begin{bmatrix} 0 \\ w \end{bmatrix} \in \text{dom}(F) \right\}$. Then $V_0$ has the graph representation (1.1.8) and $\text{dom}(B_0) = W_0$. If $\Sigma$ is semi-regular, then $V$ is closed, hence $V_0$ is closed, and $B_0$ is closed. A closed operator is continuous if and only if its domain is closed. □
The significance of the subspace \( W_0 \) of \( W \) and the subspace \( V_0 \) of \( \mathbb{X}W \) is the following. Suppose that \( \left[ \begin{array}{c} x(t) \\ w(t) \end{array} \right] \) is a classical future trajectory of \( \Sigma \) with zero initial state \( x(0) = 0 \). Then \( \left[ \begin{array}{c} \dot{x}(0) \\ 0 \\ 0 \\ w(0) \end{array} \right] \in V \), and consequently \( w(0) \in W_0 \) and \( \left[ \begin{array}{c} \dot{x}(0) \\ 0 \end{array} \right] \in V_0 \). Thus, \( W_0 \) and \( V_0 \) contain the sets of all possible signal values \( w(0) \) and all possible values of the pairs \( \left[ \begin{array}{c} \dot{x}(0) \\ w(0) \end{array} \right] \) as \( \left[ \begin{array}{c} x(t) \\ w(t) \end{array} \right] \) varies over all classical future trajectories of \( \Sigma \) with zero initial state \( x(0) = 0 \).

1.1.12. Remark. It follows from Lemma 1.1.10 that the equation (1.1.1) describing the dynamics of a s/s system which has property (ii) in Definition 1.1.9 can be simplified into the form

\[
\Sigma: \begin{cases} 
\left[ \begin{array}{c} x(t) \\ w(t) \end{array} \right] \in \text{dom}(F), \\
\dot{x}(t) = F \left( \left[ \begin{array}{c} x(t) \\ w(t) \end{array} \right] \right), \\
\end{cases} \quad t \in I,
\]

Even without property (ii) one may try to argue in the same way, but instead of getting an (standard) operator \( F \) one then ends up with a multi-valued operator \(^3\) (or relation) \( F \), and (1.1.9) must be rewritten in the form

\[
\begin{cases} 
\dot{x}(t) \in F \left( \left[ \begin{array}{c} x(t) \\ w(t) \end{array} \right] \right), \\
\left[ \begin{array}{c} x(t) \\ w(t) \end{array} \right] \in \text{dom}(F), \\
\end{cases} \quad t \in I,
\]

where the equality sign in the last line of (1.1.9) has been replaced by an inclusion. Also in this case \( F \) is closed if and only if \( V \) is closed.

1.1.13. Remark. Condition (iii) in Definition 1.1.9 can be interpreted as a “non-degeneracy” condition of the following type. It follows from Definition 1.1.9 that if \( \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \) is a classical trajectory of \( \Sigma \) on \( I \), then \( x(t) \in \mathbb{X}_0 \), \( t \in I \), where \( \mathbb{X}_0 \) is defined as in (iii). Consequently, the values of the derivative \( \dot{x}(t) \) must lie in \( \mathbb{X}_0 \). If we suppose, in addition, that every \( \left[ \begin{array}{c} z_0 \\ x_0 \\ w_0 \end{array} \right] \in V \) is “active” in the sense that \( \left[ \begin{array}{c} z_0 \\ x_0 \\ w_0 \end{array} \right] = \left[ \begin{array}{c} \dot{x}(t_0) \\ x(t_0) \\ w(t_0) \end{array} \right] \) for some classical trajectory \( \left[ \begin{array}{c} x \\ w \end{array} \right] \) defined on some interval \( I \ni t_0 \), then \( \left[ \begin{array}{c} z_0 \\ x_0 \\ w_0 \end{array} \right] \in V \) implies \( \left[ \begin{array}{c} z_0 \\ x_0 \\ w_0 \end{array} \right] \in V \cap \left( \mathbb{X}_0 \mathbb{X}_0 \mathbb{W} \right) \). In other words, \( V = V \cap \left( \mathbb{X}_0 \mathbb{X}_0 \mathbb{W} \right) \). Thus, if we simply replace \( \mathbb{X} \) by \( \mathbb{X}_0 \), then we get a new system for which condition (iii) holds, and which has the same classical and generalized trajectories as the original system. This is discussed in more detail in Lemma 1.3.16.

1.1.14. Remark. We do not a priori exclude the possibility that \( \mathbb{X} = \{0\} \) or \( W = \{0\} \) in Definition 1.1.9. In some of our examples we shall, in fact, take \( W = \{0\} \). In this case (1.1.1) can be rewritten in the form

\[
\dot{x}(t) = Fx(t), \quad t \in I,
\]

where \( F \) is a closed operator in \( \mathbb{X} \) with dense domain. In particular, we then often take \( F \) to be the generator of a \( C_0 \) semigroup.

\(^3\)See Appendix A.4 for a short introduction to the notion of a multi-valued operator.
1.1.2. **Bounded state/signal systems.** There is a particular subclass of s/s systems that is mathematically simpler to deal with than the general class that we have considered above, namely the class of bounded s/s systems. Typical examples of s/s systems from mathematical physics are not bounded, but bounded s/s systems do appear, e.g., when one wants to study numerical approximations of s/s systems. They are also relevant in the definition of the resolvent set of a s/s node in Chapter 5.

1.1.15. **Definition.** A s/s node or system $\Sigma = (V; X, W)$ is called bounded if conditions (i)–(iii) in Definition 1.1.9 hold in the following stronger form.

(i) $V$ is closed;

(ii) $V \cap \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \{0\};$

(iii) $\left[ \begin{array}{c} 0 \\ 1 \end{array} \right] V = X$; (i.e., $X_0 = X$ in the notation of Definition 1.1.9);

(iv) The set $\left[ \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{array} \right] V$ is closed in $[X; W]$ (this is the domain of the operator $F$ defined in Lemma 1.1.10).

1.1.16. **Lemma.** Every semi-regular finite-dimensional s/s system $\Sigma$ is bounded.

**Proof.** Condition (i) and (iv) in Definition 1.1.15 hold since every subspace of a finite-dimensional space is closed. Condition (ii) is the same in Definitions 1.1.9 and 1.1.15. Condition (iii) follows from condition (iii) in Definition 1.1.9 and the fact that the dimension of $X$ is finite.

The following lemma gives an alternative description of the class of bounded s/s systems.

1.1.17. **Lemma.** A subspace $V$ of the node space $K := [X; W]$ is the generating subspace of a bounded s/s node $\Sigma = (V; X, W)$ if and only if the following condition holds:

\[
\text{V has a graph representation (1.1.3) over the last two components [X; W] of } K \text{ with a continuous linear operator } F: \text{dom}(F) \subset [X; W] \to X \text{ with closed domain satisfying (1.1.11) for every } x \in X \text{ there is some } w \in W \text{ such that } [x; w] \in \text{dom}(F).}
\]

**Proof.** It is easy to see that if $V$ has a graph representation for some operator $F$ satisfying the properties listed above, then $V$ is closed and has properties (i)–(iv) in Definition 1.1.15 and hence it the generating subspace of a bounded s/s system.

Conversely, suppose that $V$ has properties (i)–(iv) in Definition 1.1.15. By Lemma 1.1.10, $V$ has a graph representation of the type (1.1.3) for some linear operator $F$. This operator $F$ is closed since $V$ is closed, and (iv) implies that $\text{dom}(F)$ is closed in $[X; W]$. By the closed graph theorem, $F$ is continuous. Finally, property (1.1.11) of $F$ is equivalent to property (iii).
1.2. Transformations of S/S Nodes (Jan 02, 2016)

In this section we describe various transformations that can be applied to s/s nodes and systems, and begin with the notion of a time reflection.

1.2.1. Time reflection of a s/s node. The idea behind the definition of the “time reflection” of a s/s system is to reverse the direction of time in equation \((1.1.1)\), i.e., to replace \(x(t)\) by \(x(-t)\) and \(w(t)\) by \(w(-t)\). Since \(\frac{d}{dt}x(-t) = -\dot{x}(-t)\) the new equation satisfied by \(x(-t)\) and \(w(-t)\) contains an extra change of sign compared to \((1.1.1)\). This motivates the following definition (see also Lemma 1.2.4 below).

1.2.1. Definition. Let \(\Sigma = (V; X, W)\) be a s/s node or system. By the time reflection of \(\Sigma\) we mean the s/s node or system \(\Sigma_R = (V_R; X, W)\) with generating subspace

\[
V_R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \end{bmatrix} V.
\]

1.2.2. Lemma. Let \(\Sigma\) be a s/s node.

(i) The time reflection \(\Sigma_R\) of \(\Sigma\) is closed, or semi-regular, or regular, or bounded if and only if \(\Sigma\) is closed, or semi-regular, or regular, or bounded, respectively.

(ii) \(\Sigma = (\Sigma \cdot R)\) i.e., \(\Sigma\) is equal to the time reflection of the time reflection of itself.

Proof. This follows directly from the relevant definitions.

In order to compare the trajectories of a s/s node \(\Sigma\) and its time reflection \(\Sigma_R\) we introduce the following time reflection operator \(R\).

1.2.3. Definition. For each interval \(I \subset \mathbb{R}\) we define the reflected interval \(I_R\) by

\[
I_R = \{ -t \mid t \in I \},
\]

and for each function \(f\) defined on some time interval \(I\) we define the time reflection \(R_f\) of \(f\) to be the function

\[
(\hat{R}_f)(t) = f(-t), \quad t \in I_R.
\]

We call \(R\) the time reflection operator.

1.2.4. Lemma. Let \(\Sigma = (V; X, W)\) be a s/s system, and let \(\Sigma_R = (V_R; X, W)\) be the time reflection of \(\Sigma\). Then \([\hat{x}]\) is a classical or generalized trajectory of \(\Sigma\) on the interval \(I\) if and only if \([\hat{x}]\) is a classical or generalized trajectory of \(\Sigma_R\) on \(I_R = \{ -t \mid t \in I \}\).

Proof. We have \([\hat{x}_R] = \begin{bmatrix} \hat{x}_R \\ \hat{w}_R \end{bmatrix} \in \begin{bmatrix} C^1(I; V_R) \\ C^1(I; W) \end{bmatrix}\) if and only if \([\hat{x}] \in \begin{bmatrix} C^1(I; X) \\ C^1(I; W) \end{bmatrix}\) and \(\frac{d}{dt}x(-t) = -\dot{x}(-t)\). Therefore

\[
\begin{bmatrix} \frac{d}{dt}(x \cdot \hat{t}) \\ (x \cdot \hat{t}) \\ \hat{w} \cdot \hat{t} \end{bmatrix} \in V_R \quad t \in I_R.
\]
if and only if
\[
\begin{bmatrix}
\dot{x}(t) \\
x(t) \\
w(t)
\end{bmatrix} \in V, \quad t \in I. \quad \square
\]

1.2.2. Time rescaling of a s/s node. In the time reflection we changed the direction of time by replacing \(x(t)\) and \(w(t)\) by \(x(-t)\) and \(w(-t)\) in (1.1.1). If we instead rescale the time by replacing \(x(t)\) and \(w(t)\) by \(x(\gamma t)\) and \(w(\gamma t)\) for some \(\gamma > 0\), then we get a time rescaled s/s system. In this case \(\frac{dx}{dt}(\gamma t) = \gamma \dot{x}(\gamma t)\) and the new equation satisfied by \(x(\gamma t)\) and \(w(\gamma t)\) contains an extra factor \(\gamma\), which replaces the factor \(-1\) in the top left corner of the right-hand side of (1.2.1).

1.2.5. Definition. Let \(\Sigma = (V; X, W)\) be a s/s node or system and let \(\gamma > 0\). By the time \(\gamma\)-rescaling of \(\Sigma\) we mean the s/s node or system \(\Sigma_\gamma = (V_\gamma; X, W)\) with generating subspace
\[
(1.2.4)
V_\gamma := \begin{bmatrix}
\gamma & 0 & 0 \\
0 & 1_X & 0 \\
0 & 0 & 1_W
\end{bmatrix} V.
\]

1.2.6. Lemma. Let \(\Sigma\) be a s/s node, and let \(\gamma > 0\).

(i) The time \(\gamma\)-rescaling \(\Sigma\) is closed, or semi-regular, or regular, or bounded if and only if \(\Sigma\) is closed, or semi-regular, or regular, or bounded, respectively.

(ii) The time \(\gamma_1\)-rescaling of the time \(\gamma_2\)-rescaling of \(\Sigma\) is equal to the time \(\gamma\)-rescaling of \(\Sigma\), where \(\gamma = \gamma_1\gamma_2\).

(iii) Time reflection and time \(\gamma\)-rescaling commute, i.e., the time \(\gamma\)-rescaling of the time reflection of \(\Sigma\) is equal to the time reflection of the time \(\gamma\)-rescaling of \(\Sigma\).

Proof. This follows directly from the relevant definitions. \(\square\)

1.2.7. Lemma. Let \(\Sigma = (V; X, W)\) be a s/s system, let \(\gamma > 0\), and let \(\Sigma_\gamma = (V_\gamma; X, W)\) be the time \(\gamma\)-rescaling of \(\Sigma\). Then \(\begin{bmatrix} x \\wedge \\
w \\wedge \end{bmatrix}\) is a classical or generalized trajectory of \(\Sigma\) on the interval \(I\) if and only if \(\begin{bmatrix} x \\gamma \\
w \gamma \end{bmatrix}\) is a classical or generalized trajectory of \(\Sigma_\gamma\) on the interval \(I_\gamma\), where \(\begin{bmatrix} x_{\gamma}(t) \\
w_{\gamma}(t)\end{bmatrix} = \begin{bmatrix} x(t) \\
w(t)\end{bmatrix} \in I_\gamma := \{\gamma t \mid t \in I\} = \{\gamma t \mid t \in I\}.

Proof. The proof is analogous to the proof of Lemma 1.2.4. \(\square\)

1.2.3. Exponentially weighted s/s nodes. The exponential weighting of an s/s node or system \(\Sigma\) is the new node or system that we get by modifying trajectories \(\begin{bmatrix} x \\wedge \\
w \\wedge \end{bmatrix}\) of \(\Sigma\) on the interval \(I\) by multiplying both \(x\) and \(w\) by the function \(e^{\alpha t}\), \(t \in I\). We denote this function by \(e_\alpha\). Since \(\frac{dx}{dt}(e^{\alpha t}(x(t)) = \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t)\), if \(\begin{bmatrix} x \\wedge \\
w \\wedge \end{bmatrix}\) is a classical trajectory of \(\Sigma\) on \(I\), then the pair \(\begin{bmatrix} x_\alpha \\
w_\alpha \end{bmatrix} := \begin{bmatrix} e_\alpha x \\
e_\alpha w \end{bmatrix}\) satisfies the equation
\[
\begin{bmatrix}
\dot{x}_\alpha(t) - \alpha x_\alpha(t) \\
x_\alpha(t) \\
w_\alpha(t)
\end{bmatrix} \in V, \quad t \in I.
\]

1.2.8. Definition. Let \(\Sigma = (V; X, W)\) be a s/s node or system, and let \(\alpha \in \mathbb{C}\). By the exponential \(\alpha\)-weighting of \(\Sigma\) we mean the s/s node or system \(\Sigma_\alpha = \begin{bmatrix} x_\alpha \\
w_\alpha \end{bmatrix} \in V, \quad t \in I.\)
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(V_α; X, W) with generating subspace

\[ V_α := \begin{bmatrix} 1_X & α & 0 \\ 0 & 1_X & 0 \\ 0 & 0 & 1_W \end{bmatrix} V. \]

**1.2.9. Lemma.** Let Σ be a s/s node.

(i) The exponential α-weighting of Σ is closed, or semi-regular, or regular, or bounded if and only if Σ is closed, or semi-regular, or regular, or bounded, respectively.

(ii) If Σ_α is the exponential α-weighting of Σ, then the exponential β-weighting of Σ_α is the exponential (α + β)-weighting of Σ.

(iii) Σ_α is the exponential α-weighting of Σ if and only if Σ is the exponential −α-weighting of Σ.

(iv) The exponential α-weighting of the time reflection of Σ is the time reflection of the (−α)-weighting of Σ.

(v) The exponential α-weighting of the time γ-scaling of Σ is the time γ-scaling of Σ the exponential α/γ-weighting of Σ.

**Proof.** This follows directly from the relevant definitions. □

**1.2.10. Lemma.** Let Σ = (V; X, W) be a s/s system, let \( α ∈ \mathbb{C} \), and let Σ_α = (V_α; X, W) be the exponential α-weighting of Σ. Then \( [x_α \ w_α] \) is a classical or generalized trajectory of Σ on some time interval I if and only if \( [Px_α \ Qw_α] \) defined by

\[ [x_α(t) \ w_α(t)] := e^{αt} [x(t) \ w(t)], \quad t ∈ I, \]

is a classical or generalized trajectory of Σ_α on I.

**Proof.** The proof is analogous to the proof of Lemma 1.2.4. □

**1.2.4. Similarity of two s/s nodes.** The idea behind the definition of “similarity” of two s/s systems Σ = (V; X, W) and Σ_1 = (V_1; X_1, W_1) is to require that there is a one-to-one correspondence between classical and generalized trajectories of Σ and Σ_1 of the following type: There should exists bicontinuous linear bijections P: X → X_1 and Q: W → W_1 such that \( [x_α] \) is a classical or generalized trajectory of Σ if and only if \( [Px_α] \) is a classical or generalized trajectory of Σ_1. In order to achieve this we require that the generating subspaces V and V_1 are related as described in the following definition (see also Lemma 1.2.13).

**1.2.11. Definition.** Let Σ = (V; X, W) and Σ_1 = (V_1; X_1, W_1) be two s/s nodes or systems.

(i) We say that Σ_1 is \( (P,Q) \)-similar to Σ if P ∈ \( \mathcal{B}(X; X_1) \) and Q ∈ \( \mathcal{B}(W; W_1) \) are bicontinuous bijections and

\[ V_1 = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} V. \]

In this case we also call Σ_1 the \( (P,Q) \)-similarity transformation of Σ. The operators P and Q in (i) are called the state similarity operator and the signal similarity operator, respectively.

(ii) Σ and Σ_1 are similar if Σ_1 is \( (P,Q) \)-similar to Σ for some P and Q.
(iii) We say that \( \Sigma_1 \) is \( P \)-similar to \( \Sigma \) if \( \mathcal{W}_1 = \mathcal{W} \) and \( \Sigma_1 \) is \((P,1_{\mathcal{W}})\)-similar to \( \Sigma \). In this case we also call \( \Sigma_1 \) the \( P \)-similarity transformation of \( \Sigma \).

1.2.12. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) and \( \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1) \) be two s/s nodes or systems.

(i) If \( \Sigma \) and \( \Sigma_1 \) are similar, then \( \Sigma_1 \) is closed, or semi-regular, or regular or bounded if and only if \( \Sigma \) is closed, or semi-regular, or regular, or bounded, respectively.

(ii) \( \Sigma_1 \) is \((P,Q)\)-similar to \( \Sigma \) if and only if \( \Sigma \) is \((P^{-1}, Q^{-1})\)-similar to \( \Sigma_1 \).

Thus, similarity of two s/s nodes or systems is an equivalence relation.

(iii) If \( \Sigma_1 \) is \((P,Q)\)-similar to \( \Sigma \), then the time reflection of \( \Sigma_1 \) is \((P,Q)\)-similar to the time reflection of \( \Sigma \).

(iv) If \( \Sigma_1 \) is \((P,Q)\)-similar to \( \Sigma \), then for each \( \alpha \in \mathbb{C} \) the exponential \( \alpha \)-weighting of the time reflection of \( \Sigma_1 \) is \((P,Q)\)-similar to the exponential \( \alpha \)-weighting of of \( \Sigma \).

Proof. This follows immediately from the relevant definitions. \( \square \)

1.2.13. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) and \( \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1) \) be two s/s nodes or systems, and suppose that \( \Sigma_1 \) is \((P,Q)\)-similar to \( \Sigma \). Then \( \left[ \begin{array}{c} \alpha \\ \bar{w} \end{array} \right] \) is a classical or generalized trajectory of \( \Sigma \) on the interval \( I \) if and only if \( \left[ \begin{array}{c} P \alpha \\ Q \bar{w} \end{array} \right] \) is a classical or generalized trajectory of \( \Sigma_1 \) on \( I \).

Proof. This follows immediately from the definitions of these notions. \( \square \)

1.2.14. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s node or system, let \( \mathcal{X}_1 \) and \( \mathcal{W}_1 \) be two \( H \)-spaces, and let \( P: \mathcal{X} \to \mathcal{X}_1 \) and \( Q: \mathcal{W} \to \mathcal{W}_1 \) be two continuous linear operators with closed domains \( \text{dom} (P) \subset \mathcal{X} \) and \( \text{dom} (Q) \subset \mathcal{W} \).

(i) By the \((P,Q)\)-image of \( \Sigma \) we mean the s/s node or system \( \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1) \) where

\[
V_1 = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \left( V \cap \begin{bmatrix} \text{dom} (P) \\ \text{dom} (P) \end{bmatrix} \right).
\]

(ii) If \( 1.2.7 \) holds and \( \text{dom} (P) = \mathcal{X} \) and \( \text{dom} (Q) = \mathcal{W} \), then we call \( \Sigma_1 \) a \((P,Q)\)-image of \( \Sigma \) with full domain.

(iii) If \( 1.2.7 \) holds and both \( P \) and \( Q \) are injective, then we call \( \Sigma_1 \) a injective \((P,Q)\)-image of \( \Sigma \).

(iv) If \( 1.2.7 \) holds and both \( P \) and \( Q \) are surjective, then we call \( \Sigma_1 \) a surjective \((P,Q)\)-image of \( \Sigma \).

Most (if not all) of our \((P,Q)\)-images will be surjective, some will be injective, and some will have full domain.

1.2.15. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) and \( \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1) \) be two s/s nodes or systems.

(i) \( \Sigma_1 \) is \((P,Q)\)-similar to \( \Sigma \) if and only if \( \Sigma_1 \) is an injective and surjective \((P,Q)\)-image of \( \Sigma \) with full domain.
(ii) If $\Sigma_1$ is the $(P,Q)$-image of $\Sigma$, then the time reflection of $\Sigma_1$ is the $(P,Q)$-image to the time reflection of $\Sigma$.

(iii) If $\Sigma_1$ is the $(P,Q)$-image of $\Sigma$, then for each $\alpha \in \mathbb{C}$ the exponential $\alpha$-weighting of $\Sigma_1$ is the $(P,Q)$-image to the exponential $\alpha$-weighting of $\Sigma$.

(iv) If $\Sigma_1$ is the $(P,Q)$-image of $\Sigma$, then for each $\gamma > 0$ the time $\gamma$-rescaling of $\Sigma_1$ is the $(P,Q)$-image to the time $\gamma$-rescaling of $\Sigma$.

**Proof.** This follows immediately from the relevant definitions. \qed

Note that we do not claim that $\Sigma_1$ is closed, or semi-regular, or regular, or bounded whenever $\Sigma$ is closed, or semi-regular, or regular, or bounded.

### 1.2.16. Lemma

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node or system, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1)$ be the $(P,Q)$-image of $\Sigma$, where $P: \mathcal{X} \rightarrow \mathcal{X}_1$ and $Q: \mathcal{W} \rightarrow \mathcal{W}_1$ are two continuous linear operators with closed domains $\text{dom}(P) \subset \mathcal{X}$ and $\text{dom}(Q) \subset \mathcal{W}$.

(i) If $[x_w]$ is a classical or generalized trajectories of $\Sigma$ on some interval $I$ satisfying $x(t) \in \text{dom}(P)$ and $w(t) \in \text{dom}(Q)$ for (almost) all $t \in I$, then $[P^x Q^w]$ is a classical respectively generalized trajectory of $\Sigma_1$ on $I$.

(ii) If both $P$ and $Q$ are injective and surjective (i.e., $\Sigma_1$ is an injective and surjective $(P,Q)$-image of $\Sigma$), then the converse is also true, i.e., to any $[x_{w_1}]$ is a classical or generalized trajectories of $\Sigma_1$ on some interval $I$ there exists a (unique) trajectory $[x_w]$ of $\Sigma$ on $I$ such that $x(t) \in \text{dom}(P)$ and $w(t) \in \text{dom}(Q)$ for (almost) all $t \in I$ and $[x_{w_1}] = [P^x Q^w]$.

**Proof.** This follows from Definitions 1.1.3, 1.1.6 and 1.2.14. \qed

### 1.2.6. Parts and projections of a s/s node

Two special types of $(P,Q)$-images are described below:

#### 1.2.17. Definition

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1)$ be two s/s nodes or systems.

(i) We call $\Sigma_1$ the part of $\Sigma$ in $[\mathcal{X}_1; \mathcal{W}_1]$ if $\mathcal{X}_1$ and $\mathcal{W}_1$ are closed subspaces of $\mathcal{X}$ respectively $\mathcal{W}$ and $\Sigma_1$ is the $(1_{\mathcal{X}_1}, 1_{\mathcal{W}_1})$-image of $\Sigma$, i.e.,

$$V_1 = V \cap \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{W}_1 \end{array} \right].$$

(ii) We call $\Sigma_1$ the static projection of $\Sigma$ onto $[\mathcal{X}_1; \mathcal{W}_1]$ along $[\mathcal{X}_2; \mathcal{W}_2]$ if $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$, and $\Sigma_1$ is the $(P_{\mathcal{X}_1}^{\mathcal{X}_2}, P_{\mathcal{W}_1}^{\mathcal{W}_2})$-image of $\Sigma$, i.e.,

$$V_1 = \begin{bmatrix} P_{\mathcal{X}_1}^{\mathcal{X}_2} & 0 & 0 \\ 0 & P_{\mathcal{X}_2}^{\mathcal{X}_1} & 0 \\ 0 & 0 & P_{\mathcal{W}_1}^{\mathcal{W}_2} \end{bmatrix} V.$$

Note, in particular, that the $(1_{\mathcal{X}_1}, 1_{\mathcal{W}_1})$-image in part (i) of Definition 1.2.17 is both injective and surjective, and that the $(P_{\mathcal{X}_1}^{\mathcal{X}_2}, P_{\mathcal{W}_1}^{\mathcal{W}_2})$-image in part (ii) has full domain.

### 1.2.18. Remark

The reason for the extra word “static” in part (ii) of Definition 1.2.8 is to differentiate between the notion in Definition 1.2.8(ii) (which is given in terms of the i/s/o node itself) and the “dynamic” notion that will be introduced in
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Definition 1.5.37 below (which is given in terms of trajectories of the i/s/o system induced by the node).

1.2.19. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s node, let \( \mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 \) and \( \mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 \).

(i) The part \( \Sigma_{\text{part}} = (V_{\text{part}}; \mathcal{X}_1; \mathcal{W}_1) \) of \( \Sigma \) in \( [\mathcal{X}_1 \mathcal{W}_1] \) has the following properties:

(a) If \( \Sigma \) is closed then \( \Sigma_{\text{part}} \) is closed (this is condition (i) in Definition 1.1.9).

(b) If \( \Sigma \) satisfies condition (ii) in Definition 1.1.9, then \( \Sigma_{\text{part}} \) satisfies the same condition.

(ii) The static projection \( \Sigma_{\text{proj}} = (V_{\text{proj}}; \mathcal{X}_1; \mathcal{W}_1) \) of \( \Sigma \) onto \( [\mathcal{X}_1 \mathcal{W}_1] \) along \( [\mathcal{X}_2 \mathcal{W}_2] \) has the following properties:

(a) If \( \Sigma \) satisfies condition (iii) in Definition 1.1.9, then \( \Sigma_{\text{part}} \) satisfies the same condition.

Proof. This follows directly from Definitions 1.1.9 and 1.2.17.

As the following lemma shows, every \( (P,Q) \)-image \( \Sigma_1 \) of a s/s node or system \( \Sigma \) can be interpreted as an \( (P_2,Q_2) \)-image with full domain of an injective and surjective \( (P_1,Q_1) \)-image \( \Sigma_2 \) of \( \Sigma \) (i.e., \( (P,Q) \)-images can be “factored” into the product of an injective and surjective \( (P_1,Q_1) \)-image and a \( (P_2,Q_2) \)-image with full domain).

1.2.20. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s node or system, and let \( \Sigma_1 = (V_1, \mathcal{X}_1, \mathcal{W}_1) \) be a \( (P,Q) \)-image of \( \Sigma \).

(i) Let \( \Sigma_2 = (V_2; \text{dom}(P), \text{dom}(Q)) \) be the part of \( \Sigma \) in \( [\text{dom}(P) \text{dom}(Q)] \), i.e.,

\[(1.2.10) V_1 = V \cap \left[ \begin{array}{c} \text{dom}(P) \\ \text{dom}(Q) \end{array} \right] \]

Then \( \Sigma_2 \) is the injective and surjective \( (1_{\mathcal{X}}|_{\text{dom}(P)}, 1_{\mathcal{W}}|_{\text{dom}(Q)}) \)-image of \( \Sigma \).

(ii) Let \( \bar{P} \in \mathcal{B}(\text{dom}(P); \mathcal{X}_1) \) and \( \bar{Q} \in \mathcal{B}(\text{dom}(Q); \mathcal{W}_1) \) be the operators that we get by interpreting \( P \) and \( Q \) as bounded linear operators with domain spaces \( \text{dom}(P) \) respectively \( \text{dom}(Q) \). Then \( \Sigma \) is the \( (\bar{P}, \bar{Q}) \)-image of \( \Sigma_2 \) with full domain.

Proof. This follows directly from Definitions 1.2.14 and 1.2.17.

1.2.7. Adding bounded inputs and output to a s/s node. All the transformations that we have considered so far have been “block diagonal” in the sense that block matrix which maps the generating subspace of one of the systems does not contain any terms which represents a direct coupling between the state or derivative of the state and the signal. We shall next define a transformation containing such terms. This transformation can be interpreted as the addition of a bounded input and a bounded output to a given s/s node.

1.2.21. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s node or system, let \( \mathcal{U}_1 \) and \( \mathcal{Y}_1 \) be \( H \)-spaces, and let \( B_1 \in \mathcal{B}(\mathcal{U}_1; \mathcal{X}), C_1 \in \mathcal{B}(\mathcal{X}; \mathcal{Y}_1), \) and \( D_1 \in \mathcal{B}(\mathcal{U}_1; \mathcal{Y}_1) \).
(i) The s/s node or system \( \Sigma_{(B,C,D)} = (V_{(B,C,D)}; \mathcal{X}, [U_0 \ W_1]) \) where

\[
V_{(B,C,D)} = \left\{ \begin{bmatrix} \frac{z}{x} \\ \frac{u_1}{w} \\ C_{1x+D_1u_1} \end{bmatrix} \in \mathcal{X} \begin{bmatrix} u_0 \\ \frac{W}{Y_1} \end{bmatrix} \bigg| \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \ u_1 \in U_1 \right\}
\]

is called the (bounded) i/o extension of \( \Sigma \) to \([U_0 \ W_1] \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( D_1 \).

(ii) The s/s node or system \( \Sigma_C = (V_C; \mathcal{X}, [U_0 \ W_1]) \) where

\[
V_C = \left\{ \begin{bmatrix} \frac{z+B_1u_1}{x} \\ \frac{u_1}{w} \\ C_{1x} \end{bmatrix} \in \mathcal{X} \begin{bmatrix} u_0 \\ \frac{W}{Y_1} \end{bmatrix} \bigg| \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \ u_1 \in U_1 \right\}
\]

is called the (bounded) output extension of \( \Sigma \) to \([U_0 \ W_1] \) with control operator \( B_1 \).

(iii) The s/s node or system \( \Sigma_3 = (V_B; \mathcal{X}, [W \ Y_1]) \) where

\[
V_B = \left\{ \begin{bmatrix} \frac{z}{x} \\ \frac{w}{C_{1x}} \end{bmatrix} \in \mathcal{X} \begin{bmatrix} u_0 \\ \frac{W}{Y_1} \end{bmatrix} \bigg| \begin{bmatrix} z \\ x \end{bmatrix} \in V \right\}
\]

is called the (bounded) i/o extension of \( \Sigma \) to \([W \ Y_1] \) with observation operator \( C_1 \).

Note that (1.2.11) can be written in the alternative forms

\[V_{(B,C,D)} = \left\{ \begin{bmatrix} \frac{z}{x} \\ \frac{u_1}{w} \\ C_{1x+D_1u_1} \end{bmatrix} \in \mathcal{X} \begin{bmatrix} U_0 \\ \frac{W}{Y_1} \end{bmatrix} \bigg| \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \ u_1 \in U_1 \right\},\]

\[V_{(B,C,D)} = \begin{bmatrix} 1 \chi & 0 & 0 & B_1 \\ 0 & 1 \chi & 0 & 0 \\ 0 & 0 & 1 \chi & 0 \\ 0 & 0 & C_1 & D_1 \end{bmatrix} \begin{bmatrix} V \\ U_1 \end{bmatrix},\]

and that also (1.2.12) and (1.2.13) can be written analogously (simply drop the last row or column). Clearly input and output extensions can be regarded as special cases of i/o extensions (take either \( Y_1 = \{0\} \) or \( U_1 = \{0\} \)). Also note that the i/o extension can be obtained by first performing an input extension of \( \Sigma \) with control operator \( B_1 \), then performing an output extension of the resulting system with observation operator \( C_1 \), and finally adding a feedthrough term \( D_1 \) from \( U_1 \) to \( Y_1 \). An alternative interpretation is also possible, where one first does the output extension with observation operator \( C_1 \) and then an input extension of the resulting system with control operator \( B_1 \), with the same addition of a feedthrough operator \( D_1 \).

1.2.22. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s node or system, and let \( \Sigma_{(B,C,D)} = (V_{(B,C,D)}; \mathcal{X}, [U_0 \ W_1]) \) be the bounded i/o extension of \( \Sigma \) to \([U_0 \ W_1] \) with control operator \( B \in \mathcal{B}(U_1; \mathcal{X}) \), observation operator \( C \in \mathcal{B}(\mathcal{X}, Y_1) \), and feedthrough operator \( D \in \mathcal{B}(U_1; Y_1) \), and let \( \Sigma_{(0,0,0)} = (V_{(0,0,0)}; \mathcal{X}, [U_0 \ W_1]) \) be the bounded i/o extension of \( \Sigma \) to \([U_0 \ W_1] \) with zero control operator, observation operator, and feedthrough operator.

(see 1.2.11).
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(i) \( V_{B,C,D} \) and \( V_{0,0,0} \) can be obtained from each other as follows:

\[
V_{(B,C,D)} = \begin{bmatrix}
1 & 0 & B_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & V_{1} & 0 & 0 \\
0 & 0 & 0 & 1 & V_{1} & 0 \\
0 & 0 & C_1 & 0 & 1 & Y_{1} \\
0 & 0 & 0 & 0 & 1 & Z_1 \\
\end{bmatrix}
\begin{bmatrix}
V_{0,0,0} \\
\end{bmatrix}
\]

(1.2.15)

\[
V_{(0,0,0)} = \begin{bmatrix}
1 & 0 & B_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & V_{1} & 0 & 0 \\
0 & 0 & 0 & 1 & V_{1} & 0 \\
0 & 0 & C_1 & 0 & 1 & Y_{1} \\
0 & 0 & 0 & 0 & 1 & Z_1 \\
\end{bmatrix}
\begin{bmatrix}
V_{B,C,D} \\
\end{bmatrix}
\]

(ii) \( \Sigma_{(B,C,D)} \) is closed, or semi-regular, or regular, or bounded if and only if \( \Sigma \) is closed, or semi-regular, or regular, or bounded, respectively.

(iii) The time reflection of \( \Sigma_{(B,C,D)} \) is the bounded i/o extension of the time reflection of \( \Sigma \) with the same control operator, observation operator, and feedthrough operator.

(iv) For each \( \alpha \in \mathbb{C} \), the exponential \( \alpha \)-weighting of \( \Sigma_{(B,C,D)} \) is the bounded i/o extension of the exponential \( \alpha \)-weighting of \( \Sigma \) with the same control operator, observation operator, and feedthrough operator.

(v) For each \( \gamma > 0 \), the time \( \gamma \)-rescaling of \( \Sigma_{(B,C,D)} \) is the bounded i/o extension of the time \( \gamma \)-rescaling of \( \Sigma \) with the control operator \( \gamma B \), observation operator \( C \), and feedthrough operator \( D \).

Proof. The easy proof is left to the reader.

Note that although the generating subspaces of the two s/s nodes \( \Sigma_1 \) and \( \Sigma_2 \) in part (i) of Lemma 1.2.22 can be obtained from each other by an invertible block matrix transformation, the two systems are still not similar to each other, due to the existence of the non-diagonal entries \( B \) and \( C \) in the transformations in (1.2.15).

As the following lemma shows, an output extension amounts to an addition of an “output component” to the signal space which depends continuously on the state of the system.

1.2.23. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system, let \( C \in \mathcal{B}(\mathcal{X}; \mathcal{Y}) \), and let \( \Sigma_C = (V_C; \mathcal{X}, [\mathcal{W}]) \) be the output extension of \( \Sigma \) to \([\mathcal{W}] \) with observation operator \( C \). Then the following claims are true:

(i) \( \Sigma \) is the static projection of \( \Sigma_C \) onto \([\mathcal{X}] \) along \([{0 \atop 0 \atop 0}] \);

(ii) \([x_w]\) is a classical or generalized trajectory of \( \Sigma \) on \( I \) if and only if \([\bar{x}_{C_{1}\mathcal{X}}]\) is a classical respectively generalized trajectory of \( \Sigma_C \) on \( I \).

Proof. This follows directly for the relevant definitions.

Analogously, an input extension amounts to the addition of a bounded “input component” to the signal space.

1.2.24. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system, let \( B \in \mathcal{B}(\mathcal{U}; \mathcal{X}) \), and let \( \Sigma_B = (V_I; \mathcal{X}, [\mathcal{U}]) \) be the input extension of \( \Sigma \) to \([\mathcal{U}] \) with control operator \( B \). Then the following claims are true:

(i) \( \Sigma \) is the part of \( \Sigma_B \) in \([\mathcal{X} \atop \{0\}] \);

(ii) \([x_w] \) is a classical or generalized trajectory of \( \Sigma \) on \( I \) if and only if \([x_0_{C_{1}\mathcal{X}}]\) is a classical respectively generalized trajectory of \( \Sigma_1 \) on \( I \).
(iii) If $\Sigma$ and $\Sigma_1$ are regular, if $[\frac{x}{w}]$ and $[\frac{x_B}{w_B}]$ are classical trajectories of $\Sigma$ respectively $\Sigma_B$ on some interval $I$, and if $[\frac{x(t)}{w(t)}] = [\frac{x_B(t)}{w_B(t)}]$ at some point $t \in I$, then $\dot{x}_B(t) = \dot{x}(t) + Bu_B(t)$.

**Proof.** This follows directly for the relevant definitions. \(\square\)

Addition of bounded inputs and outputs in an i/s/o setting will be discussed in Section 2.3.

### 1.2.8. The cross product of two s/s nodes.

Our following definition formalizes the idea of "how combined two s/s nodes into one without any loss of information".

1.2.25. **Definition.** Let $\Sigma_i = (V_i; X_i, W_i)$, $i = 1, 2$, be two s/s nodes or systems. By the cross product of $\Sigma_1$ and $\Sigma_2$ we mean the s/s node or system $\Sigma_\times = (V_\times; X_\times, W_\times)$ with state space $X_\times = [X_1, X_2]$ and signal space $W_\times = [W_1, W_2]$ with generating subspace $V_\times$ given by

$$V_\times = \left\{ \begin{bmatrix} z_i \\ x_i \\ w_i \end{bmatrix} \in \begin{bmatrix} X_1 \\ X_2 \\ W_1, W_2 \end{bmatrix} \mid x_i \in V_i, \ i = 1, 2 \right\}.$$  

We denote the cross product of $\Sigma_1$ and $\Sigma_2$ by $\Sigma_1 \times \Sigma_2$.

1.2.26. **Lemma.** Let $\Sigma_i = (V_i; X_i, W_i)$, $i = 1, 2$, be two s/s systems, and let $\Sigma_\times = \Sigma_1 \times \Sigma_2$ be the cross product of $\Sigma_1$ and $\Sigma_2$. Then $[\frac{z}{w}]$ is a classical or generalized trajectory of $\Sigma_\times$ on some interval $I$ if and only $x = [\frac{x_1}{x_2}]$ and $w = [\frac{w_1}{w_2}]$ where $[\frac{x}{w}]$ is a classical respectively generalized trajectory of $\Sigma_i$ in $I$, $i = 1, 2$.

**Proof.** This follows directly from Definitions 1.1.3, 1.1.6, and 1.2.25. \(\square\)

### 1.2.9. $(P,Q)$-interconnections of s/s nodes.

We proceed to define what we mean by the interconnection of two s/s nodes or systems.

1.2.27. **Definition.** Let $\Sigma_i = (V_i; X_i, W_i)$, $i = 1, 2$, be two s/s nodes or systems, let $X$ and $W$ be two $H$-spaces, and let $P: [X_1, X_2] \to X$ and $Q: [W_1, W_2] \to W$ be two continuous linear surjective operators with closed domains $\text{dom}(P) \subset [X_1, X_2]$ and $\text{dom}(Q) \subset [W_1, W_2]$. By the $(P,Q)$-interconnection of $\Sigma_1$ and $\Sigma_2$ we mean the $(P,Q)$-image $\Sigma = (V; X, W)$ of the cross product $\Sigma_1 \times \Sigma_2 = (V_\times; [X_1, X_2], [W_1, W_2])$ of $\Sigma_1$ and $\Sigma_2$, i.e., $V$ is given by

\[
V = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \left( V_\times \cap \begin{bmatrix} \text{dom}(P) \\ \text{dom}(P) \end{bmatrix} \right).
\]

1.2.28. **Lemma.** Let $\Sigma_i = (V_i; X_i, W_i)$, $i = 1, 2$, be two s/s nodes or systems and let $\Sigma = (V; X, W)$ be the $(P,Q)$-interconnection of $\Sigma_1$ and $\Sigma_2$, where $P: [X_1, X_2] \to X$ and $Q: [W_1, W_2] \to W$ are continuous linear surjective operators with closed domains $\text{dom}(P) \subset [X_1, X_2]$ and $\text{dom}(Q) \subset [W_1, W_2]$. If $[\frac{x_i}{w_i}]$, $i = 1, 2$ are classical or generalized trajectories of $\Sigma_i$, $i = 1, 2$ on some interval $I$ satisfying $[\frac{x_i(t)}{x_2(t)}] \in \text{dom}(P)$ and $[\frac{w_1(t)}{w_2(t)}] \in \text{dom}(Q)$ for (almost) all $t \in I$, then $[\frac{x}{w}]$, where $x = P[\frac{x_1}{x_2}]$ and $w = Q[\frac{x_2}{w_1}]$ is a classical respectively generalized trajectory of $\Sigma$ on $I$.\(\square\)
**Proof.** This follows from Lemmas 1.2.16 and 1.2.26. □

The following “short circuit” connection is a special case of the one described in Definition 1.2.27. In this connection we force the signals of the two systems \( \Sigma_i = (V_i; X_i, W) \) to be the same, and also force the states of the two systems to lie in a closed subspace of \( [X_1 \mid X_2] \), as drawn in Figure ??.

**1.2.29. Definition.** Let \( \Sigma_i = (V_i; X_i, W) \), \( i = 1, 2 \), be two s/s nodes or systems (with the same signal space), and let \( X_0 \) be a closed subspace of \( [X_1 \mid X_2] \). By the \( X_0 \)-short circuit \( \Sigma = (V; X_0 \mid X_0, W) \) of \( \Sigma_1 \) and \( \Sigma_2 \) we mean the \( (P, Q) \)-connection of \( \Sigma_1 \) and \( \Sigma_1 \), where \( P = 1_{[X_1 \mid X_2]} |_{X_0} \), \( \text{dom} \ (Q) = \{ [w] \mid w \in W \} \), and \( Q [w] = w \), \( [w] \in \text{dom} \ (Q) \), i.e.,

\[
(1.2.18) \quad V = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix} \in \begin{bmatrix} X_1 \\ X_0 \\ W \end{bmatrix} \mid \begin{bmatrix} z_i \\ x_i \\ w \end{bmatrix} \in V_i, \ i = 1, 2 \right\}.
\]

This notion will be particularly useful in the study of intertwined s/s systems (see Definition 1.5.22).
1.3. Properties of Trajectories of S/S Systems (Jan 02, 2016)

1.3.1. Basic properties of the sets of classical and generalized trajectories. Some elementary properties of the sets of all classical and all generalized trajectories of a s/s system Σ are listed in Lemma 1.3.2 below. In this lemma we use following notations:

1.3.1. Notation. The left-shift and right-shift semigroups or groups of operators acting on a functions defined on $\mathbb{R}^+$, $\mathbb{R}^-$, or $\mathbb{R}$ are denoted as follows: For all $t \in \mathbb{R}^+$ we denote
\[
(\tau_t^+ \varphi)(s) = \varphi(s + t), \quad s \in \mathbb{R}^+,
\]
\[
(\tau_t^* \varphi)(s) = \begin{cases} 
\varphi(s - t), & s \geq t, \\
0, & 0 \leq s < t,
\end{cases}
\]
\[
(\tau_t^- \varphi)(s) = \begin{cases} 
\varphi(s + t), & s \leq -t, \\
0, & -t < s \leq 0,
\end{cases}
\]
\[
(\tau_t^{*\prime} \varphi)(s) = \varphi(s - t), \quad s \in \mathbb{R}^-,
\]
and for all $t \in \mathbb{R}$ we denote
\[
(\tau_t^\prime \varphi)(s) = \varphi(s + t), \quad s \in \mathbb{R},
\]
\[
(\tau_t^{*\prime} \varphi)(s) = (\tau_t^{-\prime} \varphi)(s) = \varphi(s - t), \quad s \in \mathbb{R}.
\]

Thus, in all cases the upper index $t$ with $t > 0$ stands for a left-shift, and the upper index $*t$ with $t > 0$ stands for a right-shift.

The above sets have the following elementary properties.

1.3.2. Lemma. Let $\Sigma = (V; X, W)$ be a s/s system, and let $I \subset \mathbb{R}$ be an interval.

(i) Every classical trajectory of $\Sigma$ on $I$ is also a generalized trajectory of $\Sigma$ on $I$.

(ii) If $\Sigma$ is closed, then the set of classical trajectories of $\Sigma$ on $I$ is a closed subspace of $C^1(I; X)$ and the set of generalized trajectories of $\Sigma$ on $I$ is a closed subspace of $C(I; X)$ and the set of generalized trajectories of $\Sigma$ on $I$ is a closed subspace of $L^1_{\text{loc}}(I; W)$.

(iii) The restriction of a classical or generalized trajectory $[\bar{x}_w]$ of $\Sigma$ on $I$ to any subinterval $I'$ of $I$ is a classical respectively generalized trajectory of $\Sigma$ on $I'$.

(iv) If $[\bar{x}_w]$ is a classical or generalized trajectory of $\Sigma$ on some interval $I$, then, for all $t \in \mathbb{R}$, the shifted pair of functions $\begin{bmatrix} \tau_t^{\prime \prime} x_w \\ \tau_t^{\prime \prime} w \end{bmatrix} \begin{bmatrix} s \mapsto x(t + s) \\ s \mapsto w(t + s) \end{bmatrix}$ is a classical respectively generalized trajectory of $\Sigma$ on the shifted interval $\tau^t I := \{ s \in \mathbb{R} | t + s \in I \}$.

(v) The sets of past, future, and two-sided classical or generalized trajectories have the following invariance properties:

(a) The sets of all future classical trajectories and all future generalized trajectories of $\Sigma$ are left-shift invariant,

(b) The sets of all past classical trajectories and all past generalized trajectories of $\Sigma$ are right-shift invariant,

(c) The sets of all two-sided classical trajectories and all two-sided generalized trajectories of $\Sigma$ are both left and right shift invariant.
(vi) Let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are intervals satisfying \( I_1 \cap I_2 = \{ t_0 \} \) and \( t_0 \) is both the right end-point of \( I_1 \) and the left end-point of \( I_2 \), and let \([x_i^w]\) be a classical trajectory of \( \Sigma \) on \( I_i \), \( i = 1, 2 \). Define \([x^w]\) on \( I \) by

\[
\begin{bmatrix}
x(t) \\
w(t)
\end{bmatrix} = \begin{cases} 
\begin{bmatrix}
x_1(t) \\
w_1(t)
\end{bmatrix} & t \in I, \quad t < t_0, \\
\begin{bmatrix}
x_2(t) \\
w_2(t)
\end{bmatrix} & t \in I, \quad t \geq t_0.
\end{cases}
\]

Then \([x^w]\) is a classical trajectory of \( \Sigma \) on \( I \) if and only if

\[
\begin{bmatrix}
x_{1(t_0)} \\
w_{1(t_0)}
\end{bmatrix} = \begin{bmatrix}
x_{2(t_0)} \\
w_{2(t_0)}
\end{bmatrix}.
\]

The easy proof of this lemma is left to the reader.

1.3.2. Solvability and the uniqueness and continuation properties.

The conditions that we have imposed on a s/s node, and even on a regular s/s node, are not strong enough to imply that the set of classical trajectories of such a system is rich enough to have any interesting properties. Moreover, it need not be true that a classical trajectory on some interval \( I \) is uniquely determined by its signal component and the state evaluated at some time \( t_0 \in I \). Questions of this type arise in the study of classical trajectories on some time interval \( I \) in the forward, backward or two-sided time direction, with initial time \( t_0 \in I \) being the left or right end-point of \( I \), or an internal point of \( I \), respectively.

We begin by discussing existence and uniqueness of classical and generalized trajectories in the forward time direction. In this discussion we have dropped the word “forward”, adopting the convention that if no time direction is explicitly mentioned, then we are working in the forward time direction.

1.3.3. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system.

(i) \( \Sigma \) is \textit{solvable} if it is true that for every \( \begin{bmatrix} z^0 \\ x^0 \\ w^0 \end{bmatrix} \in V \) there is at least one classical future trajectory \([x^w]\) of \( \Sigma \) satisfying

\[
\begin{bmatrix}
x(0) \\
w(0)
\end{bmatrix} = \begin{bmatrix} z^0 \\ x^0 \\ w^0 \end{bmatrix}.
\]

(ii) \( \Sigma \) has the \textit{uniqueness property} if for every \( T > 0 \), every \( x^0 \in \mathcal{X} \), and every \( w \in C([0,T]; \mathcal{W}) \) there is at most one classical trajectory \([x^w]\) on \([0,T]\) with the given signal component \( w \) and initial state \( x(0) = x^0 \).

(iii) \( \Sigma \) is \textit{uniquely solvable} if it is both solvable and has the uniqueness property.

(iv) \( \Sigma \) has the \textit{continuation property} if it is true for every \( T > 0 \) that every generalized trajectory of \( \Sigma \) on the interval \([0,T]\) can be continued to a generalized future trajectory of \( \Sigma \).

1.3.4. Remark. The above definition of the uniqueness property refers to uniqueness of \textit{classical} trajectories. As we shall see in Lemma 1.3.22 below, for a closed s/s system it is also possible to define the same notion by using \textit{generalized} trajectories. In other words, \( \Sigma \) has the uniqueness property if and only if for every \( T > 0 \), every \( x^0 \in \mathcal{X} \), and every \( w \in L^1([0,T]; \mathcal{W}) \) there is at most one generalized trajectory \([x^w]\) on \([0,T]\) with the given signal component \( w \) and initial state \( x(0) = x^0 \).
1.3. REMARK. Later we shall also introduce the corresponding notions in the backward time direction, or in both time directions at the same time (see Definition 1.3.11). In those cases we shall replace “solvable” by forward solvable, “uniqueness and continuation properties” by forward uniqueness and continuation properties, and “uniquely solvable” by forward uniquely solvable when we refer to the properties defined in Definition 1.3.3.

1.3.6. LEMMA. Let \( \Sigma = (V; X, W) \) be a solvable s/s system.

(i) If \( I \) is an interval with left end-point \( a \in I \), then for every \( \begin{bmatrix} z_a \\ x_a \\ w_a \end{bmatrix} \in V \), there exists a classical trajectory \( \begin{bmatrix} x \\ w \end{bmatrix} \) of \( \Sigma \) on \( I \) satisfying \( \frac{dx}{da}(a) = \begin{bmatrix} z_a \\ x_a \\ w_a \end{bmatrix} \).

(ii) Every classical trajectory of \( \Sigma \) on some interval \( I \) with finite right end-point \( b \in I \) may be continued to a classical trajectory of \( \Sigma \) on \( I \cup [b, \infty) \).

PROOF. Claim (i) follows from Definition 1.3.3 and claims (iii) and (iv) in Lemma 1.3.2, and claim (ii) follows from Definition 1.3.3 and Lemma 1.3.2(vi). □

1.3.7. LEMMA. A s/s system \( \Sigma = (V; X, W) \) is solvable if and only if

\[
V = \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \mid \begin{bmatrix} x \\ w \end{bmatrix} \text{ is a classical future trajectory of } \Sigma \right\}.
\] (1.3.2)

PROOF. If \( \begin{bmatrix} x \\ w \end{bmatrix} \) is a classical future trajectory of \( \Sigma \), then by the definition of a classical trajectory, \( \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} := \begin{bmatrix} \dot{x}^{(0+)} \\ x^{(0)} \\ w^{(0)} \end{bmatrix} \in V \). Thus, the right-hand side of (1.3.2) is always contained in \( V \), and (1.3.2) is equivalent to the condition that for every \( \begin{bmatrix} z^0_a \\ x^0_a \\ w^0_a \end{bmatrix} \in V \) there is a classical trajectory of \( \Sigma \) with \( \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z^0_a \\ x^0_a \\ w^0_a \end{bmatrix} \). But this is exactly the condition for the classical solvability of \( \Sigma \) given in Definition 1.3.3(iii). □

1.3.8. COROLLARY. If the s/s system \( \Sigma = (V; X, W) \) is solvable, then the generating subspace \( V \) is uniquely determined by the set of classical future trajectories of \( \Sigma \).

PROOF. This follows from Lemma 1.3.7. □

1.3.9. LEMMA. Let \( \Sigma = (V; X, W) \) be a s/s system, and let the s/s system \( \Sigma_1 \) be \((P, Q)\)-similar to \( \Sigma \). Then the following claims are true:

(i) \( \Sigma \) is solvable if and only if \( \Sigma_1 \) is solvable.

(ii) \( \Sigma \) has the uniqueness property if and only if \( \Sigma_1 \) has the uniqueness property.

(iii) \( \Sigma \) has the continuation property if and only if \( \Sigma_1 \) has the continuation property.

PROOF. This follows immediately from the appropriate definitions. □

Lemma 1.2.13(i) has the following partial converse:

1.3.10. LEMMA. Let \( \Sigma \) and \( \Sigma_1 \) be two solvable s/s systems. If \( P \) and \( Q \) are bicontinuous linear operators, and if it is true that \( \begin{bmatrix} x \\ w \end{bmatrix} \) is a classical future trajectory of \( \Sigma \) if and only if \( \begin{bmatrix} Px \\ Qw \end{bmatrix} \) is a classical future trajectory of \( \Sigma_1 \), then \( \Sigma_1 \) is \((P, Q)\)-similar to \( \Sigma \).
Proof. This follows from Lemma 1.3.7. □

Above we have only looked at classical solvability and uniqueness in the forward time direction. It is, of course, possible to also work in the backward time direction, or in both time directions at the same time.

1.3.11. Definition. Let \( \Sigma = (V; X, W) \) be a s/s system.

(i) \( \Sigma \) is **backward solvable or two-sided solvable** if it is true that for every \( \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \in V \) there is at least one classical past respectively two-sided trajectory \( \left[ \begin{array}{c} x \\ w \end{array} \right] \) of \( \Sigma \) satisfying \( \left[ \begin{array}{c} \dot{x}(0) \\ x(0) \\ w(0) \end{array} \right] = \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \).

(ii) \( \Sigma \) has the **backward or two-sided uniqueness property** if for every finite closed interval \( I \) with right-end point zero or containing the point zero, respectively, every \( x^0 \in X \), and every \( w \in C(I; W) \) there is at most one classical trajectory \( \left[ \begin{array}{c} x \\ w \end{array} \right] \) on \( I \) with the given signal component \( w \) and initial state \( x(0) = x^0 \).

(iii) \( \Sigma \) is **backward or two-sided uniquely solvable** if it has both the respective properties in (i) and (ii).

(iv) \( \Sigma \) has the **backward continuation property** if it is true for every \( T > 0 \) that every generalized trajectory of \( \Sigma \) on the interval \( [-T, 0] \) can be continued to a generalized past trajectory of \( \Sigma \).

(v) \( \Sigma \) has the **two-sided continuation property** if it is true for every \( T > 0 \) that every generalized trajectory of \( \Sigma \) on the interval \( [-T, T] \) can be continued to a generalized two-sided trajectory of \( \Sigma \).

1.3.12. Remark. All the results that we have presented above about forward solvability or the forward uniqueness property remain valid for backward or two-sided classical solvability, and for the backward and two-sided uniqueness property with some obvious modifications. All the relevant results in the backward case can be obtained from the forward case by a time reflection (see Definition 1.2.1). The results for the two-sided case can be obtained by combining the forward and backward cases, as will be described in more detail below.

1.3.13. Lemma. A s/s system \( \Sigma \) is **two-sided solvable if and only if it is both forward and backward solvable.**

Proof. If \( \Sigma \) is two-sided solvable, then it follows from Lemma 1.3.2(ii) and Definition 1.3.3 that \( \Sigma \) is both forward and backward solvable. Conversely, suppose that \( \Sigma \) is both forward and backward solvable. Then, given any \( \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \in V \) there exists both a classical past trajectory \( \left[ \begin{array}{c} x_1 \\ w_1 \end{array} \right] \) with \( \left[ \begin{array}{c} x_1(0) \\ w_1(0) \end{array} \right] = \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \) and a classical future trajectory \( \left[ \begin{array}{c} x_2 \\ w_2 \end{array} \right] \) with \( \left[ \begin{array}{c} x_2(0) \\ w_2(0) \end{array} \right] = \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \) of \( \Sigma \). Define

\[
\begin{bmatrix}
x(t) \\
w(t)
\end{bmatrix} = \begin{cases} 
\begin{bmatrix} x_1(t) \\ w_1(t) \end{bmatrix}, & t < 0, \\
\begin{bmatrix} x_2(t) \\ w_2(t) \end{bmatrix}, & t \geq 0.
\end{cases}
\]

Then by Lemma 1.3.2 vi) \( \left[ \begin{array}{c} x \\ w \end{array} \right] \) is a two-sided classical trajectory of \( \Sigma \) with \( \left[ \begin{array}{c} x(0) \\ w(0) \end{array} \right] = \left[ \begin{array}{c} x_0 \\ w_0 \end{array} \right] \). □
1.3.14. **Lemma.** Let Σ be a s/s system, and let \( \Sigma^R \) be the time reflection of Σ.

(i) Σ is forward solvable if and only if \( \Sigma^R \) is backward solvable.

(ii) Σ has the forward uniqueness property if and only if \( \Sigma^R \) has the backward uniqueness property.

(iii) Σ has the forward continuation property if and only if \( \Sigma^R \) has the backward continuation property.

**Proof.** This follows from Definitions 1.3.3, 1.3.11, and 1.2.1. \( \square \)

Lemma 1.2.4 has the following partial converse:

1.3.15. **Lemma.** Let \( \Sigma \) be a solvable system, and let \( \Sigma_1 \) be backward a solvable system. If it is true that \([x \ w]\) is a classical future trajectory of \( \Sigma \) if and only if \([R x \ R w]\) is a classical past trajectory of \( \Sigma_1 \), then \( \Sigma_1 \) is the time reflection of \( \Sigma \).

**Proof.** This follows from Lemma 1.3.6. \( \square \)

As the following lemma shows, if Σ is a closed s/s system that satisfies condition (ii) in Definition 1.1.9 then this system may be reduced to a regular s/s system which has the same set of classical and general trajectories as the original system.

1.3.16. **Lemma.** Let \( \Sigma = (V; \mathcal{X}, W) \) be a closed s/s system which is forward or backward solvable, and define \( \lambda_0 \) as in (1.1.2). Then the following claims are true.

(i) \( V = V \cap \left[ \frac{x}{x_0} \frac{w}{w} \right] \).

(ii) \( \Sigma_0 = (V; \lambda_0, W) \) is a s/s system which satisfies condition (ii) in Definition 1.1.9.

(iii) \( \Sigma_0 \) is closed if and only if Σ is closed.

(iv) \( \Sigma_0 \) is a regular s/s system if and only if Σ is closed and V satisfies condition (iii) in Definition 1.1.9.

(v) \( \Sigma_0 \) and Σ have the same sets of classical and generalized trajectories.

(vi) \( \Sigma_0 \) has the forward or backward uniqueness property if and only if Σ has the same property.

(vii) \( \Sigma_0 \) has the forward or backward continuation property if and only if Σ has the same property.

**Proof.** See Remark 1.1.13. \( \square \)

1.3.3. **Connections between classical, generalized, and mild trajectories.** In the study of various connections between classical and generalized trajectories the following results will be useful.

1.3.17. **Lemma.** Let \([x \ w]\) be a generalized trajectory of the closed s/s system \( \Sigma = (V; \mathcal{X}, W) \) on some interval \( I \). Then, for all closed finite intervals \([t_1, t_2] \subset I \) we have

\[
\begin{bmatrix}
x(t_2) - x(t_1) \\
\int_{t_1}^{t_2} x(s) \, ds \\
\int_{t_1}^{t_2} w(s) \, ds
\end{bmatrix} \in V.
\]
As $k \to \infty$, we have
\[ \int_{t_1}^{t_2} \left[ x^k(s) \atop w^k(s) \right] ds \to \int_{t_1}^{t_2} \left[ x(s) \atop w(s) \right] ds \]
and
\[ \int_{t_1}^{t_2} \dot{x}^k(s) ds = x^k(t_2) - x^k(t_1) \to x(t_2) - x(t_1). \]
Since $V$ is closed, this gives (1.3.3). \hfill \Box

1.3.18. Definition. Let $\Sigma = (V; X, W)$ be a s/s system. We call $[\dot{x} \atop w] \in \left[ C(I; X), L^1_{\text{loc}}(I; W) \right]$ a mild trajectory of $\Sigma$ on the interval $I$ if (1.3.3) holds for all closed finite subintervals $[t_1, t_2]$ of $I$.

1.3.19. Lemma. Let $\Sigma = (V; X, W)$ be a closed s/s system. Then every generalized trajectory of $\Sigma$ on some interval $I$ is also a mild trajectory of $\Sigma$ on $I$. In particular, every classical trajectory of $\Sigma$ on $I$ is also a mild trajectory of $\Sigma$ on $I$.

Proof. See Lemma 1.3.17 and Definition 1.3.18. \hfill \Box

As will be shown in Lemma 1.3.26 below, the converse of Lemma 1.3.19 is also true when at least one of the end-points of $I$ does not belong to $I$ (i.e., $I$ is semi-open or open).

1.3.20. Lemma. Let $\Sigma = (V; X, W)$ be a s/s system, let $\Sigma_{\text{ext}} = (V_{\text{ext}}; X, [\dot{x}_{\text{ext}}])$ be the bounded input extension of $\Sigma$ with control operator $1_X$ (see Definition 1.2.21), and let $[\dot{x} \atop w] \in \left[ C(I; X), L^1_{\text{loc}}(I; W) \right]$ for some interval $I \subset \mathbb{R}$. Then the following conditions are equivalent:

(i) $[\dot{x} \atop w]$ is a mild trajectory of $\Sigma$ on $I$;
(ii) For some $t_0 \in I$ the pair $[x_{\text{ext}} \atop w_{\text{ext}}]$ is a classical trajectory of $\Sigma_{\text{ext}}$ on $I$, where $x_{\text{ext}}(t) = \int_{t_0}^{t} x(s) ds$ and $w_{\text{ext}}(t) = \left[ \int_{t_0}^{t} x(s) ds \right]$, $t \in I$.
(iii) Claim (ii) is true for all $t_0 \in I$.

Proof. (i) $\iff$ (iii): This follows from Definitions 1.1.6, 1.2.21, and 1.3.20 since $\frac{d}{dt} \int_{t_0}^{t} x(s) ds = x(t)$, $t \in I$.
(iii) $\Rightarrow$ (ii): This is trivial.
(ii) $\iff$ (i): Suppose that (ii) holds, and let $t_1 \in I$. Then for all $t \in I$ we have
\[
\begin{bmatrix}
  x(t) - x(t_0)
  \\
  \int_{t_0}^{t} x(s) ds
  \\
  \int_{t_0}^{t} w(s) ds
\end{bmatrix} \in V.
\]
Let \( t_1 \in I \). Then the above identity also holds with \( t \) replaced by \( t_1 \), and consequently
\[
\begin{pmatrix}
\int_{t_1}^{t} x(s) \, ds \\
\int_{t_1}^{t} w(s) \, ds
\end{pmatrix} = \begin{pmatrix}
\int_{t_1}^{t} x(s) \, ds \\
\int_{t_1}^{t} w(s) \, ds
\end{pmatrix} - \begin{pmatrix}
\int_{t_1}^{t} x(s) \, ds \\
\int_{t_1}^{t} w(s) \, ds
\end{pmatrix} \in V.
\]
Thus \([\frac{x}{w}]_{\text{ext}}\) is a classical trajectory of \( \Sigma_{\text{ext}} \). \( \square \)

1.3.21. Lemma. Let \([x]_{w}\) be a generalized trajectory of the closed s/s system \( \Sigma = (V; \mathcal{X}, W) \) on some interval \( I \), and let \( t_0 \in I \). If \( x(t_0) = 0 \), then \([x]_{w_1}\) defined by
\[
\begin{pmatrix}
x_1(t) \\
w_1(t)
\end{pmatrix} = \int_{t_0}^{t} \begin{pmatrix}
x(s) \\
w(s)
\end{pmatrix} \, ds, \quad t \in I,
\]
is a classical trajectory of \( \Sigma \) on \( I \) with \( x_1(t_0) = 0 \) and \( w_1(t_0) = 0 \) and with \( w_1 \in W_{1,2}^{1,I}(I; W) \) (and hence, in particular, \([x]_{w}\) is the derivative in the distribution sense of the classical trajectory \([x]_{w_1}\)).

Proof. This follows from Lemmas 1.2.24 and 1.3.17. \( \square \)

1.3.22. Lemma. A closed s/s system \( \Sigma = (V; \mathcal{X}, W) \) has the uniqueness property if and only if for every \( T > 0 \), every \( x^0 \in \mathcal{X} \), and every \( w \in L^1((0,T]; W) \) there is at most one generalized trajectory \([x]_{w}\) on \([0,T]\) with the given signal component \( w \) and initial state \( x(0) = x^0 \).

Proof. Clearly the condition given above implies that \( \Sigma \) has the uniqueness property (since every classical trajectory is also a generalized trajectory). Conversely, suppose that \( \Sigma \) has the uniqueness property, and suppose that \([x]_{w_1}\) and \([x]_{w_2}\) are two generalized trajectories of \( \Sigma \) on \([0,T]\) with \( x_1(0) = x_2(0) \) (and with the same signal component). Let \( x = x_1 - x_2 \). Then \( x(0) = 0 \) and \([x]_0\) is a generalized trajectory of \( \Sigma \) on \([0,T]\). Let \( x_3(t) = \int_{t_0}^{t} x(s) \, ds, \quad t \in [0,T] \). Then by Lemma 1.3.21 \([x]_0\) is a classical trajectory of \( \Sigma_{y=0} \) with \( x(0) = 0 \). As \( \Sigma \) has the uniqueness property, we must have \( x_3(t) = 0 \) for all \( t \in [0,T] \), and consequently also \( x(t) = \dot{x}_3(t) = 0 \) for all \( t \in [0,T] \). \( \square \)

1.3.23. Lemma. Let \( \Sigma = (V; \mathcal{X}, W) \) be a closed s/s system, and let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are intervals satisfying \( I_1 \cap I_2 = \{t_0\} \) and \( t_0 \) is both the right endpoint of \( I_1 \) and the left endpoint of \( I_2 \). Let \([x]_{w_i}\) be a mild trajectory of \( \Sigma \) on \( I_i \), \( i = 1, 2 \). Define \([x]_{w}\) on \( I \) by (1.3.1). The \([x]_{w}\) is a mild trajectory of \( \Sigma \) on \( I \) if and only if \( x_1(t_0) = x_2(t_0) \).

Proof. Clearly the condition \( x_1(t_0) = x_2(t_0) \) is necessary for \([x]_{w}\) to be a mild trajectory of \( \Sigma \) on \( I \) since \( x \) must be continuous.

Since \( x \) is a mild trajectory of \( \Sigma \) both on \( I_1 \) and on \( I_2 \) the equation (1.3.3) holds whenever both \( c \) and \( d \) belong to the same interval \( I_1 \) or \( I_2 \). In particular, (1.3.3) holds in intervals of the type \([c, t_0]\) and \([t_0, d]\), where \( c \in I_1 \) and \( d \in I_2 \). If, in addition, \( x_1(t_0) = x_2(t_0) \) then this implies that (1.3.3) also holds if \( c \in I_1 \) and \( d \in I_2 \). Thus \([x]_{w}\) is a mild trajectory of \( \Sigma \) on \( I \). \( \square \)

1.3.24. Lemma. The following claims are true for every closed s/s system \( \Sigma = (V; \mathcal{X}, W)\):
(i) Let $I$ be an interval with right end-point $+\infty$, and let $[\frac{x}{w}]$ be a generalized trajectory of $\Sigma$ on $I$. For each $n \in \mathbb{N}$, define $[\frac{x_n}{w_n}]$ by

\begin{equation}
\begin{bmatrix}
  x_n(t) \\
  w_n(t)
\end{bmatrix} := n \int_t^{t+1/n} \begin{bmatrix}
  x(s) \\
  w(s)
\end{bmatrix} ds, \quad t \in I.
\end{equation}

Then $[\frac{x_n}{w_n}]$ is a classical trajectory of $\Sigma$ on $I$, and $[\frac{x_n}{w_n}] \to [\frac{x}{w}]$ in $C(I;X)$ as $n \to \infty$.

(ii) Let $I$ be an interval with finite right end-point $b$, and let $[\frac{x}{w}]$ be a generalized trajectory of $\Sigma$ on $I \cup [b,b+\epsilon]$ for some $\epsilon > 0$. For each $n > 1/\epsilon$, define $[\frac{x_n}{w_n}]$ by (1.3.4). Then $[\frac{x_n}{w_n}]$ is a classical trajectory of $\Sigma$ on $I$, and $[\frac{x_n}{w_n}] \to [\frac{x}{w}]$ in $\mathcal{L}_{loc(I;W)}$ as $n \to \infty$.

(iii) Let $I$ be an interval with left end-point $-\infty$, and let $[\frac{x}{w}]$ be a generalized trajectory of $\Sigma$ on $I$. For each $n \in \mathbb{N}$, define $[\frac{x_n}{w_n}]$ by

\begin{equation}
\begin{bmatrix}
  x_n(t) \\
  w_n(t)
\end{bmatrix} := n \int_{t-1/n}^t \begin{bmatrix}
  x(s) \\
  w(s)
\end{bmatrix} ds, \quad t \in I.
\end{equation}

Then $[\frac{x_n}{w_n}]$ is a classical trajectory of $\Sigma$ on $I$, and $[\frac{x_n}{w_n}] \to [\frac{x}{w}]$ in $\mathcal{L}_{loc(I;W)}$ as $n \to \infty$.

(iv) Let $I$ be an interval with finite left end-point $a$, and let $[\frac{x}{w}]$ be a generalized trajectory of $\Sigma$ on $[a-a, a] \cup I$ for some $\epsilon > 0$. For each $n > 1/\epsilon$, define $[\frac{x_n}{w_n}]$ by (1.3.5). Then $[\frac{x_n}{w_n}]$ is a classical trajectory of $\Sigma$ on $I$, and $[\frac{x_n}{w_n}] \to [\frac{x}{w}]$ in $\mathcal{L}_{loc(I;W)}$ as $n \to \infty$.

**Proof.** (i) Clearly, each $[\frac{x_n}{w_n}] \in \left[ C^1(\mathbb{R}^+;X), C(\mathbb{R}^+;W) \right]$ and $\dot{x}_n(t) = n[x(t + 1/n) - x(t)]$ for all $t \in I$. By Lemma 1.3.17, $\begin{bmatrix}
  x_n(t) \\
  x_n(t)
\end{bmatrix} \in V$ for all $t \in I$, and hence $[\frac{x_n}{w_n}]$ is a classical trajectory of $\Sigma$ on $I$. The final claim in part (i) of Lemma 1.3.24 follows from standard properties of approximate identities (the scalar versions of these results are found in many places, such as [Gripenberg et al. 1990, p. 67], and the vector-valued versions can be proved in the same way).

(ii)–(iv) The proofs of (ii)–(iv) are analogous to the proof of (i). \hfill \square

1.3.25. **Corollary.** Let $\Sigma = (V;X,W)$ be a closed s/s system, and let $I$ be an infinite interval. Then a pair of function $[\frac{x}{w}] \in \left[ C^1(I;X), \mathcal{L}_{loc(I;W)} \right]$ is a generalized trajectory of $\Sigma$ on $I$ if and only if there exists a sequence $[\frac{x_n}{w_n}]$ of classical trajectories of $\Sigma$ which converges to $[\frac{x}{w}]$ in $\left[ C^1(I;X), \mathcal{L}_{loc(I;W)} \right]$ as $n \to \infty$.

**Proof.** This follows from Definition 1.1.6 and Lemma 1.3.24. \hfill \square

1.3.26. **Lemma.** Let $\Sigma = (V;X,W)$ be a closed s/s system, and let $I$ be interval for which at least one of the end point of $I$ does not belong to $I$ (i.e., $I$ is unbounded or semi-open or open). Then every mild trajectory $[\frac{x}{w}]$ of $\Sigma$ on $I$ is also a generalized trajectory of $\Sigma$ on $I$. (That the converse is also true follows from Lemma 1.3.19.)

**Proof.** We start by considering the case where $I$ is unbounded to the right. For each $n \in \mathbb{N}$ we define $[\frac{x_n}{w_n}]$ by (1.3.4). Then $x_n \in C^1(I;X)$ and $w_n \in C(I;W)$,
and \( \dot{x}_n(t) = n(x(t + 1/n) - x(t)), \ t \in I \). Since \([x^n]\) is a mild trajectory of \( \Sigma \) on \( I \) we therefore get
\[
\begin{bmatrix}
\dot{x}_n(t) \\
x_n(t) \\
w_n(t)
\end{bmatrix} = n
\begin{bmatrix}
x(t + 1/n) - x(t) \\
\int_{t}^{t+1/n} x(s) \, ds \\
\int_{t}^{t+1/n} w(s) \, ds
\end{bmatrix}
\in \mathcal{V}, \ t \in I, \ n \in \mathbb{N}.
\]
Thus, \([x^n]\) is a classical trajectory of \( \Sigma \) on \( I \). As \( n \to \infty \) we have \( x_n \to x \in C(I; \mathcal{X}) \) and \( w_n \to w \in L^1_{\text{loc}}(I; \mathcal{W}) \) (cf. the proof of Lemma 1.3.24), and thus \([w]\) a generalized trajectory of \( \Sigma \) on \( I \).

We next look at the case where the right end-point \( t_1 \) of \( I \) is finite, but \( t_1 \notin I \). Let \( \epsilon > 0 \) be small enough so that \( t_1 - \epsilon \) is an interior point of \( I \). Then for all \( n \in \mathbb{N} \), \( n > 1/\epsilon \) we may define \([x^n]\) as above, and by repreating the above argument we find that \([x^n]\) is a classical trajectory of \( \Sigma \) on the interval \( I_\epsilon = \{t \in I \mid t \leq \epsilon\} \), and that \( x_n \to x \in C(I_\epsilon; \mathcal{X}) \) and \( w_n \to w \in L^1_{\text{loc}}(I_\epsilon; \mathcal{W}) \). This implies that \([w]\) is a generalized trajectory of \( \Sigma \) on \( I \).

The case where the left end-point of \( I \) does not belong to \( I \) is proved in an analogous way (or it can be derived from the above argument by using a time reflection).

The following modified version of Lemma 1.3.26 is valid in the case where the interval \( I \) is closed and finite.

1.3.27. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a closed s/s system, and let \( I = [t_0, t_1] \) be a finite closed interval. If \([x]\) is a mild trajectory of \( \Sigma \) on \( I \), and in addition, \([x]\) can be extended to a mild trajectory on the interval \([t_0, t_1 + \epsilon]\) or the interval \([t_0 - \epsilon, t_1]\), then \([w]\) is a generalized trajectory of \( \Sigma \) on \( I \).

Proof. The proof is analogous to the proof of Lemma 1.3.26.

1.3.28. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a closed s/s system, and let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are intervals satisfying \( I_1 \cap I_2 = \{t_0\} \) and \( t_0 \) is both the right end-point of \( I_1 \) and the left end-point of \( I_2 \). Let \([x_{w_1}]\) be a generalized trajectory of \( \Sigma \) on \( I_i, i = 1, 2 \), and suppose that at least one of the following conditions hold:

(i) At least one of the intervals \( I_1 \) and \( I_2 \) is infinite or semi-open (i.e., the left end-point of \( I_1 \) does not belong to \( I_1 \), or the right end-point of \( I_2 \) does not belong to \( I_2 \)).

(ii) \([x_{w_1}]\) can be continued to a generalized trajectory on the interval \([t_1 - \epsilon, t_0]\) or \([x_{w_2}]\) can be continued to a generalized trajectory on the interval \([t_0, t_2 + \epsilon]\) for some \( \epsilon > 0 \).

Define \([w]\) on \( I \) by \((1.3.1)\). The \([w]\) is a generalized trajectory of \( \Sigma \) on \( I \) if and only if \( x_1(t_0) = x_2(t_0) \).

Proof. This follows from Lemmas 1.3.19, 1.3.23, 1.3.26 and 1.3.27.

1.3.29. Lemma. The following claims are true for every closed s/s system \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \):

(i) Let \( I \) be an interval with finite left-end point \( a \in I \). Then every generalized trajectory \([x]\) of \( \Sigma \) on \( I \) satisfying \( x(a) = 0 \) can be extended to a generalized trajectory of \( \Sigma \) on \( (-\infty, a] \cup I \) by taking \( \frac{x(t)}{w(t)} = 0 \) for \( t < a \).
(ii) Let \( [x_n^w] \) be the extended trajectory of \( \Sigma \) on \((−∞, a] \cup I \) considered in (i) which vanishes on \((−∞, a] \). For each \( n \in \mathbb{N} \), define \( [x_n^w] \) by (1.3.5) with \( I \) replaced by \((−∞, a] \cup I \). Then \( [x_n^w] \) is a classical trajectory of \( \Sigma \) on \((−∞, a] \cup I \). \( x_n(t) = 0 \) for all \( t \leq a \) and all \( n \in \mathbb{N} \), and \( [x_n^w] \rightarrow [x^w] \) in \( L^1([−∞, a]∪I;X) \) as \( n \rightarrow ∞ \).

(iii) Let \( I \) be an interval with finite right end-point \( b \in I \). Then every generalized trajectory \( [x_n^w] \) of \( \Sigma \) on \( I \) satisfying \( x(b) = 0 \) can be extended to a generalized trajectory of \( \Sigma \) on \( I \cup [b, ∞) \) by taking \( [x_n^w] \rightarrow [x_n^w] \) as \( t \rightarrow ∞ \).

(iv) Let \( [x_n^w] \) be the extended trajectory of \( \Sigma \) on \( I \cup [b, ∞) \) considered in (i) which vanishes on \([b, ∞) \). For each \( n \in \mathbb{N} \), define \( [x_n^w] \) by (1.3.4) with \( I \) replaced by \( I \cup [b, ∞) \). Then \( [x_n^w] \) is a classical trajectory of \( \Sigma \) on \( I \cup [b, ∞) \), \( x_n(t) = 0 \) for all \( t \geq b \) and all \( n \in \mathbb{N} \), and \( [x_n^w] \rightarrow [x^w] \) in \( L^1(I;X;W) \) as \( n \rightarrow ∞ \).

**Proof.** This follows from Lemma 1.3.28 by taking either \( [x_1^w] \) to be zero on \( I_1 \) or \( [x_2^w] \) to be zero on \( I_1 \). □

The following extension of Lemma 1.3.29 vi) is valid for the class of generalized trajectories:

**Lemma.** Let \( [x^w] \) be a generalized trajectory of the closed s/s system \( \Sigma = (V;X,W) \) on the interval \([a, b]\).

(i) If for some \( t \in [a, b] \) both

\[
\begin{align*}
z_t &= \lim_{h \to 0^+} \frac{1}{h} \left( x(t+h) - x(t) \right) \quad \text{and} \quad w_t = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} w(s) \, ds \end{align*}
\]

exist, then \( [z_t^w] \in V \).

(ii) If for some \( t \in [a, b] \) both

\[
\begin{align*}
z_t &= \lim_{h \to 0^+} \frac{1}{h} \left( x(t) - x(t-h) \right) \quad \text{and} \quad w_t = \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} w(s) \, ds \end{align*}
\]

exist, then \( [z_t^w] \in V \).

**Proof.** (i) Fix some \( t \in [a, b] \) and define \( [x_n^w] \) by (1.3.4) for all \( n > 1/(b-t) \) and \( t \in [a, b-1/n] \). Then \( [x_n^w] \) is a classical trajectory of \( \Sigma \) on \([a, b-1/n]\) with

\[
\begin{align*}
\begin{bmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{bmatrix} &= \frac{1}{n} \begin{bmatrix} x(t) \\ x(t+1/n) - x(t) \\ \int_{t}^{t+1/n} w(s) \, ds \end{bmatrix}.
\end{align*}
\]

If the limits in (1.3.6) exist, then the right-hand side of (1.3.8) tends to \( [z_t^w] \) as \( n \to ∞ \), and since \( V \) is closed, it then follows that \( [z_t^w] \in V \).

(ii) The proof of (ii) is essentially the same as the proof of (i), where integrals over the interval \([t, t+1/n]\) has been replaced by integrals over \([t-1/n, t]\). □
1.3.31. Theorem. Let $\Sigma = (V; X, W)$ be a closed s/s system. Then a generalized trajectory $[z]$ of $\Sigma$ on some interval $I$ is classical if and only if $[z] \in C^1(I; X) \cap C(I; W)$ (i.e., if and only if it has the necessary smoothness in order to be a classical trajectory).

Proof. By Definition 1.1.6(i), $[z]$ cannot be a classical trajectory of $\Sigma$ unless $[z] \in C^1(I; X) \cap C(I; W)$. Conversely, suppose that $[z] \in C^1(I; X) \cap C(I; W)$. Then (1.3.6) holds for all $t \in I$, possibly with the exception of the left end-point, and (1.3.7) holds for all $t \in I$, possibly with the exception of the right end-point, and in both cases $z_t = \dot{x}(t)$ and $w_t = w(t)$. Thus, by Lemma 1.3.30, $[\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix}] \in V$ for all $t \in I$, and hence $[z]$ is a classical trajectory of $\Sigma$ on $I$. \qed

1.3.32. Corollary. The generating subspace $V$ of a closed solvable s/s system $\Sigma = (V; X, W)$ is uniquely determined by the set of generalized future trajectories of $\Sigma$.

Proof. By Theorem 1.3.31, the set of generalized future trajectories of $\Sigma$ determines the set of classical future trajectories of $\Sigma$ uniquely. If $\Sigma$ is closed and solvable, then by Corollary 1.3.8 $V$ is uniquely determined by the set of classical future trajectories of $\Sigma$. \qed
1.4. Some Simple Examples (Jan 02, 2016)

In the finite-dimensional setting all s/s systems are two-sided uniquely solvable, and as will be shown in Corollary 2.2.28 all bounded s/s systems have the same property. However, as the following examples show, not all regular s/s systems are, for example, forward uniquely solvable. None of the examples below is bounded.

1.4.1. Example. Let $X = L^2(\mathbb{R}^+) \text{ and } W = \{0\}$. We denote the space of all locally absolutely continuous functions in $L^1(\mathbb{R}^+)$ whose derivative also belongs to $L^1(\mathbb{R}^+)$, and let $\Sigma = (V; X, W)$ be the s/s node with generating subspace

$$V = \left\{ \begin{bmatrix} \varphi' \\ \varphi \\ 0 \end{bmatrix} \middle| \varphi \in W^{1,2}(\mathbb{R}^+) \right\}.$$ 

If we ignore the signal component (which must be identically zero), then $V$ is the graph of the operator $A \varphi = \varphi'$, $\text{dom}(A) = W^{1,2}(\mathbb{R}^+)$. As is well-known, this operator is the generator of the left-shift semigroup $\tau_+^{t}$ on $L^1(\mathbb{R}^+)$ (cf. Notations 1.3.1). We shall discuss the notion of a $C_0$ semigroup and its generator in Section 4.1, but for the moment we refer the reader Staffans [2005] for a discussion of this example (see, in particular, Examples 2.3.2(ii) and 3.2.3(ii) in Staffans [2005]). Since $A$ is the generator of a $C_0$ semigroup, $A$ is closed and $\text{dom}(A)$ is dense in $X = L^2(\mathbb{R}^+)$. From this follows that $\Sigma$ is a regular s/s node.

From the above facts and the general theory about the connection between a $C_0$ semigroups and its generator we can conclude that Example 1.4.1 has the following properties. Each $\varphi \in W^{1,2}(\mathbb{R}^+)$ is a possible initial state of a classical future trajectory $[\xi]_0^t$ of the system $\Sigma$ in Example 1.4.1. This trajectory is determined uniquely by $\varphi$, and it is given by

$$(x(t))(\xi) = \varphi(t - \xi), \quad t \in \mathbb{R}^+.$$ 

Thus, $\Sigma$ is uniquely forward solvable. It is also true that for every $\varphi \in L^1(\mathbb{R}^+)$ there exists unique generalized future trajectory of $\Sigma$, given by the same formula with $\varphi \in L^1(\mathbb{R}^+)$. The situation is quite different if instead of studying future classical and generalized trajectories of the system $\Sigma$ in Example 1.4.1 we look at past classical and generalized trajectories of $\Sigma$. It is not difficult to see that $[\xi]_0^t$ is a classical past trajectory of $\Sigma$ if and only if $x$ is of the form

$$(x(t))(\xi) = \begin{cases} \varphi(\xi + t), & \xi \geq -t, \\ \psi(\xi + t), & 0 \leq \xi < -t, \end{cases}.$$ 

where $\varphi := x(0) \in W^{1,2}(\mathbb{R}^+)$ and $\psi$ is an arbitrary function in $W^{1,2}_{\text{loc}}(\mathbb{R}^-)$ satisfying $\psi(0) = \varphi(0)$. The same description is valid also for all generalized future trajectories of $\Sigma$ with $W^{1,2}$ replaced by $L^1$ and without the condition $\psi(0) = \varphi(0)$. Thus, this system is backward solvable but not uniquely.

Summarizing the above discussion we find that the s/s system $\Sigma$ in Example 1.4.1 is two-sided solvable and it has the forward uniqueness property, but not the backward uniqueness property.
1.4.2. Example. Let $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \{0\}$. We denote the space of all locally absolutely continuous functions in $L^1(\mathbb{R}^+)$ whose derivative also belongs to $L^1(\mathbb{R}^+)$ by $W^{1,2}(\mathbb{R}^+)$, and denote the space of all functions in $W^{1,2}(\mathbb{R}^+)$ which vanish at zero by $W^{1,2}_0(\mathbb{R}^+)$. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be the $s/s$ node with generating subspace

$$ V = \left\{ \begin{bmatrix} -\varphi' \\ \varphi \\ 0 \end{bmatrix} \middle| \varphi \in W^{1,2}_0(\mathbb{R}^+) \right\}, $$

where $\varphi'$ is the derivative of $\varphi$.

If again we ignore the signal component (which must be indentically zero), then $V$ is the graph of the operator

$$(1.4.2) \quad A\varphi = -\varphi', \quad \text{dom}(A) = W^{1,2}_0(\mathbb{R}^+).$$

This operator is the adjoint of the operator $A$ in (1.4.1), and $A$ is the generator of the right-shift semigroup $\tau^+_t$ on $L^1(\mathbb{R}^+)$ (see [Staffans, 2005, Example 3.5.11]). This implies that $A$ is closed, that the domain of $A$ is dense in $\mathcal{X} = L^2(\mathbb{R}^+)$, and that $\Sigma$ is a regular $s/s$ system.

From the general theory about the connection between a $C_0$ semigroups and its generator we can conclude that Example 1.4.2 has the following properties. Each $\varphi \in W^{1,2}_0(\mathbb{R}^+)$ is a possible initial state of a classical future trajectory $[\xi]$ of the system $\Sigma$ in Example 1.4.2. This trajectory is determined uniquely by $\varphi$, and it is given by

$$(x(t))(\xi) = \begin{cases} \varphi(\xi - t), & \xi \geq t, \\ 0, & 0 \leq \xi < t. \end{cases}$$

Thus, $\Sigma$ is uniquely forward solvable. It is also true that for every $\varphi \in L^1(\mathbb{R}^+)$ there exists unique generalized future trajectory of $\Sigma$, given by the same formula with $\varphi \in L^1(\mathbb{R}^+)$ instead of $\varphi \in W^{1,2}_0(\mathbb{R}^+)$. The situation is again different if we instead of looking at future classical and generalized trajectories of the system $\Sigma$ in Example 1.4.2 look at past classical and generalized trajectories of $\Sigma$. It is not difficult to see that a function $\varphi \in W^{1,2}_0(\mathbb{R}^+)$ is the initial state of a classical trajectory of $\Sigma$ on a time interval $[-T, 0]$ if and only if $\varphi(\xi) = 0$ for all $\xi \in [0, T]$, and in this case the state $x(t)$ for $t \in [-T, 0]$ of the corresponding unique trajectory of $\Sigma$ is given by

$$(x(t))(\xi) = \varphi(\xi - t), \quad \xi \in \mathbb{R}^-.$$ 

Consequently, $\Sigma$ has only one classical past trajectory, namely the zero trajectory. The same statement is true for generalized trajectories with $W^{1,2}_0(\mathbb{R}^+)$ replaced by $L^1(\mathbb{R}^+)$. Consequently, this system is not backward solvable, but it does have the backward uniqueness property.

Summarizing the above discussion we find that the $s/s$ system $\Sigma$ in Example 1.4.2 has the two-sided uniqueness property and is forward solvable, but not backward solvable.

1.4.3. Example. Let $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \{0\}$. We denote the space of all locally absolutely continuous functions in $L^1(\mathbb{R}^+)$ whose derivative also belongs to $L^1(\mathbb{R}^+)$ by $W^{1,2}(\mathbb{R}^+)$, and denote the space of all functions in $W^{1,2}(\mathbb{R}^+)$ which
vanish at zero by $W^{1,2}_0(\mathbb{R}^+)$. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be the s/s node with generating subspace

$$V = \left\{ \begin{bmatrix} \varphi' \\ \varphi \\ 0 \end{bmatrix} \mid \varphi \in W^{1,2}_0(\mathbb{R}^+) \right\},$$

where $\varphi'$ is the derivative of $\varphi$.

This is the time reflection of Example 1.4.2. Consequently, it follows from our analysis of Example 1.4.2 that this example has the two-sided uniqueness property and is backward solvable, but not forward solvable.

1.4.4. Example. Let $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \{0\}$. We denote the space of all locally absolutely continuous functions in $L^1(\mathbb{R}^+)$ whose derivative also belongs to $L^1(\mathbb{R}^+)$ by $W^{1,2}(\mathbb{R}^+)$, and let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be the s/s node with generating subspace

$$V = \left\{ \begin{bmatrix} -\varphi' \\ \varphi \\ 0 \end{bmatrix} \mid \varphi \in W^{1,2}(\mathbb{R}^+) \right\}.$$

This is the time reflection of Example 1.4.1. Consequently, it follows from our analysis of Example 1.4.1 that this example is two-sided solvable and it has the backward uniqueness property, but not the forward uniqueness property.

1.4.5. Example. Let $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \mathbb{C}$. We denote the space of all locally absolutely continuous functions in $L^1(\mathbb{R}^+)$ whose derivative also belongs to $L^1(\mathbb{R}^+)$ by $W^{1,2}(\mathbb{R}^+)$, and let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be the s/s node with generating subspace

$$(1.4.3) \quad V = \left\{ \begin{bmatrix} \varphi' \\ \varphi \\ \varphi(0) \end{bmatrix} \mid \varphi \in W^{1,2}(\mathbb{R}^+) \right\}.$$

This example can be interpreted as a well-posed i/s/o system with input space $\mathcal{U} = \{0\}$ and output space $\mathcal{Y} = \mathcal{W}$ (well-posed i/s/o systems will be discussed in more detail in Chapter 3), and it is a special case of Staffans [2005, Example 2.6.5(ii)] with $U = \{0\}$, $Y = W$, and $\mathcal{D} = 0$ (see also Examples 3.2.3(ii) and 4.4.3 in Staffans [2005]). The set of all generalized future trajectories $[\varphi]_X$ of this system can be characterized as follows: for each $\varphi \in L^1(\mathbb{R}^+)$ the system $\Sigma$ in Example 1.4.5 has the unique generalized future trajectory $[\varphi]$, where

$$(x(t))(\xi) = \varphi(t + \xi), \quad w(t) = x(t), \quad t \in \mathbb{R}^+.$$ 

The same characterization is valid for classical future trajectories of $\Sigma$ with the condition $\varphi \in L^1(\mathbb{R}^+)$ replaced by $\varphi \in W^{1,2}(\mathbb{R}^+)$. Thus, this s/s system is uniquely forward solvable.

Before saying anything about the classical and generalized past trajectories of this system we first take a closer look at the following example.

1.4.6. Example. Let $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \mathbb{C}$. We denote the space of all locally absolutely continuous functions in $L^1(\mathbb{R}^+)$ whose derivative also belongs to $L^1(\mathbb{R}^+)$ by $W^{1,2}(\mathbb{R}^+)$, and let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be the s/s node with generating subspace

$$(1.4.4) \quad V = \left\{ \begin{bmatrix} -\varphi' \\ \varphi \\ \varphi(0) \end{bmatrix} \mid \varphi \in W^{1,2}(\mathbb{R}^+) \right\}.$$
This is the time reflection of Example 1.4.5, so by our analysis of Example 1.4.5, the s/s system Σ in Example 1.4.6 is backward uniquely solvable.

To study how Example 1.4.6 behaves in the forward time direction we may again use the theory of well-posed i/s/o systems from Staffans [2005]. We begin by mapping the s/s system Σ in (1.4.6) into another s/s system with the help of a similarity transform. We take
\[ X_1 = L^1(\mathbb{R}^-), \]
interpret the operator \( R \) in (1.2.3) as a unitary similarity operator from \( X \) to \( X_1 \), and let \( \Sigma_1 = (V_1; X_1; C) \) be the s/s node which is \( (R, 1_C) \)-similar to \( \Sigma \). In other words,
\[
\begin{bmatrix}
\psi_1 \\
\varphi_1 \\
\varphi_1(0)
\end{bmatrix} \in V_1 \text{ if and only if } \begin{bmatrix}
-\varphi'_1 \\
-\psi_1 \\
\varphi_1(0)
\end{bmatrix} \in V,
\]
where \( \varphi_1 \in W^{1,2}(\mathbb{R}^+) \) and \( \psi_1 \in W^{1,2}(\mathbb{R}^-) \).

Thus
\[
(1.4.5) \quad V_1 = \left\{ \begin{bmatrix}
\varphi_1 \\
\varphi_1 \\
\varphi_1(0)
\end{bmatrix} : \varphi_1 \in W^{1,2}(\mathbb{R}^-) \right\}.
\]

The system \( \Sigma_1 \) with this generating subspace can be interpreted as a well-posed i/s/o system with input space \( U = W \) and output space \( Y = \{0\} \), and it is a special case of Staffans [2005] Example 2.6.5(i) with \( U = W \), \( Y = \{0\} \), and \( D = 0 \) (see also Examples 3.2.3(i) and 4.2.6(i) in Staffans [2005]). A pair of functions \( [x, w] \) is a classical future trajectory of \( \Sigma_1 \) if and only if \( w \in W^{1,2}(\mathbb{R}^+) \) and \( x \) is of the following form. Denote \( \varphi := x(0) \). Then \( \varphi \in W^{1,2}(\mathbb{R}^-) \), \( w(0) = \varphi(0) \), and
\[
(x(t))(\xi) = \begin{cases}
\varphi(\xi + t), & \xi \leq -t, \\
w(\xi + t), & -t < \xi \leq 0.
\end{cases}
\]

The same description is valid also for all generalized future trajectories of \( \Sigma \) with \( W^{1,2} \) replaced by \( L^1 \) and without the condition \( w(0) = \varphi(0) \).

Since \( \Sigma_1 \) is \( (R, 1_C) \)-similar to the s/s system \( \Sigma \) in Example 1.4.6 we find that a pair of functions \( [x, w] \) is a classical future trajectory of \( \Sigma_1 \) if and only if \( w \in W^{1,2}_{\text{loc}}(\mathbb{R}^+) \) and \( x \) is of the following form. Denote \( \varphi := x(0) \). Then \( \varphi \in W^{1,2}(\mathbb{R}^-) \), \( w(0) = \varphi(0) \), and
\[
(x(t))(\xi) = \begin{cases}
\varphi(\xi - t), & \xi \geq t, \\
w(t - \xi), & 0 < \xi \leq t.
\end{cases}
\]

The same description is valid also for all generalized future trajectories of \( \Sigma \) with \( W^{1,2} \) replaced by \( L^1 \) and without the condition \( w(0) = \varphi(0) \). Thus, both the s/s system \( \Sigma_1 \) and the s/s system in Example 1.4.6 are uniquely forward solvable.

Summarizing the above discussion, we find that both the s/s system \( \Sigma_1 \) above and the s/s system \( \Sigma \) in Example 1.4.6 are two-sided uniquely solvable.

We now return once more to Example 1.4.5. Since that example is the time reflection of Example 1.4.6 we conclude that also the s/s system in Example 1.4.5 is two-sided uniquely solvable.
1.5. Dynamical Properties of State/Signal Systems (Jan 02, 2016)

In this section we discuss a number of properties of s/s systems which are defined with the help of the set of classical and generalized trajectories of the system, such as controllability, observability, intertwinements, restrictions, projections, and compressions. These properties are allowed to depend on the direction of time, i.e., they need not be preserved under a time reflection.

1.5.1. Controllability and observability of state/signal systems.

1.5.1. Definition. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system.

(i) A state vector $x^0 \in \mathcal{X}$ is called (forward) **classically reachable** if there exists a classical past trajectory $[\mathcal{X}]$ of $\Sigma$ with compact support such that $x(0) = x^0$.

(ii) A state vector $x^0 \in \mathcal{X}$ is called (forward) **exactly reachable** if there exists a generalized past trajectory $[\mathcal{X}]$ of $\Sigma$ with compact support such that $x(0) = x^0$.

(iii) A classical or generalized future trajectory $[\mathcal{X}]$ of $\Sigma$ is called (forward) **unobservable** if its signal part $\mathcal{W}$ is identically zero.

(iv) A state vector $x^0 \in \mathcal{X}$ is called (forward) **classically unobservable** if there exists a classical future unobservable trajectory $[\mathcal{X}]$ of $\Sigma$ with $x(0) = x^0$.

(v) A state vector $x^0 \in \mathcal{X}$ is called (forward) **unobservable** if there exists a generalized future unobservable trajectory $[\mathcal{X}]$ of $\Sigma$ with $x(0) = x^0$.

1.5.2. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system. Then each of the following sets is a subspace of $\mathcal{X}$:

(i) the set of all classically reachable states of $\Sigma$;
(ii) the set of all exactly reachable states of $\Sigma$;
(iii) the set of all classically unobservable states of $\Sigma$;
(iv) the set of all unobservable states of $\Sigma$.

Proof. This follows from Definition 1.5.1 and Lemma 1.3.2.

1.5.3. Definition. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system.

(i) The subspace of all classically reachable states of $\Sigma$ is called the **classically reachable subspace** of $\Sigma$ and it is denoted by $R_{\Sigma}^{\text{class}}$.

(ii) The subspace of all exactly reachable states of $\Sigma$ is called the **exactly reachable subspace** of $\Sigma$ and it is denoted by $R_{\Sigma}^{\text{exact}}$.

(iii) The closure of $R_{\Sigma}^{\text{class}}$ is called the **(approximately) reachable subspace** of $\Sigma$ and it is denoted by $R_{\Sigma}$.

(iv) The subspace of all classically unobservable states of $\Sigma$ is called the **classically unobservable subspace** of $\Sigma$ and it is denoted by $U_{\Sigma}^{\text{class}}$.

(v) The subspace of all unobservable states of $\Sigma$ is called the **unobservable subspace** of $\Sigma$, and it is denoted by $U_{\Sigma}$.

1.5.4. Lemma. If $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is closed, then the subspaces defined above have the following properties:

(i) $R_{\Sigma}^{\text{class}} \subset R_{\Sigma}^{\text{exact}} \subset R_{\Sigma}$. Thus $R_{\Sigma}$ is also the closure of $R_{\Sigma}^{\text{exact}}$.

(ii) $U_{\Sigma}^{\text{class}} \subset U_{\Sigma}$ and $U_{\Sigma} \subset U_{\Sigma}^{\text{class}}$.

As we shall see later in Chapter 9, if $\Sigma$ is a well-posed s/s system, then $U_{\Sigma}$ is closed, and hence $U_{\Sigma} = U_{\Sigma}^{\text{class}}$. (This follows, e.g., from Lemma 9.1.19.)
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PROOF OF LEMMA 1.5.4. The inclusions $\mathcal{R}_\Sigma^{\text{class}} \subset \mathcal{R}_\Sigma^{\text{exact}}$ and $\mathcal{U}_\Sigma^{\text{class}} \subset \mathcal{U}_\Sigma$ follow directly from Definition 1.5.3.

Let $x^0 \in \mathcal{R}_\Sigma^{\text{exact}}$. Then there exists a generalized past trajectory $[\tilde{x}_w]_0$ of $\Sigma$ with compact support satisfying $x(0) = x^0$. For each integer $n > 0$ we define

$$[x_n(t) 
 w_n(t)] = n \int_{t-1/n}^t [x(s) 
 w(s)] \, ds, \quad t \in \mathbb{R}^-.$$ 

Then by Lemma 1.3.24(ii), $[\tilde{x}_w]_0$ is a classical past trajectory of $\Sigma$ with compact support, and hence $x_n(0) \in \mathcal{R}_\Sigma^{\text{class}}$. Since $x_n(0) \to x(0) = x^0$ as $n \to \infty$ we have $x^0 \in \mathcal{R}_\Sigma^{\text{class}} = \mathcal{R}_\Sigma$.

Let $x^0 \in \mathcal{U}_\Sigma$. Then there exists an unobservable generalized future trajectory $[\tilde{x}_w]_0$ of $\Sigma$ with $x(0) = x^0$. For each integer $n > 0$ we define

$$x_n(t) = n \int_t^{t+1/n} x(s) \, ds, \quad t \in \mathbb{R}^+.$$ 

Then by Lemma 1.3.24(i), $[\tilde{x}_w]_0$ is a classical future unobservable trajectory of $\Sigma$, and hence $x_n(0) \in \mathcal{U}_\Sigma^{\text{class}}$. Since $x_n(0) \to x(0) = x^0$ as $n \to \infty$ we have $x^0 \in \mathcal{U}_\Sigma^{\text{class}}$. □

1.5.5. LEMMA. For each s/s system $\Sigma = (V; X, W)$ the following claims are true:

(i) $x^0 \in \mathcal{R}_\Sigma^{\text{class}}$ if and only if there exists a classical trajectory $[\tilde{x}_w]$ of $\Sigma$ on some interval $[0, T]$ with $x(0) = 0$ and $w(0) = 0$ such that $x^0 = x(T)$.

(ii) If $\Sigma$ is closed, then $x^0 \in \mathcal{R}_\Sigma^{\text{exact}}$ if and only if there exists a generalized trajectory $[\tilde{x}_w]$ of $\Sigma$ on some interval $[0, T]$ with $x(0) = 0$ such that $x^0 = x(T)$.

PROOF. (i) Let $x^0 \in \mathcal{R}_\Sigma^{\text{class}}$. Then there exists a classical past trajectory $[\tilde{x}_w]$ of $\Sigma$ with support contained in some finite interval $[-T, 0]$ and with $x(0) = x^0$. By Lemma 1.3.2(iv), the right-translated pair of functions $[\tau_{-T}^{-} \tilde{x}_w]$ is a classical trajectory of $\Sigma$ on $[0, T]$ with $(\tau_{-T}^{-} x)(0) = x(-T) = 0$ and $(\tau_{-T}^{-} w)(0) = w(-T) = 0$ and with $(\tau_{-T}^{-} x)(T) = x(0) = x^0$.

Conversely, suppose that $[\tilde{x}_w]$ is a classical trajectory of $\Sigma$ on $[0, T]$ satisfying $x(0) = 0$ and $w(0) = 0$. By Lemma 1.3.2(vi), it is possible to extend $[\tilde{x}_w]$ to a classical trajectory of $\Sigma$ on $(-\infty, T]$ which vanishes on $\mathbb{R}^-$, and by Lemma 1.3.2(iv), the left-translated pair of functions $[\tau_{-T}^{+} \tilde{x}_w]$ is a generalized trajectory of $\Sigma$ on $\mathbb{R}^-$ whose support is contained in $[-T, 0]$ and satisfies $(\tau_{-T}^{+} x)(0) = x(T)$. This implies that $x(T) \in \mathcal{R}_\Sigma^{\text{class}}$.

(ii) The proof of claim (ii) is essentially the same as the one above, with Lemma 1.3.2(vi) replaced by Lemma 1.3.29(i). □

1.5.6. LEMMA. Let $\Sigma = (V; X, W)$ be a solvable s/s system. Then $x^0 \in \mathcal{R}_\Sigma^{\text{class}}$ if and only if there exists an exact future trajectory $[\tilde{x}_w]$ of $\Sigma$ with $x(0) = 0$ and $w(0) = 0$ such that $x^0 = x(t)$ for some $t \in \mathbb{R}^+$.

PROOF. This follows from Lemmas 1.3.6(ii) and 1.5.5(i). □

1.5.7. LEMMA. For every closed s/s system $\Sigma = (V; X, W)$ with reachable subspace $\mathcal{R}_\Sigma$ the following claims are true:
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(i) $\mathcal{R}_\Sigma$ is the closure of the set of all $x(T)$, where $x \in C^1([0, T]; \mathcal{X})$ is the state component of some classical trajectory $[\tilde{x}_1, \tilde{x}_w]$ of $\Sigma$ on the interval $[0, T]$ satisfying $x(0) = 0$ and $w(0) = 0$.

(ii) $\mathcal{R}_\Sigma$ is also the closure of the set of all $x(T)$, where $x \in C([0, T]; \mathcal{X})$ is the state component of some generalized trajectory $[\tilde{x}_1, \tilde{x}_w]$ of $\Sigma$ on the interval $[0, T]$ satisfying $x(0) = 0$.

**Proof.** This follows from Lemmas 1.5.4 and Lemma 1.5.5.

1.5.8. Definition. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system.

(i) A subspace $\mathcal{X}$ of $\mathcal{X}$ is strongly invariant for $\Sigma$ if the following condition holds: If $[\tilde{x}_w]$ is a generalized trajectory of $\Sigma$ on any interval $[0, T]$ with $T > 0$ satisfying $x(0) \in \mathcal{X}$, then $x(T) \in \mathcal{X}$.

(ii) A subspace $\mathcal{X}$ of $\mathcal{X}$ is unobservably invariant for $\Sigma$ if the following condition holds: For every $x^0 \in \mathcal{X}$ there exists a generalized future trajectory $[\tilde{x}_1, \tilde{x}_w]$ of $\Sigma$ satisfying $x(0) = x^0$ and $x(t) \in \mathcal{X}$ for all $t > 0$.

1.5.9. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a closed s/s system with exactly reachable subspace $\mathcal{R}_\Sigma^{\text{exact}}$ and unobservable subspace $\mathcal{U}_\Sigma$.

(i) $\mathcal{R}_\Sigma^{\text{exact}}$ is the minimal strongly invariant subspace for $\Sigma$, i.e., $\mathcal{R}_\Sigma^{\text{exact}}$ is strongly invariant for $\Sigma$, and it is contained in every other strongly invariant subspace for $\Sigma$.

(ii) $\mathcal{U}_\Sigma$ is the maximal unobservable invariant subspace for $\Sigma$, i.e., $\mathcal{U}_\Sigma$ is unobservable invariant for $\Sigma$, and it contains every other unobserving invariant subspace for $\Sigma$.

**Proof.** (i) Let $x^0 \in \mathcal{R}_\Sigma^{\text{exact}}$. Then by Definitions 1.5.1 and 1.5.3, there exists a generalized past trajectory $[\tilde{x}_1, \tilde{x}_w]$ of $\Sigma$ with compact support satisfying $x_1(0) = x^0$. Let $[\tilde{x}_2, \tilde{x}_w]$ be a generalized trajectory on some interval $[0, T]$ for some $T > 0$ satisfying $x_2(0) = x^0 = x_1(0)$. To prove that $\mathcal{R}_\Sigma^{\text{exact}}$ is strongly invariant we need to show that $x_2(T) \in \mathcal{R}_\Sigma^{\text{exact}}$. This can be done as follows. By Lemma 1.3.28 if we define $[\tilde{x}_w]$ by (1.3.1) with $I_1 = \mathbb{R}^+$ and $I_2 = [0, T]$, then $[\tilde{x}_w]$ is a generalized trajectory of $\Sigma$ on $(-\infty, T]$. Consequently, by Lemma 1.3.2, $[\tilde{r}_T x, \tilde{r}_T w]$ is a generalized past trajectory of $\Sigma$ satisfying $(\tau^T x)(0) = x_2(T)$. This implies that $x_2(T) \in \mathcal{R}_\Sigma^{\text{exact}}$. Thus, $\mathcal{R}_\Sigma^{\text{exact}}$ is strongly invariant, as claimed.

To show that $\mathcal{R}_\Sigma^{\text{exact}}$ is the minimal strongly invariant subspace for $\Sigma$ we let $\mathcal{X}$ be an arbitrary strongly invariant subspace for $\Sigma$, and let $x^0 \in \mathcal{R}_\Sigma^{\text{exact}}$. By Lemma 1.5.5(ii), there exists some generalized trajectory $[\tilde{x}_w]$ of $\Sigma$ on some interval $[0, T]$ with $x(0) = 0$ and $w(0) = 0$ such that $x^0 = x(T)$. On the other hand, since $x(0) = 0 \in \mathcal{X}$ the strong invariance of $\mathcal{X}$ implies that $x(T) = x^0 \in \mathcal{X}$. Thus $\mathcal{R}_\Sigma^{\text{exact}} \subset \mathcal{X}$.

(ii) That $\mathcal{U}_\Sigma$ contains every unobservable invariant subspace for $\Sigma$ and is itself an unobservable invariant subspace for $\Sigma$ follows directly from Definitions 1.5.3 and 1.5.8.

Under the present set of assumptions it is not known if the approximately reachable subspace $\mathcal{R}_\Sigma$ is strongly invariant or not. However, as the following lemma shows, the existence of a minimal closed strongly invariant subspace is always guaranteed. See also Lemma 8.2.4.

1.5.10. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a closed s/s system.
(i) For each closed subspace \( Z \) of \( \mathcal{X} \) there exists a unique minimal closed strongly invariant subspace \( Z_{\min} \) for \( \Sigma \) which contains \( Z \), i.e., \( Z_{\min} \) is closed and strongly invariant for \( \Sigma \) and \( Z_{\min} \) is contained in every other closed strongly invariant closed subspace for \( \Sigma \) which contains \( Z \). Moreover, \( R_\Sigma \subset Z_{\min} \), where \( R_\Sigma \) is the (approximately) reachable subspace of \( \Sigma \).

(ii) For each subspace \( Z \) of \( \mathcal{X} \) there exists a unique maximal unobservably invariant subspace \( Z_{\max} \) for \( \Sigma \) which is contained in \( Z \), i.e., \( Z_{\max} \) is unobservably invariant for \( \Sigma \) and \( Z_{\max} \) contains every other unobservably invariant subspace for \( \Sigma \) which is contained in \( Z \). Moreover, \( Z_{\max} \subset U_\Sigma \), where \( U_\Sigma \) is the unobservable subspace \( \Sigma \).

\[\begin{align*}
\text{Proof.} \quad (i) & \text{ Let } \{A_\alpha\}_{\alpha \in A} \text{ be the collection of all closed strongly invariant subspaces for } \Sigma \text{ which contains } Z. \text{ This collection is nonempty since the state space } \mathcal{X} \text{ itself is strongly invariant. Define } Z_{\min} = \cap_{\alpha \in A} Z_\alpha. \text{ Then } Z \subset Z_{\min}, Z_{\min} \text{ is closed, and it is contained in every closed strongly invariant subspace for } \Sigma \text{ which contains } Z. \text{ We claim that } Z_{\min} \text{ is strongly invariant for } \Sigma. \text{ To prove this we let } I \text{ be an interval of the type } I = [0, T] (\text{where } T > 0) \text{ or } I = \mathbb{R}^+, \text{ and let } [x_\alpha] \text{ be a generalized trajectory of } \Sigma \text{ on } I \text{ with } x(0) \in Z. \text{ The } x(0) \in Z_\alpha \text{ for all } \alpha \in A, \text{ and from the strong invariance of } Z_\alpha \text{ we get } x(t) \in Z_\alpha, \alpha \in A. \text{ Thus, } x(t) \in \cap_{\alpha \in A} Z_\alpha = Z, \quad t \in I. \text{ This proves that } Z \text{ is strongly invariant.}

\text{By Lemma } 1.5.9 \ R_\Sigma^{\text{exact}} \subset Z_{\min}. \text{ Since } Z_{\max} \text{ is closed, it therefore also contains the closure of } R_\Sigma^{\text{exact}}, \text{ which by definition is equal to } U_\Sigma.

(ii) \text{ Let } \{Z_\alpha\}_{\alpha \in A} \text{ be the collection of all unobservably invariant subspaces for } \Sigma \text{ which are contained in } Z. \text{ This collection is nonempty since it contains } \{0\}. \text{ Define } Z_{\max} = \text{span}_{\alpha \in A} Z_\alpha. \text{ Then } Z_{\max} \subset Z, \text{ and it contains every unobservably invariant subspace for } \Sigma \text{ which is contained in } Z. \text{ Each vector } x^0 \text{ in } Z_{\max} \text{ is a finite sum of vectors } x_\alpha^0, \text{ where each } x_\alpha^0 \in Z_\alpha. \text{ Each such vector is the initial state of an unobservable future trajectory } [x_\alpha^0] \text{ satisfying } x_\alpha(t) \in Z \text{ for all } t \in \mathbb{R}^+. \text{ The corresponding sum of these trajectories is an unobservable future trajectory } [x_0^0] \text{ satisfying } x(t) \in Z \text{ for all } t \in \mathbb{R}^+. \text{ Thus, } Z_{\max} \text{ is unobservable invariant. As we observed above, } Z_{\max} \subset U_\Sigma. \quad \square

1.5.11. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system.

(i) \( \Sigma \) is (forward approximately) controllable if \( R_\Sigma = \mathcal{X} \).

(ii) \( \Sigma \) is called (forward) observable if \( U_\Sigma = \{0\} \).

1.5.12. Corollary. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a closed s/s system.

(i) If \( \Sigma \) is controllable, then \( \mathcal{X} \) does not contain any proper closed strongly invariant subspace (proper means that this subspace is strictly contained in \( \mathcal{X} \)).

(ii) If \( \Sigma \) is observable, then \( \mathcal{X} \) does not contain any nonzero unobservably invariant subspace.

\[\begin{align*}
\text{Proof.}\ &\text{This follows from Lemma 1.5.10 and Definition 1.5.11.} \quad \square
\end{align*}\]

1.5.13. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system. If \( \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1) \) is \((P, Q)\)-similar to \( \Sigma \), and if we denote the classically reachable subspaces of \( \Sigma \) and \( \Sigma_1 \) by \( R_\Sigma^{\text{class}} \) and \( R_{\Sigma_1}^{\text{class}} \) and the classically unobservable subspaces by \( U_\Sigma^{\text{class}} \) and \( U_{\Sigma_1}^{\text{class}} \), respectively, then \( R_\Sigma^{\text{class}} = PR_{\Sigma_1}^{\text{class}} \) and \( U_\Sigma^{\text{class}} = PU_{\Sigma_1}^{\text{class}} \). In particular, \( \Sigma_1 \) is controllable or observable if and only if \( \Sigma \) is controllable respectively observable.
Proof. This follows from the Definitions 1.5.1, 1.5.3, and 1.5.11.

However, the reachable subspace and the unobservable subspace of a s/s system are in general not preserved under time reflection. See Examples 1.5.15–1.5.20 below.

1.5.14. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system.

(i) By the backward reachable subspace and the backward unobservable subspace of \( \Sigma \) we mean the reachable subspace and the unobservable subspace, respectively, of the time reflection \( \Sigma^R \) of \( \Sigma \).

(ii) A subspace \( \mathcal{Z} \) of \( \mathcal{X} \) is backward strongly invariant or backward unobservably invariant for \( \Sigma \) if \( \mathcal{Z} \) is strongly invariant respectively unobservably invariant for the time reflection \( \Sigma^R \) of \( \Sigma \).

(iii) \( \Sigma \) is backward controllable or backward observable if the time reflection \( \Sigma^R \) of \( \Sigma \) is controllable respectively observable.

For systems in finite-dimensional spaces all the notions introduced in the above definitions are reasonable and useful. For systems in infinite-dimensional spaces they can lead to strange results unless we add some additional conditions on the system, such as forward classical solvability or uniqueness, as the following examples show.

1.5.15. Example. In Example 1.4.1 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{W} = \{0\} \) both the forward and the backward unobservable subspace is \( \mathcal{X} \), so \( \Sigma \) is neither forward nor backward observable. The forward reachable subspace is \( \{0\} \), so this example is not forward controllable. However, the backward reachable subspace is \( \mathcal{X} \), so this example is backward controllable, in spite of the fact that the signal space is \( \{0\} \). Recall that this example does not have the backward uniqueness property.

1.5.16. Example. In Example 1.4.2 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{W} = \{0\} \) both the forward and the backward reachable subspace is \( \{0\} \), so \( \Sigma \) is neither forward nor backward controllable. The forward unobservable subspace is \( \mathcal{X} \), so this example is not forward observable. However, the only generalized past trajectory of this example is the zero trajectory \( [\hat{x}] = [0] \), so the backward unobservable subspace is \( \{0\} \). This means that this example is backward observable, in spite of the fact that its signal space is \( \{0\} \). Recall that this example is not backward solvable.

1.5.17. Example. Example 1.4.3 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{W} = \{0\} \) is the time reflection of Example 1.5.16, so this example is neither forward nor backward controllable and it is not backward observable, but it is forward observable, in spite of the fact that its signal space is \( \{0\} \). Recall that this example is not forward solvable.

1.5.18. Example. Example 1.4.4 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{W} = \{0\} \) is the time reflection of Example 1.5.15, so this example is neither forward nor backward observable and it is not backward controllable, but it is forward controllable, in spite of the fact that its signal space is \( \{0\} \). Recall that this example does not have the forward uniqueness property.

1.5.19. Example. In Example 1.4.5 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{W} = \mathbb{C} \) both the forward reachable subspace and the forward unobservable subspace is \( \{0\} \), and both the backward unobservable subspace and the backward reachable subspace is \( \mathcal{X} \). This system is forward observable and backward controllable, but it is not forward
controllable and not backward observable. Recall that this example is two-sided uniquely solvable.

1.5.20. Example. Example 1.4.6 with $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \mathbb{C}$ is the time reflection of Example 1.5.19 so this example is forward controllable and backward observable, but it is not forward observable and not backward controllable. Recall that this example is two-sided uniquely solvable.

Because of the above examples, we shall usually apply the definitions related to forward or backward controllability and observability only to i/s/o systems that are forward respectively backward uniquely solvable.

1.5.2. Intertwinements and compressions of state/signal systems. In this subsection we define a number of additional time domain notions in the theory of s/s systems in continuous time. We do not claim that these notions are necessarily meaningful for all possible subclasses of s/s systems, but as we shall see in Chapter 11 they are meaningful at least for the class of well-posed s/s system, and hence also for the class of passive s/s systems discussed in Chapter 4.

1.5.21. Definition. Two s/s system $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), i = 1, 2,$ (with the same signal space) are externally equivalent if they satisfy the following condition for all intervals $I$ of the type $I = [0, T]$ (where $T > 0$) as well as for $I = \mathbb{R}^+: \Sigma_1$ has a generalized trajectory $[x^1_w]$ on $I$ with zero initial state $x_1(0) = 0$ and signal $w \in L^1_{loc}(I; \mathcal{W})$ if and only if $\Sigma_2$ has a generalized trajectory $[x^2_w]$ on $I$ with zero initial state $x_2(0) = 0$ and the same signal $w$.

In the following definition (and many other places) we shall need the notion of a linear multi-valued operator from one $H$-space $\mathcal{X}_1$ into another $H$-spaces $\mathcal{X}_2$. We denote the set of all such operators by $\mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ and abbreviate $\mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ to $\mathcal{ML}(\mathcal{X})$. A short review of the theory of linear multi-valued operators is given in Appendix A.4.

1.5.22. Definition. Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), i = 1, 2,$ be two s/s system (with the same signal space), and let $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$. We say that $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$ if the following two conditions holds for all intervals $I$ of the type $I = [0, T]$ (where $T > 0$) and for $I = \mathbb{R}^+$:

(i) If $[x^1_w]$ is a generalized trajectory of $\Sigma_1$ on $I$ with $x_1(0) \in \text{dom}(P)$, then for every $x^0_1 \in Px_1(0)$ there exists a generalized trajectory $[x^2_w]$ of $\Sigma_2$ on $I$ satisfying $x_2(0) = x^0_1$ and $x_2(t) \in Px_1(t)$ for all $t \in I$.

(ii) Condition (i) above also holds if we interchange $\Sigma_1$ and $\Sigma_2$ and replace $P$ by $P^{-1}$. In other words, if $[x^1_w]$ is a generalized trajectory of $\Sigma_2$ on $I$ with $x_1(0) \in \text{rng}(P)$, then for every $x^0_1 \in P^{-1}x_2(0)$ there exists a generalized trajectory $[x^1_w]$ of $\Sigma_1$ on $I$ satisfying $x_1(0) = x^0_1$ and $x_2(t) \in Px_1(t)$ for all $t \in I$.

1.5.23. Definition. Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), i = 1, 2,$ be two s/s systems (with the same signal space). We say that $\Sigma_1$ and $\Sigma_2$ are pseudo-similar if they are intertwined by a closed injective (single-valued) linear operator $P: \mathcal{X} \to \mathcal{X}_1$ with dense domain and dense range, called the pseudo-similarity operator.

1.5.24. Lemma. Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), i = 1, 2,$ be two s/s systems (with the same signal space).
(i) $\Sigma_1$ and $\Sigma_2$ are intertwined by $P \in M\ell(X_1; X_2)$ if and only if $\Sigma_2$ and $\Sigma_1$ are intertwined by $P^{-1}$.
(ii) $\Sigma_1$ and $\Sigma_2$ are pseudo-similar with pseudo-similarity operator $P$ if and only if $\Sigma_2$ and $\Sigma_1$ are pseudo-similar with pseudo-similarity operator $P^{-1}$.

**Proof.** This follows directly from Definition 1.5.23.

1.5.25. **Lemma.** Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2, 3$, be three s/s systems. If $\Sigma_1$ and $\Sigma_2$ are intertwined by $P \in M\ell(X_1; X_2)$ and $\Sigma_2$ and $\Sigma_3$ are intertwined by $P \in M\ell(X_2; X_3)$, then $\Sigma_1$ and $\Sigma_3$ are intertwined by $P_3 := P_2 P_1$.

**Proof.** Claim (i) follows immediately from Definition 1.5.22 and the definition of the composition of two multi-valued operators.

1.5.26. **Lemma.** Let $\Sigma_i = (V_i; X_i, W)$ be two solvable s/s systems, and let $P \in B(X_i; X_2)$ have an inverse in $B(X_2; X_1)$. Then $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$ if and only if $\Sigma_1$ and $\Sigma_2$ are $(P, 1_W)$-similar (see Definition 1.2.11).

**Proof.** This follows from Lemmas 1.2.13 and 1.3.10 and Definition 1.5.22.

1.5.27. **Lemma.** Let $\Sigma_i = (V_i; X_i, W)$ be two closed s/s systems with exactly reachable subspaces $R_{\Sigma_i}^{\text{exact}}$ and unobservable subspaces $U_{\Sigma_i}$, $i = 1, 2$. If $\Sigma_1$ and $\Sigma_2$ are intertwined by $P \in M\ell(X_1; X_2)$, then the following claims hold:

(i) $\Sigma_1$ and $\Sigma_2$ are externally equivalent.
(ii) $\text{dom} (P)$ is strongly invariant for $\Sigma_1$. In particular, $R_{\Sigma_1}^{\text{exact}} \subset \text{dom} (P)$.
(iii) $\ker (P)$ is unobservably invariant for $\Sigma_1$. In particular, $\ker (P) \subset U_{\Sigma_1}$.
(iv) $\text{rng} (P)$ is strongly invariant for $\Sigma_2$. In particular, $R_{\Sigma_2}^{\text{exact}} \subset \text{rng} (P)$.
(v) $\text{mul} (P)$ is unobservably invariant for $\Sigma_2$. In particular, $\text{mul} (P) \subset U_{\Sigma_2}$.

**Proof.** (i) Claim (i) follows immediately from Definitions 1.5.21 and 1.5.22.
(ii) Claim (ii) follows immediately from Definitions 1.5.8 and 1.5.22 and Lemma 1.5.9.
(iii) Let $x_1^0 \in \ker (P)$. Then $0 \in P x_1^0$. Since $\left[ x_1^0 \right] = \left[ 0 \right]$ is a generalized future trajectory of $\Sigma_2$, it follows from condition (ii) in Definition 1.5.22 that $\Sigma_1$ has a generalized future trajectory $\left[ x_2^0 \right]$ satisfying $x_1(0) = x_1^0$ and $0 = x_2(t) \in P x_1(t)$ for all $t \in \mathbb{R}^+$, i.e., $x_1(t) \in \ker (P)$ for all $t \in \mathbb{R}^+$. Thus, $\ker (P)$ is unobservably invariant for $\Sigma_1$. The last claim follows from Lemma 1.5.9.
(iv)–(v) Claims (iv) and (v) follow from (ii) and (iii) if we interchange $\Sigma_1$ and $\Sigma_2$ and replace $P$ by $P^{-1}$.

1.5.28. **Definition.** Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two s/s nodes (with the same signal space), where $X_1$ is a closed subspace of $X_2$, and let $Z_1$ be a direct complement to $X_1$ in $X_2$. We call $\Sigma_1$ a compression of $\Sigma_2$ onto $X_1$ along $Z_1$, and we call $\Sigma_2$ a dilation of $\Sigma_1$ along $Z_1$, if the following two conditions hold for all intervals $I$ of the type $I = [0, T]$ (where $T > 0$) and for $I = \mathbb{R}^+$:

(i) If $\left[ x_2^0 \right]$ is a generalized trajectory of $\Sigma_2$ on $I$ with $x_2(0) \in X_1$, then $\left[ P x_2^0 \right]$ is a generalized trajectory of $\Sigma_1$ on $I$.
(ii) For each generalized trajectory $\left[ x_1^0 \right]$ of $\Sigma_1$ on $I$ there exists some generalized trajectory $\left[ x_2^0 \right]$ of $\Sigma_2$ on $I$ with $x_2(0) = x_1(0) \in X_1$ such that $x_1 = P^{x_2^0} x_2$. 


1.5.29. **Lemma.** If the s/s system $\Sigma_1 = (V_1; \mathcal{X}_1; \mathcal{W})$ is the compression of the s/s system $\Sigma_2 = (V_2; \mathcal{X}_2; \mathcal{W})$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, then $\Sigma_1$ and $\Sigma_2$ are externally equivalent.

**Proof.** This follows immediately from Definitions 1.5.21 and 1.5.28. □

1.5.30. **Lemma.** Let $\Sigma_i = (V_i; \mathcal{X}_i; \mathcal{U}, \mathcal{Y})$ be three s/s systems. If $\Sigma_2$ is the compression of $\Sigma_3$ onto $\mathcal{X}_2$ along $\mathcal{Z}_2$ and $\Sigma_1$ is the compression of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, then $\Sigma_1$ is the compression of $\Sigma_3$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1 + \mathcal{Z}_2$.

**Proof.** Let $[x_3^0]$ be a generalized trajectory of $\Sigma_3$ on $I$ with $x_3(0) \in \mathcal{X}_1$, and define $x_2 = P_{\mathcal{X}_2} x_3$. Then $[x_2^0]$ is a generalized trajectory of $\Sigma_3$ on $I$ with $x_3(0) \in \mathcal{X}_1$. Define $x_1 = P_{\mathcal{X}_1} x_2 = P_{\mathcal{X}_1}^{x_2^0} z_3 x_3$. Then $[x_1^0]$ is a generalized trajectory of $\Sigma_1$ on $I$. Thus, condition (i) in Definition 1.5.28 holds with $\mathcal{X}_3$ replaced by $\mathcal{X}_3$ and $\mathcal{Z}_1$ replaced by $\mathcal{Z}_1 + \mathcal{Z}_2$. That also condition (ii) holds in proved in an analogous way. □

1.5.31. **Lemma.** Let $\Sigma_2 = (V_2; \mathcal{X}_2; \mathcal{W})$ be a closed s/s system, and let $\mathcal{X}_2 = \mathcal{X}_1 + \mathcal{Z}_1$.

(i) $\Sigma_2$ has at most one closed solvable compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$.

(ii) If $\Sigma_2$ has the uniqueness property, then every closed compression of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ has the uniqueness property.

**Proof.** (i) Claim (i) follows from Corollary 1.3.32 and the fact that the set of all generalized future trajectories of $\Sigma_1$ is determined uniquely by the set of all generalized future trajectories of $\Sigma_2$.

(ii) Suppose that $\Sigma_1 = (V_1; \mathcal{X}, \mathcal{W})$ is a compression of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, and let $[x_2^0]$ be a generalized trajectory of $\Sigma_1$ on $[0, T]$. Then there exists some generalized trajectory $[x_2^0]$ of $\Sigma_2$ on $[0, T]$ satisfying $x_2(0) = x_1(0)$ and $x_1(t) = P_{\mathcal{X}_1}^{x_2^0} x_2(t)$ for all $t \in [0, T]$. By Lemma 1.3.22, $x_2$ is determined uniquely by its initial state $x_2(0) = x_1(0)$ and the signal $w$. Since $x_1 = P_{\mathcal{X}_1}^{x_2^0} x_2$, also $x_1$ is determined uniquely by $x_1(0)$ and $w$. Applying Lemma 1.3.22 once more we find that $\Sigma_1$ has the uniqueness property. □

1.5.32. **Lemma.** Let the closed s/s system $\Sigma_1$ be the compression of the closed s/s system $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$. For $i = 1, 2$ we denote the unobservable subspace of $\Sigma_i$ by $\mathfrak{U}_{\Sigma_i}$, the exactly reachable subspace of $\Sigma_i$ by $\mathcal{R}_{\Sigma_i}^{\text{exact}}$, and the reachable subspace of $\Sigma_i$ by $\mathcal{R}_{\Sigma_i}$. Then

(i) $\mathfrak{U}_{\Sigma_1} = \mathfrak{U}_{\Sigma_2} \cap \mathcal{X}_1$. In particular, $\Sigma_1$ is observable whenever $\Sigma_2$ is observable.

(ii) $\mathcal{R}_{\Sigma_1}^{\text{exact}} = P_{\mathcal{X}_1}^{\mathfrak{U}_{\Sigma_2}} \mathcal{R}_{\Sigma_2}^{\text{exact}}$ and $\mathfrak{U}_{\Sigma_1} = P_{\mathcal{X}_1}^{\mathfrak{U}_{\Sigma_2}} \mathcal{R}_{\Sigma_2}$. In particular, $\Sigma_1$ is controllable whenever $\Sigma_2$ is controllable.

**Proof.** (i) If $x_1^0 \in \mathfrak{U}_{\Sigma_1}$, then there exists a generalized future unobservable trajectory $[x_1^0]$ of $\Sigma_1$ with $x_1(0) = x_1^0$. This implies that there exists a generalized future unobservable trajectory $[x_2^0]$ with $x_2(0) = x_1(0) = x_1^0$. Thus $x_1^0 \in \mathfrak{U}_{\Sigma_2} \cap \mathcal{X}_1$. Conversely, if $x_1^0 \in \mathfrak{U}_{\Sigma_2} \cap \mathcal{X}_1$, then there exists a generalized future unobservable trajectory $[x_1^0]$ of $\Sigma_2$ with $x_1(0) = x_1^0$, and $[x_1^{x_1^0} x_2^0]$ is a generalized future unobservable trajectory of $\Sigma_1$. This means that $P_{\mathcal{X}_1}^{x_1^0} x_2(0) = x_1^0 \in \mathfrak{U}_{\Sigma_1}$.
Lemma 1.3.31, it is even a classical trajectory of $\Sigma^1$ and hence a compression of $\Sigma^2$, $[x^2_0,w^2_{\Sigma^2}]$ is a generalized trajectory of $\Sigma^1$ on $[0,T]$. Consequently, again by Lemma 1.5.5, $P_{X^1}x_2(t) = P_{X^1}x_2(0) \in \mathcal{R}_{\Sigma^1}$. Thus $P_{X^1}\mathcal{R}_{\Sigma^1} \subset \mathcal{R}_{\Sigma^1}$. The proof of the converse inclusion $\mathcal{R}_{\Sigma^1} \subset P_{X^1}\mathcal{R}_{\Sigma^1}$ is analogous. Thus $\mathcal{R}_{\Sigma^1} = P_{X^1}\mathcal{R}_{\Sigma^1}$. This combined with Lemma 1.5.4 implies that $\mathcal{R}_{\Sigma^1} = \mathcal{R}_{\Sigma^1}$. 

1.5.5. DEFINITION. Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two s/s systems (with the same signal space), where $X_i$ is a closed subspace of $X_i$. We call $\Sigma_i$ a restriction of $\Sigma_2$ to $X_i$ if the following two conditions hold for all intervals $I$ of the type $I = [0,T]$ (where $T > 0$) and for $I = \mathbb{R}^+$:

(i) Every generalized trajectory of $\Sigma_1$ on $I$ is also a generalized trajectory of $\Sigma_2$ on $I$.

(ii) If $[x^2_0,w^2]$ is a generalized trajectory of $\Sigma_2$ on $I$ with $x_2(0) \in X_1$, then $x_2(t) \in X_1$ for all $t \in I$, and $[x^2_0,w^2]$ is also a generalized trajectory of $\Sigma_1$ on $I$.

1.5.34. LEMMA. Let $\Sigma_i = (V_i; X_i, W)$ be three s/s systems. If $\Sigma_2$ is the restriction of $\Sigma_3$ to $X_2$ and $\Sigma_1$ is the restriction of $\Sigma_2$ to $X_1$, then $\Sigma_1$ is the restriction of $\Sigma_3$ to $X_1$.

PROOF. The proof is analogous to the proof of Lemma 1.5.30.

1.5.35. LEMMA. Let the s/s system $\Sigma_1 = (V_1; X_1, W)$ be the restriction to $X_1$ of the s/s system $\Sigma = (V; X, W)$, and suppose that $\Sigma$ is closed and solvable and that $\Sigma_1$ is closed. Then $\Sigma_1$ is solvable and

\[
(1.5.1) \quad V_1 = V \cap \left[ \frac{X_1}{W} \right] = V \cap \left[ \frac{X^*}{W} \right].
\]

Thus, in particular, $\Sigma_1$ is the part of $\Sigma$ in $\left[ \frac{X_1}{W} \right]$.

PROOF. Let $\left[ \begin{array}{c} x^0 \\ w^0 \end{array} \right] \in V$ with $x \in X_1$. Since $\Sigma$ is solvable, there exists a classical future trajectory $[x^*]_w$ of $\Sigma$ satisfying $\left[ \begin{array}{c} x(0) \\ w(0) \end{array} \right] = \left[ \begin{array}{c} x^0 \\ w^0 \end{array} \right]$. Since $\Sigma_1$ is the restriction of $\Sigma$ to $X_1$, $x(t) \in X_1$ for all $t \in I$ and $[x^*]_w$ is a generalized trajectory of $\Sigma_1$, and by Lemma 1.3.31, it is even a classical trajectory of $\Sigma_1$. Hence $\left[ \begin{array}{c} x^0 \\ w^0 \end{array} \right] \in V \subset \left[ \frac{X_1}{W} \right]$. Thus, $V \cap \left[ \frac{X_1}{W} \right] \subset V_1$ and $V \cap \left[ \frac{X_1}{W} \right] = V \cap \left[ \frac{X_1}{W} \right]$.

We next prove that $\Sigma_1$ is solvable. Let $\left[ \begin{array}{c} x^0 \\ w^0 \end{array} \right] \in V_1$. Since $V_1 \subset V_2$ and $\Sigma_2$ is solvable, $\Sigma_2$ has a classical future trajectory $[x^*]_w$ satisfying $\left[ \begin{array}{c} x(0) \\ w(0) \end{array} \right] = \left[ \begin{array}{c} x^0 \\ w^0 \end{array} \right]$. By the strong invariance of $X_1$ and the fact that $x(0) \in X_1$ we have $x(t) \in X_1$ for all $t \in \mathbb{R}^+$, and hence $\dot{x}(t) \in X_1$ for all $t \in \mathbb{R}^+$. This means that $[x^*]_w$ is a classical future trajectory of $\Sigma_1$, and we conclude that $\Sigma_1$ is solvable.
To prove the remaining inclusion $V_1 \subset V \cap \left[X_1 \right]_W$, we let $\left[ \begin{array}{c} x_1(0) \\ x_2(0) \\ w(0) \end{array} \right] \in V_1$. Since $\Sigma_1$ is solvable, there exists a classical future trajectory $\left[ \begin{array}{c} x_1 \\ x_2 \\ w \end{array} \right]$ of $\Sigma_1$ satisfying $\left[ \begin{array}{c} x_1(0) \\ x_2(0) \\ w(0) \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$. Since $\Sigma_1$ is the restriction of $\Sigma$ to $\mathcal{X}_1$, $\left[ \begin{array}{c} x \end{array} \right]$ is also a generalized trajectory of $\Sigma$, and consequently $\left[ \begin{array}{c} x_1(0) \\ x_2(0) \\ w(0) \end{array} \right] \in V \cap \left[X_1 \right]_W$. Thus $V_1 \subset V \cap \left[X_1 \right]_W$. □

1.5.36. LEMMA. Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i)$, $i = 1, 2$, be two closed s/s systems (with the same signal space), where $\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_2$. Let $Z_1$ be a direct complement to $\mathcal{X}_1$ in $\mathcal{X}$. Then the following conditions are equivalent:

(i) $\Sigma_1$ is a restriction of $\Sigma_2$ to $\mathcal{X}_1$.
(ii) $\Sigma_1$ and $\Sigma_2$ are intertwined by the embedding operator $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$.
(iii) $\mathcal{X}_1$ is a strongly invariant subspace for $\Sigma_2$, and $\Sigma_1$ is the compression of $\Sigma_2$ onto $\mathcal{X}_1$ along $Z_1$.

PROOF. Throughout this proof we let $I$ be an interval of the type $I = [0, T]$ (where $T > 0$) or $I = \mathbb{R}^+$.

(i) $\Rightarrow$ (ii): Suppose first that $\Sigma_1$ is the restriction of $\Sigma_2$ to $\mathcal{X}_1$. If $\left[ \begin{array}{c} x_2 \end{array} \right]$ is a generalized trajectory of $\Sigma_2$ on $I$ with $x_2(0) \in \mathcal{X}_1$, then $\left[ \begin{array}{c} x_2 \end{array} \right]$ is also a generalized trajectory of $\Sigma_1$ on $I$. Thus, condition (ii) in Definition 1.5.22 holds with $P$ equal to the embedding operator $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$. If instead $\left[ \begin{array}{c} x_2 \end{array} \right]$ is a generalized trajectory of $\Sigma_1$ on $I$, then $\left[ \begin{array}{c} x_1 \end{array} \right]$ is also a generalized trajectory of $\Sigma_2$ on $I$, and hence also condition (i) in Definition 1.5.22 holds for the same operator $P$.

(ii) $\Rightarrow$ (i): Suppose that $\Sigma_1$ and $\Sigma_2$ are intertwined by the embedding operator $P: \mathcal{X}_1 \hookrightarrow \mathcal{X}_2$. If $\left[ \begin{array}{c} x_2 \end{array} \right]$ is a generalized trajectory of $\Sigma_2$ on $I$, then it follows from condition (ii) in Definition 1.5.22 that $\Sigma_2$ has a generalized trajectory $\left[ \begin{array}{c} x_2 \end{array} \right]$ on $I$ satisfying $x_2(t) = x_1(t)$ for all $t \in I$, i.e., $\left[ \begin{array}{c} x_2 \end{array} \right]$ is also a generalized trajectory of $\Sigma_2$ on $I$. Thus condition (i) in Definition 1.5.22 is satisfied. If instead $\left[ \begin{array}{c} x_2 \end{array} \right]$ is a generalized trajectory of $\Sigma_2$ on $I$ satisfying $x_2(0) \in \mathcal{X}_1$, then it follows from condition (ii) in Definition 1.5.22 that $\Sigma_1$ has a generalized trajectory $\left[ \begin{array}{c} x_1 \end{array} \right]$ on $I$ which satisfies $x_1(t) = x_2(t)$ for all $t \in I$, i.e., $\left[ \begin{array}{c} x_1 \end{array} \right]$ is also a generalized trajectory of $\Sigma_1$ on $I$. Thus, also condition (ii) in Definition 1.5.33 holds.

(i) $\Leftrightarrow$ (iii): This follows directly from Definitions 1.5.28 and 1.5.33 □

1.5.37. DEFINITION. Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i)$, $i = 1, 2$, be two s/s systems (with the same signal space), where $\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_2$, and let $Z_1$ be a direct complement to $\mathcal{X}_1$ in $\mathcal{X}_2$. We call $\Sigma_1$ a (dynamic) projection of $\Sigma_2$ onto $\mathcal{X}_1$ along $Z_1$ if the following two condition holds for all intervals $I$ of the type $I = [0, T]$ (where $T > 0$) and for $I = \mathbb{R}^+$:

(i) If $\left[ \begin{array}{c} x_2 \end{array} \right]$ is a generalized trajectory of $\Sigma_2$ on $I$, then $\left[ \begin{array}{c} P_{\mathcal{X}_1} x_2 \end{array} \right]$ is a generalized trajectory of $\Sigma_1$ on $I$.
(ii) If $\left[ \begin{array}{c} x_1 \end{array} \right]$ is a generalized trajectory of $\Sigma_1$ on $I$, then for each $x_2^0 \in \mathcal{X}_2$ satisfying $P_{\mathcal{X}_1} x_2^0 = x_1(0)$ there exists a generalized trajectory $\left[ \begin{array}{c} x_2 \end{array} \right]$ of $\Sigma_2$ on $I$ satisfying $x_2(0) = x_2^0$ and $P_{\mathcal{X}_1} x_2 = x_1$. 
1.5.38. **Lemma.** Let \( \Sigma_1 = (V_1; \mathcal{X}_1, W) \) be three s/s systems. If \( \Sigma_2 \) is the projection of \( \Sigma_3 \) onto \( \mathcal{X}_2 \) along \( \mathcal{Z}_2 \) and \( \Sigma_1 \) is the projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \), then \( \Sigma_1 \) is the projection of \( \Sigma_3 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 + \mathcal{Z}_2 \).

**Proof.** The proof is analogous to the proof of Lemma 1.5.30. □

1.5.39. **Lemma.** Let the s/s system \( \Sigma_1 = (V_1; \mathcal{X}_1, W) \) be the projection onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) of the s/s system \( \Sigma = (V; \mathcal{X}, W) \), and suppose that \( \Sigma \) is solvable and \( \Sigma_1 \) is closed. Then

\[
V_1 \subset \begin{bmatrix}
P_{\mathcal{X}_1}\mathcal{Z}_1 & 0 & 0 \\
0 & P_{\mathcal{X}_1}\mathcal{Z}_1 & 0 \\
0 & 0 & 1_{W}
\end{bmatrix} V.
\]

**Proof.** The proof is analogous to the first half of the proof of Lemma 1.5.39. □

1.5.40. **Lemma.** Let \( \Sigma_i = (V_i; \mathcal{X}_i, W), i = 1, 2, \) be two closed s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), and let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) is a projection of \( \Sigma_2 \) to \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(ii) \( \Sigma_2 \) and \( \Sigma_1 \) are intertwined by the projection operator \( P_{\mathcal{X}_1}\mathcal{Z}_1 \).

(iii) \( \mathcal{Z}_1 \) is a unobservably invariant subspace for \( \Sigma_2 \), and \( \Sigma_1 \) is the compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

**Proof.** Throughout this proof we let \( I \) be an interval of the type \( I = [0, T] \) (where \( T > 0 \)) or \( I = \mathbb{R}^+ \).

(i) \( \Rightarrow \) (ii): Suppose first that \( \Sigma_1 \) is the projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \). If \( \mathcal{Z}_1 \) is a generalized trajectory of \( \Sigma_2 \) on \( I \), then \( P_{\mathcal{X}_1}\mathcal{Z}_1 \mathcal{Z}_2 \) is a generalized trajectory of \( \Sigma_1 \) on \( I \). Thus condition (i) in Definition 1.5.22 holds for \( P = P_{\mathcal{X}_1}\mathcal{Z}_1 \) if we interchange \( \Sigma_1 \) and \( \Sigma_2 \) with each other. If instead \( \mathcal{Z}_1 \) is a generalized trajectory of \( \Sigma_1 \) on \( I \) and \( x_2^0 \in \mathcal{X}_2 \) satisfies \( P_{\mathcal{X}_1}\mathcal{Z}_1 x_2^0 = x_1(0) \), then there exists a future trajectory \( \mathcal{Z}_1 \) of \( \Sigma_2 \) on \( I \) with initial state \( x_2(0) = x_2^0 \) such that \( x_1 = P_{\mathcal{X}_1}\mathcal{Z}_1 x_2 \), and hence also condition (ii) in Definition 1.5.22 holds with the same replacements. Thus (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (i): Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by the projection operator \( P_{\mathcal{X}_1}\mathcal{Z}_1 \). If \( \mathcal{Z}_1 \) is a generalized trajectory of \( \Sigma_2 \) on \( I \), then it follows from condition (i) in Definition 1.5.22 (where \( \Sigma_1 \) and \( \Sigma_2 \) have been interchanged) that \( P_{\mathcal{X}_1}\mathcal{Z}_1 \mathcal{Z}_2 \) is a generalized trajectory of \( \Sigma_1 \) on \( I \). Thus condition (i) in Definition 1.5.27 holds. If instead \( \mathcal{Z}_1 \) is a generalized trajectory of \( \Sigma_1 \) on \( I \), then it follows from condition (ii) in Definition 1.5.22 that for each \( x_2^0 \in \mathcal{X}_2 \) satisfying \( P_{\mathcal{X}_1}\mathcal{Z}_1 x_2^0 = x_1(0) \), \( \Sigma_2 \) has a generalized trajectory \( \mathcal{Z}_1 \) on \( I \) satisfying \( x_2(0) = x_2^0 \) and \( P_{\mathcal{X}_1}\mathcal{Z}_1 x_2(t) = x_1(t) \) for all \( t \in I \). Thus, also condition (ii) in Definition 1.5.27 holds.

(i)&(ii) \( \Leftrightarrow \) (iii): It follows immediately from Definitions 1.5.28 and 1.5.37 that \( \Sigma_1 \) is a compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \). That, in addition, \( \mathcal{Z}_1 \) is unobservably invariant follows from (ii) and Lemma 1.5.27, since \( \ker(P_{\mathcal{X}_1}\mathcal{Z}_1) = \mathcal{Z}_1 \).

(iii) \( \Rightarrow \) (ii): Suppose that (iii) holds. We first prove that this implies that condition (i) in Definition 1.5.37 holds. Let \( \mathcal{Z}_1 \) be a generalized trajectory of \( \Sigma_2 \) on \( I \). Define \( x_1^0 = P_{\mathcal{X}_1}\mathcal{Z}_1 x_2(0) \) and \( z_1^0 = P_{\mathcal{Z}_1}\mathcal{Z}_1 x_2(0) \). Since \( \mathcal{Z}_1 \) is unobservably invariant for \( \Sigma_2 \) there exists a generalized trajectory \( \mathcal{Z}_1 \) of \( \Sigma_2 \) on \( I \) satisfying \( z_2(0) = z_1^0 \) and
$z_2(t) \in Z_1$ for all $t \in I$. Define $x_3 = x_2 - z_2$. Then $x_3(0) = x_0^1 \in X_1$. Since $\Sigma_1$ is the compression of $\Sigma_2$ onto $X_1$ along $Z_1$ it follows that $[P_{X_1^1}^{z_1^1}x_3]$ is a future trajectory of $\Sigma_1$ on $I$. However, $P_{X_1^1}^{z_1^1}x_3 = P_{X_1^1}^{z_1^1}x_2$ since $z_2(t) \in Z_1$, and thus $[P_{X_1^1}^{z_1^1}x_3]$ is a generalized trajectory of $\Sigma_1$ on $I$. This shows that condition (i) in Definition 1.5.37 holds.

We finally show that if (iii) holds, then also condition (ii) in Definition 1.5.37 holds. Let $[x_w^2]$ be a generalized trajectory of $\Sigma_1$ on $I$, and let $x_2^0 \in X_2$ satisfy $P_{X_1^1}^{z_1^1}x_2^0 = x_1(0)$. By condition (ii) in Definition 1.5.28 there exists some generalized trajectory $[x_w^2]$ if $\Sigma_2$ satisfying $x_2(0) = x_1(0)$ and $P_{X_1^1}^{z_1^1}x_2(t) = x_1(t)$ for all $t \in I$. In addition, since $Z_1$ is unobservably invariant, there exists some (unobservable) trajectory $[x_0^2]$ of $\Sigma_1$ satisfying $x_2(0) = P_{X_1^1}^{z_1^1}x_2^0 = x_0^2 - P_{X_1^1}^{z_1^1}x_0^2 = x_0^2 - x_1(0)$ and $P_{X_1^1}^{z_1^1}x_2(t) = 0$ for all $t \in I$. This implies that $[x_w^{z_1^1+x_0^1}]$ is a generalized trajectory of $\Sigma_2$ with initial state $x_2^0$ satisfying $P_{X_1^1}^{z_1^1}(x_2(t) + x_3(t)) = x_1(t)$ for all $t \in I$. This shows that also condition (ii) in Definition 1.5.37 holds.

1.5.41. Definition.

(i) A s/s system $\Sigma = (V; X, W)$ is minimal if it does not have any non-trivial compression.

(ii) By a minimal compression of a s/s system $\Sigma$ we mean a compression $\Sigma_1$ of $\Sigma$ which is minimal (i.e., $\Sigma_1$ does not have any further non-trivial compressions).

Existence of minimal compressions will be proved later for three special classes of s/s systems, namely bounded i/s/o systems (see Chapter 3), semi-bounded i/s/o systems (see Chapter 4), and well-posed s/s systems (see Chapter 9).

In all the notions that we have defined in this subsection we have been working in the forward time direction in the sense that we interpret zero as the initial time, and study properties of trajectories of one or two systems on some interval $[0, T]$ or on $\mathbb{R}^+$. The same notions can of course also be studied in the backward time direction.

1.5.42. Definition. Let $\Sigma = (V; X, W)$ and $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be s/s systems. The following “backward” notions are defined by applying the corresponding “forward” definitions to the time reflected systems $\Sigma$ and $\Sigma_i$ of $\Sigma$ respectively $\Sigma_i$, $i = 1, 2$:

(i) backward external equivalence of $\Sigma_1$ and $\Sigma_2$ (cf. Definition 1.5.21);
(ii) backward intertwinement of $\Sigma_1$ and $\Sigma_2$ (cf. Definition 1.5.22);
(iii) backward bounded intertwinement and backward pseudo-similarity of $\Sigma_1$ and $\Sigma_2$ (cf. Definition 1.5.23);
(iv) backward compression of $\Sigma_2$ onto $X_1$ along $Z_1$ (cf. Definition 1.5.28);
(v) backward restriction of $\Sigma_2$ onto $X_1$ (cf. Definition 1.5.33);
(vi) backward projection of $\Sigma_2$ onto $X_1$ along $Z_1$ (cf. Definition 1.5.33);
(vii) backward minimality of $\Sigma$ (cf. Definition 1.5.41).

It is possible to prove “backward” results which are analogous to the “forward” results presented above. We leave the formulation and proofs of these results to the readers.

In general a “forward” result does not imply the corresponding “backward” result and vice versa, as was noticed in Examples 1.5.15-1.5.20 above. However,
for bounded s/s nodes the forward and backward results are equivalent, as we shall see in Chapter 3.

1.5.3. Consequences of the continuation property. In several of our earlier definitions we have required that generalized trajectories of a s/s system Σ on the interval I have certain properties when I is an interval of the type I = [0, T] (with T > 0) or I = R+. As we shall see below, if Σ has the continuation property, then it is often enough to check these conditions for future generalized trajectories, i.e., for trajectories defined on I = R+.

1.5.43. Definition. By the future behavior of a s/s system Σ = (V; X, W) we mean the set of all w ∈ L^1_{loc}(R^+; W) for which there exists some x ∈ C_0(R^+; X) (with x(0) = 0) such that [x w] is a generalized future trajectory of Σ.

It easy to see that the future behavior is a subspace of L^1_{loc}(R^+; W), and it follows from Lemma 1.3.29 that this subspace is right-shift invariant in L^1_{loc}(R^+; W).

1.5.44. Lemma. Let Σ = (V; X, W) and Σ_i = (V_i; X_i, W), i = 1, 2, be two closed s/s systems with the continuation property.

(i) x^0 ∈ R^n_{X^0} if and only if there exists a generalized future trajectory [x^0 w] of Σ with x(0) = 0 such that x^0 = x(t) for some t ∈ R^+.

(ii) A subspace Z of X is strongly invariant for Σ if and only if every generalized future trajectory [x w] of Σ with x(0) ∈ Z satisfies x(t) ∈ Z for all t ∈ R^+.

(iii) Σ_1 and Σ_2 are externally equivalent if and only if they have the same future behavior.

Proof. This follows more or less directly from the relevant definitions. □

1.5.45. Lemma. Let Σ_i = (V_i; X_i, W), i = 1, 2, be two closed s/s systems, and let P ∈ MC(X_1; X_2).

(i) If Σ_1 and Σ_2 are intertwined by P, then the following claims are true:

(a) If Σ_1 has the continuation property, then every generalized trajectory [x^1 w^1] of Σ_2 on some interval [0, T] with x^1(0) ∈ rng(P) can be continued to a generalized trajectory of Σ_1 on R^+, and conversely, if Σ_2 has the continuation property, then every generalized trajectory [x^2 w^2] of Σ_1 on some interval [0, T] with x^2(0) ∈ dom(P) can be continued to a generalized trajectory of Σ_2 on R^+.

(b) If Σ_1 has the continuation property and rng(P) = X_2, then Σ_2 has the continuation property, and conversely, if Σ_2 has the continuation property and dom(P) = X_1, then Σ_1 has the continuation property.

(ii) If both Σ_1 and Σ_2 have the continuation property, then Σ_1 and Σ_2 are intertwined by some operator P ∈ MC(X_1; X_2) if and only if conditions (i) and (ii) in Definition 1.5.22 hold for I = R^+.

Proof. (i)(a) Suppose that Σ_2 has the continuation property, and let [x^1 w^1] be a generalized trajectory of Σ_1 on some interval [0, T] with x^1(0) ∈ dom(P). Since Σ_1 and Σ_2 are intertwined by P, this implies that for every x^2 ∈ P x^1(0) there exists a generalized trajectory [x^2 w^2] of Σ_2 on [0, T] with x^2(T) ∈ P x^1(T) and w^1 = w^2. Since Σ_2 is assumed to have the continuation property, this trajectory can be continued to a generalized trajectory of Σ_1 on R^+, which we still denote by [x^2 w^2]. Since Σ_1 and Σ_2 are intertwined by P, there exists a generalized trajectory
[\mathcal{Z}_1] of \Sigma_1 on \([T, \infty)\) with \(x_3(T) = x_1(T)\). By Lemma \ref{lem1.3.28} if we extend \([x_1^w]_w\) to \(\mathbb{R}^+\) by defining \(x_1(t) = \left[\begin{array}{c} x_3(t) \\ w_2(t) \end{array}\right]\) for \(t > T\), then the extended pair of functions \([x_1^w]_w\) is a generalized trajectory of \(\Sigma\) on \(\mathbb{R}^+\). The converse claim is proved in the same way by interchanging \(\Sigma_1\) and \(\Sigma_2\) and replacing \(P\) by \(P^{-1}\).

(i)(b) That (i)(b) holds follows directly from (i)(a) and Definition \ref{def1.3.3}.

(ii) This follows from Definitions \ref{def1.3.3} and \ref{def1.5.22}. \hfill \Box

1.5.46. Lemma. Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) and \(\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i), i = 1, 2\), be two closed s/s systems, where \(\mathcal{X}_1\) is a closed subspace of \(\mathcal{X}_2\) with a direct complement \(\mathcal{Z}_1\).

(i) If \(\Sigma_2\) has the continuation property and \(\Sigma_1\) is a restriction of \(\Sigma_2\) to \(\mathcal{X}_1\), or a projection or compression of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\), then \(\Sigma_1\) has the continuation property.

(ii) If both \(\Sigma_1\) and \(\Sigma_2\) have the continuation property, then

(a) \(\Sigma_1\) is the compression of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\) if and only if conditions (i) and (ii) in Definition \ref{def1.5.28} hold for \(I = \mathbb{R}^+\).

(b) \(\Sigma_1\) is the restriction of \(\Sigma_2\) to \(\mathcal{X}_1\) if and only if conditions (i) and (ii) in Definition \ref{def1.5.28} hold for \(I = \mathbb{R}^+\).

(c) \(\Sigma_1\) is the projection of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\) if and only if conditions (i) and (ii) in Definition \ref{def1.5.28} hold for \(I = \mathbb{R}^+\).

Proof. (i) Since every restriction and projection is also a compression, it suffices to prove (i) in the case where \(\Sigma_1\) is a compression of \(\Sigma_2\). Let \([x_1^w]_w\) be a generalized trajectory of \(\Sigma_1\) on the interval \([0, T]\). Since \(\Sigma_2\) is a dilation of \(\Sigma_1\), there exists a trajectory \([x_2^w]_w\) of \(\Sigma_2\) on \([0, T]\) satisfying \(x_2(0) = x_1(0)\) and \(x_1(t) = P_{\mathcal{X}_1}^2 x_2(t)\) for all \(t \in [0, T]\). Since \(\Sigma_2\) has the continuation property, the trajectory \([x_2^w]_w\) can be extended to a generalized trajectory defined on all of \(\mathbb{R}^+\), which we still denote by \([x_2^w]_w\). If we extend \(x_1\) to \(\mathbb{R}^+\) by defining \(x_1(t) = P_{\mathcal{X}_1}^2 x_2(t)\) for \(t > T\), then the extended pair of functions \([x_1^w]_w\) satisfies \(x_1(0) = x_2(0)\) and \(x_1(t) = P_{\mathcal{X}_1}^2 x_2(t)\) for all \(t \in \mathbb{R}^+\), and since \(\Sigma_1\) is a compression of \(\Sigma_2\), this implies that \([x_1^w]_w\) is a generalized trajectory of \(\Sigma_1\) on \(\mathbb{R}^+\). This proves that \(\Sigma_1\) has the continuation property.

(ii): The claims in (ii) follow from the relevant definitions. \hfill \Box

For a s/s system with the continuation property the following extension of Lemma \ref{lem1.3.28} is valid.

1.5.47. Lemma. Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a closed s/s system which has either the forward or the backward continuation property. Let \(I_1 \cup I_2\), where \(I_1\) and \(I_2\) are intervals satisfying \(I_1 \cap I_2 = \{t_0\}\) and \(t_0\) is both the right end-point of \(I_1\) and the left end-point of \(I_2\), and let \([x_i^w]_w\) be a generalized trajectory of \(\Sigma\) on \(I_i\), \(i = 1, 2\). Define \([x_i^w]_w\) on \(I\) by \ref{eq1.3.3}. The \([x_i^w]_w\) is a generalized trajectory of \(\Sigma\) on \(I\) if and only if \(x_1(t_0) = x_2(t_0)\).

Proof. This follows from Lemmas \ref{lem1.3.28} \hfill \Box

1.5.48. Lemma. Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a closed s/s system with the continuation property, and let \(\mathcal{X}_1\) be a closed strongly invariant subspace of \(\mathcal{X}\). Let
\( \Sigma_1 = (V_1; X_1, W) \) be the part of \( \Sigma \) in \( [X_1/W] \), i.e.,

\[ V_1 = V \cap \begin{bmatrix} X_1 \\ W \end{bmatrix}. \tag{1.5.3} \]

Then \( \Sigma_1 \) is closed, \( \Sigma_1 \) is a restriction of \( \Sigma \) to \( X_1 \), and \( \Sigma_1 \) has the continuation property. If, in addition, \( \Sigma \) is solvable, then \( \Sigma_1 \) is solvable, and \( \Sigma_1 \) is the unique closed solvable restriction of \( \Sigma \) to \( X_1 \).

Note that in this lemma we do not claim that \( \Sigma_1 \) is a regular s/s system even if \( \Sigma \) is a regular s/s system (it need not satisfy condition (ii) in Definition 1.1.6).

Proof of Lemma 1.5.48. Clearly \( V_1 \) is closed since both \( V \) and \( X_1 \) are closed.

In the rest of the proof we let \( I \) be an interval of the type \( I = [0, T] \), \( T > 0 \), or \( I = \mathbb{R}^+ \).

That condition (i) in Definition 1.5.33 holds follows from Lemma 1.2.16. To show that also condition (ii) in Definition 1.5.33 holds we let \( [x\,w] \) be a generalized trajectory of \( \Sigma \) on \( I \) with \( x(0) \in X_1 \). Since \( \Sigma \) has the continuation property, this trajectory can be extended to a trajectory on \( \mathbb{R}^+ \) (in the case where \( I = [0, T] \)). Since \( X_1 \) is strongly invariant for \( \Sigma \) the extended trajectory satisfies \( x(t) \in X_1 \) for all \( t \in \mathbb{R}^+ \). Define

\[ \left[ \begin{array}{c} x_n(t) \\ w_n(t) \end{array} \right] = n \int_t^{t+n} \left[ \begin{array}{c} x(s) \\ w(s) \end{array} \right] ds, \quad t \in \mathbb{R}^+. \]

Then by Lemma 1.3.24 \([x_n/w_n] \) is a classical future trajectory of \( \Sigma \), and \([x_n/w_n] \to [x/w] \) in \( \left[ C(\mathbb{R}^+; X) \right]_{L^1_{loc}(\mathbb{R}^+; W)} \) as \( n \to \infty \). Since \( x(t) \in X_1 \) for all \( t \in \mathbb{R}^+ \) we have \( x_n(t) \in X_1 \) for all \( t \in \mathbb{R}^+ \), and consequently also \( x_n(t) \in X_1 \) for all \( t \in \mathbb{R}^+ \). This implies that \( \left[ \begin{array}{c} x_n(t) \\ x_n(t) \\ w_n(t) \end{array} \right] \in V_1 \) for all \( t \in \mathbb{R}^+ \), i.e., \([x_n/w_n]\) is a classical future trajectory of \( \Sigma_1 \).

Moreover, since \( x_n(t) \in X_1 \) for all \( t \in \mathbb{R}^+ \) and \( X_1 \) is closed, we have \([x_n/w_n] \to [x/w] \) in \( \left[ C(\mathbb{R}^+; X_1) \right]_{L^1_{loc}(\mathbb{R}^+; W)} \) as \( n \to \infty \). This implies that \([x/w] \) is a generalized trajectory of \( \Sigma_1 \) on \( \mathbb{R}^+ \), and in the case where \( I = [0, T] \) the restriction of \( [x/w] \) to \( I \) (which is equal to the originally given trajectory) is a generalized trajectory of \( \Sigma \) on \( I \). Thus, also condition (ii) in Definition 1.5.33 is satisfied. This proves that \( \Sigma_1 \) is a restriction of \( \Sigma \) to \( X_1 \). That \( \Sigma_1 \) has the continuation property follows from Lemma 1.5.46.

That \( \Sigma_1 \) is solvable whenever \( \Sigma \) is solvable follows from Lemma 1.5.35, and that there exists at most one solvable restriction of \( \Sigma \) to \( X_1 \) follows from Lemmas 1.5.31 and 1.5.36. \( \square \)
1.6. The Characteristic Node, Signal/State, and Signal Bundles (Jan 02, 2016)

As in the standard i/s/o systems theory, s/s systems are studied not only in the time domain as we have done up to now, but also in the frequency domain. To pass to the frequency domain from the time domain one takes (formal) Laplace transforms of classical (time domain) trajectories to derive the (frequency domain) equations that their Laplace transforms must satisfy. In this way arises the two main frequency domain characteristics of a s/s system, namely the characteristic node bundle and the characteristic signal bundle of a s/s system. The characteristic node bundle is a family of subspaces of the node space $R$, and the characteristic signal bundle is a family of subspaces of the signal space $W$, both of which are parameterized by a complex frequency domain parameter $\lambda \in \mathbb{C}$. In the case of the characteristic node bundle these subspaces are closed, and they depend analytically on the frequency domain parameter $\lambda$ at every point $\lambda \in \mathbb{C}$. Under additional assumptions the same statement will be true for the characteristic signal bundle in some appropriate subset of $\mathbb{C}$.

In this section the basic properties of the characteristic node and signal bundles will be presented. A general frequency domain theory for state/signal systems will be developed in Chapters 5 and 7.

1.6.1. The characteristic node bundle. Every s/s node $\Sigma = (V; X; W)$ has a frequency domain characteristic which we call the characteristic node bundle. The motivation for introducing this notion is the following.

We say that a function $u \in L^1_{loc}$ with values in some $H$-space is Laplace transformable if the Laplace transform $\hat{u}$ of $u$ defined by

$$\hat{u}(\lambda) = \int_0^{\infty} e^{-\lambda t} x(t) \, dt$$

converges absolutely for all $\lambda$ in some right half-plane. Let $[x]$ of $\Sigma$ be a classical future trajectory of $\Sigma$, and suppose that $x, \dot{x}$, and $w$ are Laplace transformable. By taking $I = \mathbb{R}^+$ in (1.1.1), denoting $x(0) = x^0$, multiplying (1.1.1) by $e^{-\lambda t}$, integrating over $\mathbb{R}^+$, and taking into account that

$$\int_0^{\infty} e^{-\lambda t} \dot{x}(t) \, dt = \lambda \hat{x}(\lambda) - x^0$$

we find that for all those $\lambda \in \mathbb{C}$ for which the Laplace transforms converge we have

$$\begin{bmatrix} \lambda \hat{x}(\lambda) - x^0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V$$

(1.6.2)

This condition can be rewritten in the form

$$\begin{bmatrix} x^0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \hat{E}(\lambda),$$

(1.6.3)

where $\hat{E}(\lambda)$ is the subspace of the node space $R$ defined by

$$\hat{E}(\lambda) = \left\{ \begin{bmatrix} x^0 \\ x_\lambda \\ w_\lambda \end{bmatrix} \in \hat{R} \mid \begin{bmatrix} \lambda x_\lambda - x^0 \\ x_\lambda \\ w_\lambda \end{bmatrix} \in V \right\}, \quad \lambda \in \mathbb{C}.$$
Thus,
\[
\hat{E}(\lambda) = \begin{bmatrix}
-1_x & \lambda & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix}, \quad V = \begin{bmatrix}
-1_x & \lambda & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix} \hat{E}(\lambda), \quad \lambda \in \mathbb{C}.
\]

The family \( \hat{E} = \{ \hat{E}(\lambda) \}_{\lambda \in \mathbb{C}} \) is an analytic family of subspaces of the node space \( \mathfrak{K} \), which we call the characteristic node bundle. We refer to each of the subspaces \( \hat{E}(\lambda) \) as a fiber of \( \hat{E} \). (See Appendix A.3 for a short review of analytic vector bundles.)

1.6.1. Definition. The characteristic node bundle of a s/s node \( \Sigma = (V; X, W) \) is the family of subspaces \( \hat{E} = \{ \hat{E}(\lambda) \}_{\lambda \in \mathbb{C}} \) of the node space \( \mathfrak{K} \), where each fiber \( \hat{E}(\lambda) \) is defined by (1.6.4).

1.6.2. Lemma. Let \( \Sigma = (V; X, W) \) be a s/s node with characteristic node bundle \( \hat{E} \), and let \( \lambda, \mu \in \mathbb{C} \). Then
\[
\hat{E}(\lambda) = \begin{bmatrix}
-1_x & \lambda & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix} V, \quad V = \begin{bmatrix}
-1_x & \lambda & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix} \hat{E}(\lambda),
\]
\[
\hat{E}(\lambda) = \begin{bmatrix}
1_x & \lambda - \mu & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix} \hat{E}(\mu).
\]

Proof. This follows from (1.6.4) and the fact that
\[
\begin{bmatrix}
-1_x & \lambda & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix}^{-1} = \begin{bmatrix}
-1_x & \lambda & 0 \\
0 & 1_x & 0 \\
0 & 0 & 1_w
\end{bmatrix}.
\]

1.6.3. Remark. Formulas (1.6.6) show that any one of the fibers \( \hat{E}(\lambda) \) together with the value of \( \lambda \) determines the generating subspace \( V \) and the characteristic node bundle \( \hat{E} \) uniquely. Of course, the generating subspace \( V \) itself also determines \( \hat{E} \) uniquely.

1.6.4. Lemma. The characteristic node bundle \( \hat{E} \) of a closed s/s node \( \Sigma = (V; X, W) \) is analytic at every (finite) point in the complex plane.

Proof. The analyticity of \( \hat{E} \) at every finite point \( \lambda \in \mathbb{C} \) follows from Lemma 1.6.2 by taking the space \( \mathcal{U} \) in Definition A.3.2 to be equal to the generating subspace \( V \) (this can be interpreted as a Hilbert space since it is a closed subspace of \( \mathfrak{K} \)).

The analyticity of \( \hat{E} \) at infinity is studied in Proposition 5.3.14 below. (The characteristic node bundle of a bounded s/s node is analytic at infinity.)

We already gave an interpretation of the characteristic node bundle in terms of Laplace transforms of classical trajectories of the system on \( \mathbb{R}^+ \). It is also possible to give some other interpretations (depending on the value of the parameter \( \lambda \). Below we list three such interpretations:

(i) If \( \left[ \begin{array}{c} x \\ \dot{x} \\ \ddot{x} \end{array} \right] \) is a classical Laplace transformable future trajectory of the closed s/s system \( \Sigma \) with initial state \( x(0) = x^0 \), then the Laplace transform \( \left[ \begin{array}{c} \hat{x} \\ \hat{\dot{x}} \\ \hat{\ddot{x}} \end{array} \right] \) of \( \left[ \begin{array}{c} x \\ \dot{x} \\ \ddot{x} \end{array} \right] \) satisfies
\[
\begin{bmatrix}
x^0 \\
\hat{x}(\lambda) \\
\hat{\dot{x}}(\lambda)
\end{bmatrix} \in \hat{E}(\lambda)
\]

for all those \( \lambda \in \mathbb{C} \) for which the Laplace transform converges.
(ii) If $[x_w]$ is a classical past time Laplace transformable trajectory of the closed s/s system $\Sigma$ on $\mathbb{R}^-$ with final state $x(0) = x^0$, then the past time Laplace transform $\hat{[x_w]}$ of $[x_w]$ satisfies
\begin{equation}
\begin{bmatrix}
-x^0 \\
\hat{x}(\lambda) \\
\hat{w}(\lambda)
\end{bmatrix} \in \hat{E}(\lambda)
\end{equation}
for all those $\lambda \in \mathbb{C}$ for which the Laplace transform converges.

(iii) If $[x_w]$ is a classical bilaterally time Laplace transformable trajectory of the closed s/s system $\Sigma$ on $\mathbb{R}$, except for a possible jump discontinuity at the origin, then the bilateral Laplace transform $\hat{[x_w]}$ of $[x_w]$ satisfies \(1.6.8\) with $x^0$ replaced by $x(0+) - x(0-)$ for all those $\lambda \in \mathbb{C}$ for which the two-sided Laplace transform converges.

The difference between the three versions of the Laplace transform that we mention above is that in the standard Laplace transform one integrates over $\mathbb{R}^+$, in the past time Laplace transform one integrates over $\mathbb{R}^-$, and in the two-sided Laplace transform one integrates over the full real line $\mathbb{R}$ and takes into account that $x$ has a jump discontinuity at zero. As is well-known, if the standard Laplace transform of some function converges at some point, then it converges (absolutely) in some right half-plane, if the past time Laplace transform of some function converges at some point, then it converges (absolutely) in some left half-plane, and if the two-sided time Laplace transform of some function converges at some point, then it converges (absolutely) in some vertical strip (i.e., in the intersection of some left half-plane with some right half-plane). Thus, the sets of all the points $\lambda \in \mathbb{C}$ where the conclusions of (i), (ii), and (iii) are valid are some left half-plane, some right half-plane, and some vertical strip in $\mathbb{C}$, respectively.

1.6.2. **The characteristic signal/state bundle.** If a classical future trajectory $[x_w]$ of $\Sigma$ satisfies $x(0) = x^0 = 0$ and $w(0) = 0$, then \(1.6.2\) takes the form
\begin{equation}
\begin{bmatrix}
\lambda \hat{x}(\lambda) \\
\hat{x}(\lambda) \\
\hat{w}(\lambda)
\end{bmatrix} \in V
\end{equation}
and \(1.6.3\) becomes
\begin{equation}
\begin{bmatrix}
0 \\
\hat{x}(\lambda) \\
\hat{w}(\lambda)
\end{bmatrix} \in \hat{E}(\lambda).
\end{equation}
This special case is important enough to deserve a name of its own.

1.6.5. **Definition.** The *characteristic signal/state bundle* of a s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is the family of subspaces $\hat{\mathcal{G}} = \{\hat{\mathcal{G}}(\lambda)\}_{\lambda \in \mathbb{C}}$ of $[x_w]$; where each fiber $\hat{\mathcal{G}}(\lambda)$ is defined by
\begin{equation}
\hat{\mathcal{G}}(\lambda) = \left\{ \begin{bmatrix} x_{\lambda} \\ w_{\lambda} \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \left| \begin{bmatrix} \lambda x_{\lambda} \\ x_{\lambda} \end{bmatrix} \in V \right. \right\}, \quad \lambda \in \mathbb{C}.
\end{equation}
By comparing this definition to Definition 1.6.1 we see that

\[(1.6.13) \qquad \hat{\mathcal{G}}(\lambda) = \begin{bmatrix} 0 & 1_x & 0 \\ 0 & 0 & 1_w \end{bmatrix} \left( \hat{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X}_W \end{bmatrix} \right), \quad \lambda \in \mathbb{C}, \]

where \(\hat{\mathcal{E}}\) is the characteristic node bundle of \(\Sigma\).

One way to arrive at the characteristic signal/state bundle of \(\Sigma\) is to take Laplace transforms of classical future trajectories of \(\Sigma\) with initial state \(x(0) = 0\), as we saw above. Another way to arrive at the same characteristic signal/state bundle is to study the existence of classical two-sided trajectories of \(\Sigma\) of the following special type:

1.6.6. Lemma. Let \(\begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \in X_W\) and \(\lambda \in \mathbb{C}\), and define \(\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}, \quad t \in \mathbb{R}\). Then the following conditions are equivalent:

(i) \(\begin{bmatrix} x \\ w \end{bmatrix}\) is a classical two-sided trajectory of \(\Sigma\);
(ii) \(\begin{bmatrix} \lambda x_0 \\ x_0 \end{bmatrix} \in V\);
(iii) \(\begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \in \hat{\mathcal{G}}(\lambda)\).

Proof. This follows from Definitions 1.1.6 and 1.6.1 since \(\dot{x}(t) = \lambda x(t)\) for all \(t \in \mathbb{R}\). \(\square\)

Note that each fiber \(\hat{\mathcal{G}}(\lambda)\) of \(\hat{\mathcal{G}}\) is closed. The analyticity of the characteristic signal/state bundle will be discussed later (see Lemmas 3.4.3 and 5.3.25).

1.6.3. The characteristic signal bundle. From the characteristic signal/state bundle \(\hat{\mathcal{G}}\) of a s/s system \(\Sigma\) it is possible to derive another frequency domain characteristic of \(\Sigma\), called the characteristic signal bundle of \(\Sigma\) by simply projecting \(\hat{\mathcal{G}}\) onto its signal component. We denote this projection by \(\hat{\mathcal{F}}(\lambda)\). Thus, if we denote the characteristic node and signal/state bundles of \(\Sigma\) by \(\hat{\mathcal{E}}\) respectively \(\hat{\mathcal{G}}\), then

\[(1.6.14) \qquad \hat{\mathcal{F}}(\lambda) = \begin{bmatrix} 0 & 1_w \end{bmatrix} \hat{\mathcal{G}}(\lambda)
\]

Equivalently,

\[(1.6.15) \qquad \hat{\mathcal{F}}(\lambda) = \left\{ w \in W \mid \begin{bmatrix} z \\ w \end{bmatrix} \in \hat{\mathcal{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\},\quad \lambda \in \mathbb{C}.\]

1.6.7. Definition. The characteristic signal bundle of the s/s system \(\Sigma = (V; \mathcal{X}, W)\) is the family of subspaces \(\hat{\mathcal{F}} = \{\hat{\mathcal{F}}(\lambda)\}_{\lambda \in \mathbb{C}}\) of \(W\), where each fiber \(\hat{\mathcal{F}}(\lambda)\) is defined by \(1.6.15\), or equivalently, by \(1.6.14\).

The fibers \(\hat{\mathcal{F}}(\lambda)\) of \(\hat{\mathcal{F}}\) need not be closed for all values of \(\lambda\). The analyticity of the characteristic signal bundle will be discussed later (see Lemmas 3.4.3 and 5.3.25).
1.6.4. The characteristic node and signal bundles of transformed s/s systems.

1.6.8. Lemma. Let $\Sigma = (V;X,W)$ be a s/s node with characteristic node, signal/state, and signal bundles $\hat{\mathcal{E}}, \hat{\mathcal{G}},$ and $\hat{\mathcal{F}}$, respectively, and let $\Sigma^R = (V^R;X,W)$ be the time reflected s/s node with characteristic node bundle $\hat{\mathcal{E}}^R$ and characteristic signal bundle $\hat{\mathcal{F}}^R$. Then

\begin{equation}
\hat{\mathcal{E}}^R(\lambda) = \begin{bmatrix} -1_X & 0 & 0 \\ 0 & 1_X & 0 \\ 0 & 0 & 1_W \end{bmatrix} \hat{\mathcal{E}}(-\lambda), \quad \lambda \in \mathbb{C},
\end{equation}

\begin{equation}
\hat{\mathcal{G}}^R(\lambda) = \hat{\mathcal{G}}(-\lambda), \quad \lambda \in \mathbb{C},
\end{equation}

\begin{equation}
\hat{\mathcal{F}}^R(\lambda) = \hat{\mathcal{F}}(-\lambda), \quad \lambda \in \mathbb{C}.
\end{equation}

Proof. By (1.2.1) and (1.6.6), applied both to the original system $\Sigma$ and the time reflected system $\Sigma^R$, for all $\lambda \in \mathbb{C}$,

$$
\hat{\mathcal{E}}^R(\lambda) = \begin{bmatrix} -1_X & 0 & 0 \\ 0 & 1_X & 0 \\ 0 & 0 & 1_W \end{bmatrix} \hat{\mathcal{E}}(-\lambda).
$$

This proves (1.6.16a). Formulas (1.6.16b) and (1.6.16c) follow from (1.6.16a) and Definitions 1.6.5 and 1.6.7.

1.6.9. Lemma. Let $\Sigma = (V;X,W)$ be a s/s node or system with characteristic node, signal/state, and signal bundles $\hat{\mathcal{E}}, \hat{\mathcal{G}},$ and $\hat{\mathcal{F}}$, respectively, and let $\Sigma_1 = (V;X,W)$ be the $(P,Q)$-image of $\Sigma$, where $P: X \to X_1$ and $Q: W \to W_1$ are two continuous linear operators with closed domains $\text{dom}(P) \subset X$ and $\text{dom}(Q) \subset W$. Denote the node, signal/state, and signal bundles of $\Sigma_1$ by $\hat{\mathcal{E}}_1, \hat{\mathcal{G}}_1,$ and $\hat{\mathcal{F}}_1$, respectively.

(i) $\hat{\mathcal{E}}_1$ is given by

\begin{equation}
\hat{\mathcal{E}}_1(\lambda) = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \left( \hat{\mathcal{E}}(\lambda) \cap \left[ \frac{\text{dom}(P)}{\text{dom}(P)} \right] \right), \quad \lambda \in \mathbb{C}.
\end{equation}

(ii) The following inclusion always holds:

\begin{equation}
\hat{\mathcal{G}}_1(\lambda) \supset \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left( \hat{\mathcal{G}}(\lambda) \cap \left[ \frac{\text{dom}(P)}{\text{dom}(Q)} \right] \right), \quad \lambda \in \mathbb{C}.
\end{equation}

If $P$ is injective, then

\begin{equation}
\hat{\mathcal{G}}_1(\lambda) = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left( \hat{\mathcal{G}}(\lambda) \cap \left[ \frac{\text{dom}(P)}{\text{dom}(Q)} \right] \right), \quad \lambda \in \mathbb{C}.
\end{equation}
Lemma 1.6.2, and the fact that the two operators

\[(1.6.19c)\]

mutate.

**Proof of (i):** That (i) holds follows from Definitions 1.2.14 and 1.6.7, and the fact that the two operators \([P \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]\) and \([Q \ 0 \ 0 \ 0]\) commute.

**Proof of (ii):** Let \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\). Then \([0 \ x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\), and by \[(1.6.17)\], \([0 \ p_{x_{\lambda}}] \in \hat{\mathcal{E}}(\lambda)\). Conversely, suppose that \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\). Then \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\). By \[(1.6.17)\], there exists some \([z_{\lambda}] \in \hat{\mathcal{E}}(\lambda) \cap \frac{\text{dom}(P)}{\text{dom}(Q)}\) such that \(0 = p_{x_{\lambda}}\) and \([x_{\lambda}] = [p_{x_{\lambda}} w_{\lambda}]\).

If \(P\) is injective, then \(z_{\lambda} = 0\), and consequently \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\). Thus \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda) \cap \frac{\text{dom}(P)}{\text{dom}(Q)}\).

(iii) Let \(w_{\lambda} \in \hat{\mathcal{S}}(\lambda) \cap \text{dom}(Q)\). Then there exists a vector \(x_{\lambda}\) such that \([0 \ x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\), and by \[(1.6.17)\], \([0 \ p_{x_{\lambda}}] \in \hat{\mathcal{E}}(\lambda)\), and consequently \(Qw_{\lambda} \in \hat{\mathcal{S}}(\lambda)\). Conversely, let \(w_{\lambda} \in \hat{\mathcal{S}}(\lambda)\). Then there exists a vector \(x_{\lambda} \in \mathcal{X}\) such that \([0 \ x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\).

By \[(1.6.17)\], there exists some \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda) \cap \frac{\text{dom}(P)}{\text{dom}(Q)}\) such that \(0 = p_{x_{\lambda}}\) and \([x_{\lambda}] = [p_{x_{\lambda}} w_{\lambda}]\). If \(P\) is injective, then \(z_{\lambda} = 0\), and consequently \([x_{\lambda}] \in \hat{\mathcal{E}}(\lambda)\) and \(w_{\lambda} \in \hat{\mathcal{S}}(\lambda)\). Thus \(w_{\lambda} \in \hat{\mathcal{S}}(\lambda) \cap \text{dom}(Q)\) and \([p_{x_{\lambda}} w_{\lambda}] = Qw_{\lambda}\). \(\square\)

1.6.10. **Lemma.** Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a s/s node with characteristic node, signal/state, and bundles \(\hat{\mathcal{E}}, \hat{\mathcal{S}}, \) and \(\hat{\mathcal{E}}\), respectively, and let \(\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2\) and \(\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2\).

(i) The characteristic node, signal/state, and signal bundles \(\hat{\mathcal{E}}\), \(\hat{\mathcal{S}}\), and \(\hat{\mathcal{E}}\) of the part \(\Sigma_\text{part} = (V_\text{part}; \mathcal{X}_1; \mathcal{W}_1)\) of \(\Sigma\) in \([\mathcal{X}_1 \ \mathcal{W}_1]\) satisfy

\[(1.6.20a)\]

\(\hat{\mathcal{E}}\) \(\text{part}(\lambda) = \hat{\mathcal{E}}(\lambda) \cap \frac{\mathcal{X}_1}{\mathcal{W}_1}\), \(\lambda \in \mathcal{C}\),

\[(1.6.20b)\]

\(\hat{\mathcal{S}}\) \(\text{part}(\lambda) = \hat{\mathcal{S}}(\lambda) \cap \frac{\mathcal{X}_1}{\mathcal{W}_1}\), \(\lambda \in \mathcal{C}\),

\[(1.6.20c)\]

\(\hat{\mathcal{E}}\) \(\text{part}(\lambda) \subset \hat{\mathcal{S}}(\lambda) \cap \mathcal{W}_1\), \(\lambda \in \mathcal{C}\).

If \(\mathcal{X}_1 = \mathcal{X}\) (and hence \(\mathcal{X}_2 = \{0\}\)), then

\[(1.6.20d)\]

\(\hat{\mathcal{S}}\) \(\text{part}(\lambda) = \hat{\mathcal{S}}(\lambda) \cap \mathcal{W}_1\), \(\lambda \in \mathcal{C}\).
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(ii) The characteristic node, signal/state, and signal bundles $\hat{\mathcal{E}}_{\text{proj}}$, $\hat{\mathcal{G}}_{\text{proj}}$, and $\hat{\mathcal{S}}_{\text{proj}}$ of the static projection $\Sigma_{\text{proj}} = (V_{\text{proj}}; X_1; W_1)$ of $\Sigma = (X_1; W_1)$ along $[X_2 \ 1]$, and let $\hat{\mathcal{S}}$ and signal bundles $\hat{\mathcal{E}}$ and $\hat{\mathcal{S}}$ satisfy

\begin{align}
\hat{\mathcal{E}}_{\text{proj}}(\lambda) &= \begin{bmatrix} P_{X_2 X_1} \ 0 \\ 0 \ P_{X_2 W_1} \end{bmatrix} \hat{\mathcal{E}}(\lambda), \ \lambda \in \mathbb{C}, \\
\hat{\mathcal{G}}_{\text{proj}}(\lambda) &= \begin{bmatrix} P_{X_2 X_1} \ 0 \\ 0 \ P_{W_2 W_1} \end{bmatrix} \hat{\mathcal{G}}(\lambda), \ \lambda \in \mathbb{C}, \\
\hat{\mathcal{S}}_{\text{proj}}(\lambda) &= P_{W_2 W_1} \hat{\mathcal{S}}(\lambda), \ \lambda \in \mathbb{C}.
\end{align}

If $X_1 = \mathcal{X}$ (and hence $X_2 = \{0\}$), then

\begin{align}
\hat{\mathcal{E}}_{\text{proj}}(\lambda) &= \begin{bmatrix} P_{X_2 X_1} \ 0 \\ 0 \ P_{W_2 W_1} \end{bmatrix} \hat{\mathcal{E}}(\lambda), \ \lambda \in \mathbb{C}, \\
\hat{\mathcal{G}}_{\text{proj}}(\lambda) &= \begin{bmatrix} P_{X_2 X_1} \ 0 \\ 0 \ P_{W_2 W_1} \end{bmatrix} \hat{\mathcal{G}}(\lambda), \ \lambda \in \mathbb{C},
\end{align}

\begin{align}
\hat{\mathcal{S}}_{\text{proj}}(\lambda) &= P_{W_2 W_1} \hat{\mathcal{S}}(\lambda), \ \lambda \in \mathbb{C}.
\end{align}

**Proof.** This follows from Lemma 1.6.9. $\square$

1.6.11. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with characteristic node bundle $\hat{\mathcal{E}}$, and let $\Sigma_{\text{ext}} = (V_1; \mathcal{X}, [\mathcal{W}, Y_1])$ be the bounded i/o extension of $\Sigma$ with control operator $B_1 \in \mathcal{B}(U_1; \mathcal{X})$, observation operator $C_1 \in \mathcal{B}(\mathcal{X}, Y_1)$, and feedthrough operator $D_1 \in \mathcal{B}(U_1; Y_1)$ (see 1.2.11). Then the characteristic node bundle $\hat{\mathcal{E}}_{\text{ext}}$ of $\Sigma_{\text{ext}}$ is given by

\begin{equation}
\hat{\mathcal{E}}_{\text{ext}}(\lambda) = \begin{bmatrix} 1_\mathcal{X} \ 0 \\ 0 \ 1_\mathcal{X} \ 0 \ -B_1 \ 0 \\ 0 \ 0 \ 0 \ 1_{U_1} \ 0 \\ 0 \ 0 \ 1_{\mathcal{W}} \ 0 \ 0 \\ 0 \ 0 \ C_1 \ 0 \ D_1 \end{bmatrix} \hat{\mathcal{E}}(\lambda), \ \lambda \in \mathbb{C}.
\end{equation}

**Proof.** This follows from 1.2.15 and 1.6.9. $\square$

1.6.12. Lemma. Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i)$, $i = 1, 2$, be two s/s systems with characteristic node, signal/state, and signal bundles $\hat{\mathcal{E}}_i$, $\hat{\mathcal{G}}_i$, and $\hat{\mathcal{S}}_i$, respectively, and let $\Sigma_x = \Sigma_1 \times \Sigma_2$ be the cross product of $\Sigma_1$ and $\Sigma_2$. Then the node, signal/state, and signal bundles $\hat{\mathcal{E}}_x$, $\hat{\mathcal{G}}_x$, and $\hat{\mathcal{S}}_x$ of $\Sigma_x$ are given by

\begin{align}
\hat{\mathcal{E}}_x(\lambda) &= \left\{ \begin{bmatrix} z_i \\ x_i \\ w_i \end{bmatrix} \right\} \in \begin{bmatrix} X \ |
\mathcal{X} \ |
\mathcal{W} \end{bmatrix} \ 
\hat{\mathcal{E}}_i(\lambda), \ i = 1, 2 \right\}, \ \lambda \in \mathbb{C}, \\
\hat{\mathcal{G}}_x(\lambda) &= \left\{ \begin{bmatrix} z_i \\ x_i \\ w_i \end{bmatrix} \right\} \in \begin{bmatrix} X \ |
\mathcal{X} \ |
\mathcal{W} \end{bmatrix} \ 
\hat{\mathcal{G}}_i(\lambda), \ i = 1, 2 \right\}, \ \lambda \in \mathbb{C}, \\
\hat{\mathcal{S}}_x(\lambda) &= \begin{bmatrix} \hat{\mathcal{S}}_1(\lambda) \\ \hat{\mathcal{S}}_2(\lambda) \end{bmatrix}, \ \lambda \in \mathbb{C}.
\end{align}

**Proof.** This follows from Definitions 1.2.25 and 1.6.7 and Lemma 1.6.2. $\square$
1.6.13. **Lemma.** Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i), i = 1, 2, \) be two s/s nodes or systems and let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be the interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) through \((P, Q)\), where \( P \) and \( Q \) are continuous linear surjective operators with closed domains \( \text{dom}(P) \subset [\mathcal{X}_2] \) and \( \text{dom}(Q) \subset [\mathcal{W}_1] \). Denote the characteristic node, signal/state, and signal bundles of the cross product \( \Sigma_x = \Sigma_1 \times \Sigma_2 \) by \( \hat{\mathcal{E}}, \hat{\mathcal{G}}_x, \hat{\mathcal{F}}_x \), respectively.

(i) The characteristic node bundle \( \hat{\mathcal{E}} \) of \( \Sigma \) is given by

\[
\hat{\mathcal{E}}(\lambda) = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \left( \hat{\mathcal{E}}_x(\lambda) \cap \left[ \text{dom}(P) \right] \right), \quad \lambda \in \mathbb{C}.
\]

(ii) The characteristic signal/state bundle \( \hat{\mathcal{G}} \) of \( \Sigma \) satisfies

\[
\hat{\mathcal{G}}(\lambda) \supset \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \left( \hat{\mathcal{G}}_x(\lambda) \cap \left[ \text{dom}(P) \right] \right), \quad \lambda \in \mathbb{C}.
\]

If \( P \) is injective, then

\[
\hat{\mathcal{G}}(\lambda) = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \left( \hat{\mathcal{G}}_x(\lambda) \cap \left[ \text{dom}(P) \right] \right), \quad \lambda \in \mathbb{C}.
\]

(iii) If \( \text{dom}(P) = [\mathcal{X}_2] \), then the characteristic signal bundle \( \hat{\mathcal{F}} \) of \( \Sigma \) satisfies

\[
\hat{\mathcal{F}}(\lambda) \subset Q(\hat{\mathcal{F}}_x(\lambda) \cap \text{dom}(Q)), \quad \lambda \in \mathbb{C},
\]

and if \( P \) is injective, then

\[
\hat{\mathcal{F}}(\lambda) \subset Q(\hat{\mathcal{F}}_x(\lambda) \cap \text{dom}(Q)), \quad \lambda \in \mathbb{C}.
\]

Thus, if \( \text{dom}(P) = [\mathcal{X}_2] \) and \( P \) is injective, then

\[
\hat{\mathcal{F}}(\lambda) = Q(\hat{\mathcal{F}}_x(\lambda) \cap \text{dom}(Q)), \quad \lambda \in \mathbb{C}.
\]

**Proof.** This follows from Lemmas 1.6.9 and 1.6.12 □

1.6.14. **Lemma.** Let the s/s node \( \Sigma_{P,Q} = (V_{P,Q}; \mathcal{X}_1, \mathcal{W}_1) \) be \((P, Q)\)-similar to the s/s node \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \). Denote the characteristic node, state/signal, and signal bundles of \( \Sigma \) by \( \mathcal{E}, \mathcal{G}, \) and \( \mathcal{F} \) and the node, state/signal, and signal bundles of \( \Sigma_{P,Q} \) by \( \mathcal{E}_{P,Q}, \mathcal{G}_{P,Q}, \) and \( \mathcal{F}_{P,Q} \). Then

\[
\mathcal{E}_{P,Q}(\lambda) = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \hat{\mathcal{E}}(\lambda), \quad \lambda \in \mathbb{C},
\]

\[
\mathcal{G}_{P,Q}(\lambda) = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \hat{\mathcal{G}}(\lambda), \quad \lambda \in \mathbb{C},
\]

\[
\mathcal{F}_{P,Q}(\lambda) = Q\hat{\mathcal{F}}(\lambda), \quad \lambda \in \mathbb{C}.
\]

**Proof.** This follows from Lemma 1.6.13 □

In Lemmas 1.6.8–1.6.14 we have throughout assumed some relationships between two s/s nodes, and claimed that the characteristic node bundles of the two s/s nodes are related in the same (or a similar) way. Since there is a one-to-one correspondence between the generating subspace of a s/s node and each of the the fibers of the characteristic the “converse” claim will also be true.

1.6.15. **Lemma.** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}), \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W}_1), \) and \( \Sigma_2 = (V_2; \mathcal{X}_2, \mathcal{W}_2) \) be three s/s nodes with characteristic node bundles \( \mathcal{E}, \mathcal{E}_1, \) and \( \mathcal{E}_2 \), respectively.
(i) $\Sigma_1$ is the time reflection of $\Sigma$ if and only if $1.6.16a$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$) with $\mathcal{E}$ replaced by $\mathcal{E}_1$.

(ii) $\Sigma_1$ is the $(P,Q)$-image of $\Sigma$ if and only if $1.6.17$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$).

(iii) $\Sigma_1$ is the part of $\Sigma$ in $X_1$ if and only if $1.6.20a$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$) with $\mathcal{E}_{\text{part}}$ replaced by $\mathcal{E}_1$.

(iv) $\Sigma_1$ is the static projection of $\Sigma$ onto $X_1$ along $X_2$ if and only if $1.6.21a$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$) with $\mathcal{E}_{\text{proj}}$ replaced by $\mathcal{E}_1$.

(v) $\Sigma_1$ is the bounded i/o extension of $\Sigma$ with control operator $B_1 \in \mathcal{B}(U_1; X)$, observation operator $C_1 \in \mathcal{B}(X, Y_1)$, and feedthrough operator $D_1 \in \mathcal{B}(U_1; Y_1)$ if and only if $1.6.22$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$) with $\mathcal{E}_{\text{ext}}$ replaced by $\mathcal{E}_1$.

(vi) $\Sigma$ is the cross product of $\Sigma_1$ and $\Sigma_2$ if and only if $1.6.23a$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$) with $\mathcal{E}_{\times}$ replaced by $\mathcal{E}$.

(vii) $\Sigma$ is the $(P,Q)$-connection of $\Sigma_1$ and $\Sigma_2$ if and only if $1.6.24$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$).

(viii) $\Sigma_1$ is $(P,Q)$-similar of $\Sigma$ if and only if $1.6.27a$ holds for some $\lambda \in \mathbb{C}$ (and hence for all $\lambda \in \mathbb{C}$) with $\mathcal{E}_{P,Q}$ replaced by $\mathcal{E}_1$.

Proof. This follows from Lemma 1.6.2 and Lemmas 1.6.8, 1.6.14. □
1.7. Notes and Comments (Jan 02, 2016)

All results in this chapter remain valid if we throughout replace $L^1$ by $L^p$ where $1 \leq p < \infty$. This will be important in Chapters 8-11 where we shall work in $L^2$ and $L^p_{\text{loc}}$ instead of $L^1$ and $L^1_{\text{loc}}$.

For an interval $I$ which is infinite or not closed the space $C(I; \mathcal{Z})$ in Notation 1.1.5 is a Fréchet space, i.e., $C(I; \mathcal{Z})$ is locally convex and the topology of $C(I; \mathcal{Z})$ is induced by a complete invariant metric (see Rudin, 1973, Definition 1.8, p. 8). For infinite intervals $I$ also the space $L^p_{\text{loc}}(I; \mathcal{Z})$ in Notation 1.1.5 is a Fréchet space. Thus, it is possible to define convergence and continuity, etc., in $C(I; \mathcal{Z})$ and $L^p_{\text{loc}}(I; \mathcal{Z})$ by using a metric. However, in practice it is more convenient to write $I$ as a countable union $I = \bigcup_{n \in \mathbb{N}} I_n$ of an increasing sequence of finite closed intervals $I_n$ and to work with the countable family of seminorms $\|z\|_n = \sup_{s \in I_n} \|z(s)\|_\mathcal{Z}$ in $C(I; \mathcal{Z})$ and the family of seminorms $\|z\|_n = \left( \int_{I_n} \|z(s)\|^p_\mathcal{Z} \, ds \right)^{1/p}$ in $L^p_{\text{loc}}(I; \mathcal{Z})$.

This is equivalent to using the notion of convergence used in Notation 1.1.5.

All results in this chapter remain valid if we allow the state spaces and the signal spaces of the s/s systems to be $B$-spaces instead of $H$-spaces.
A significant part of the present theory for s/s systems is based on the fact that every such system has at least one i/s/o representation (and usually infinity many). Each one of these representation is an i/s/o system whose trajectories are in one-to-one correspondence with the trajectories of a s/s system in a natural way. We begin this chapter by defining what we mean by an i/s/o (input/state/output) system and study their basic properties, and we also explain the basic connection between s/s systems and i/s/o systems. The main difference between an i/s/o system and a s/s system is that instead of having a signal space $\mathcal{W}$ which enables a s/s system to interact with the outside world we now have a designated input through which the outside world can influence the i/s/o system, and also a designated output through which the i/s/o system can influence the outside world. The classes of i/s/o systems that we study in this chapter are very general, and they correspond to the very general classes of s/s systems that we studied in Chapter 1. More specific subclasses of i/s/o systems will be studied in later chapters.
2.1. Input/State/Output Systems (Jan 02, 2016)

2.1.1. Regular Input/state/output nodes and systems. I/s/o systems can be treated in many different settings. In the main classes of systems studied in this book the evolution of data (i.e., the state $x$, the input $u$, and the output $y$) is described by an equation of the form

$$
\begin{align*}
\Sigma: & \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \text{dom}(S), \\
& \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},
\end{align*}
$$

Here $I$ is a nontrivial interval, $x$, $u$, and $y$ take their values in the $H$-spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$, respectively, and $S$ is a linear operator $[\mathcal{X} \cup \mathcal{Y}] \to [\mathcal{X} \mathcal{Y}]$. The spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are called the state space, the input space, and the output space, respectively. If $I$ has a finite end-point which belongs to $I$, then at this point we replace the two-sided derivative $\dot{x}(t)$ in (2.1.1) by the appropriate one-sided derivative (see Remark 1.1.1).

2.1.1. Definition (cf. Definitions 1.1.6 and 1.1.9).

(i) By a regular i/s/o node we mean a colligation $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are $H$-spaces, and $S: [\mathcal{X} \cup \mathcal{Y}] \to [\mathcal{X} \mathcal{Y}]$ is a closed linear operator with dense domain. The spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are called the state space, the input space, and the output space, respectively. The operator $S$ is called the system operator of $\Sigma$.

(ii) By a classical trajectory generated by $\Sigma$ on the closed interval $I$ we mean a triple of functions $\begin{bmatrix} x \atop u \atop y \end{bmatrix} \in [C^1(I; \mathcal{X}) \cap C(I; \mathcal{U}) \cap C(I; \mathcal{Y})]$ satisfying (2.1.1) (for all $t \in I$).

If $S$ is bounded, then it is possible to write $S$ in block matrix form $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in B(\mathcal{X})$, $B \in B(\mathcal{U}; \mathcal{X})$, $C \in B(\mathcal{X}; \mathcal{Y})$ and $D \in B(\mathcal{U}; \mathcal{Y})$. Even in the case where $S$ is unbounded it is possible to define “formal” versions of the same operators.

Below we shall often need to combine two or more operators into a “block matrix” operator. By this we mean the following.

2.1.2. Notation.

(i) Let $A: \mathcal{X} \to Z$ and $C:\mathcal{X} \to \mathcal{Y}$ be linear operators with domains $\text{dom} (A)$ respectively $\text{dom} (C)$. By the block matrix operator $[\begin{bmatrix} A \\ C \end{bmatrix}]$ we mean the linear operator

$$
[\begin{bmatrix} A \\ C \end{bmatrix}] x = \begin{bmatrix} Ax \\ Cx \end{bmatrix}, \quad x \in \text{dom} \left( \begin{bmatrix} A \\ C \end{bmatrix} \right) := \text{dom} (A) \cap \text{dom} (C).
$$

(ii) Let $A: \mathcal{X} \to Z$ and $B: \mathcal{U} \to Z$ be linear operators with domains $\text{dom} (A)$ respectively $\text{dom} (B)$. By the block matrix operator $[\begin{bmatrix} A & B \end{bmatrix}]$ we mean the linear operator

$$
[\begin{bmatrix} A & B \end{bmatrix}] \begin{bmatrix} x \\ u \end{bmatrix} = Ax + Bu, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} \left( \begin{bmatrix} A & B \end{bmatrix} \right) := \begin{bmatrix} \text{dom} (A) \\ \text{dom} (B) \end{bmatrix}.
$$
(iii) Let $A : \mathcal{X} \to Z$, $B : \mathcal{U} \to Z$, $C : \mathcal{X} \to \mathcal{Y}$, and $D : \mathcal{U} \to \mathcal{Y}$ be linear operators with domains $\text{dom}(A)$, $\text{dom}(B)$, $\text{dom}(C)$, and $\text{dom}(D)$, respectively. By the block matrix operator $[\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ we mean the linear operator

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ Cx + Du \end{bmatrix},
$$

(2.1.4)

$$
\begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) := \left[ \text{dom}(A) \cap \text{dom}(C) \right] \cap \left[ \text{dom}(B) \cap \text{dom}(D) \right].
$$

2.1.3. DEFINITION. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular $i/s/o$ node.

(i) The main operator $A : \mathcal{X} \to \mathcal{X}$ of $\Sigma$ (or of $S$) is defined by

$$
Ax := [I_\mathcal{X} \quad 0] S \begin{bmatrix} 0 \\ \hat{\alpha} \end{bmatrix}, \quad x \in \text{dom}(A) := \{ x \in \mathcal{X} \mid \begin{bmatrix} 0 \\ \hat{\alpha} \end{bmatrix} \in \text{dom}(S) \}.
$$

(ii) The classical observation operator $C : \mathcal{X} \to \mathcal{U}$ of $\Sigma$ has the same domain as the main operator $A$, and it is defined by

$$
Cx := [0 \quad 1_\mathcal{Y}] S \begin{bmatrix} 0 \\ \hat{\alpha} \end{bmatrix}, \quad x \in \text{dom}(C) = \text{dom}(A).
$$

(iii) The classical control operator $B : \mathcal{U} \to \mathcal{X}$ of $\Sigma$ is defined by

$$
Bu := [I_\mathcal{X} \quad 0] S \begin{bmatrix} 0 \\ \hat{\beta} \end{bmatrix}, \quad u \in \text{dom}(B) := \{ u \in \mathcal{U} \mid \begin{bmatrix} 0 \\ \hat{\beta} \end{bmatrix} \in \text{dom}(S) \}.
$$

(iv) The classical feedthrough operator $D : \mathcal{U} \to \mathcal{Y}$ of $\Sigma$ has the same domain as the control operator $B$, and it is defined by

$$
Du := [0 \quad 1_\mathcal{Y}] S \begin{bmatrix} 0 \\ \hat{\gamma} \end{bmatrix}, \quad u \in \text{dom}(D) = \text{dom}(B).
$$

The most important of these operators is the main operator $A$, which will play a central role in the theory of the $i/s/o$ systems generated by the $i/s/o$ node (the notion of an $i/s/o$ system is defined in Definition 2.1.7 below). The second most important operator is the classical observation operator $C$, whereas the control operator $B$ and the observation operator $D$ will be important mainly for the special classes of “bounded” and “semi-bounded” $i/s/o$ systems. We have included the word “classical” in the definitions of $C$, $B$, and $D$ since it is often convenient to extend these operators to a bigger domain (in the case of the control operator $B$ the extension will not map $\mathcal{U}$ into $\mathcal{X}$ but into a larger space, and these extensions need not be unique).

It is tempting to try to write (2.1.1) in the form

$$
\Sigma : \begin{cases} 
  x(t) \in \text{dom}(A) = \text{dom}(C), \\
  u(t) \in \text{dom}(B) = \text{dom}(D), \\
  \dot{x}(t) = Ax(t) + Bu(t), \\
  y(t) = Cx(t) + Du(t),
\end{cases}
\quad t \in I.
$$

(2.1.9)

where $A$, $B$, $C$, and $D$ are the operators in Definition 2.1.3. This is possible if and only if it is true that $S \begin{bmatrix} \hat{\gamma} \\ \hat{\alpha} \end{bmatrix} = [A \quad B] \begin{bmatrix} \hat{\gamma} \\ \hat{\alpha} \end{bmatrix} = [C \quad D] x + [B \quad D] u$ for all $\begin{bmatrix} \hat{\gamma} \\ \hat{\alpha} \end{bmatrix} \in \text{dom}(S)$.

2.1.4. LEMMA. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular $i/s/o$ system, and let $A$, $B$, $C$, and $D$ be the operators in Definition 2.1.3.

(i) $[\begin{bmatrix} A \\ C \end{bmatrix}] : \text{dom}(A) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ is continuous if and only if $\text{dom}(A)$ is closed in $\mathcal{X}$.

(ii) $[\begin{bmatrix} B \\ C \end{bmatrix}] : \text{dom}(B) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ is continuous if and only if $\text{dom}(B)$ is closed in $\mathcal{U}$.

(iii) $S = [\begin{bmatrix} A \\ B \end{bmatrix}]$ if and only if $\text{dom}(S) = \left[ \begin{bmatrix} \text{dom}(A) \\ \text{dom}(B) \end{bmatrix} \right]$. 
(iv) If \( S = [A \ B \ C \ D] \), then equation (2.1.1) is equivalent to equation (2.1.9).

Proof. Proofs of (i) and (ii): Both \([A \ C]\) and \([B \ D]\) are closed since \( S \) is closed, and hence they are continuous if and only if their domains are closed.

Proof of (iii): If \( S = [A \ B \ C \ D] \), then dom \((S) = \text{dom}((A \ B \ C \ D)) = \text{dom}(A) \cap \text{dom}(D)\). Conversely, suppose that dom \((S) = \text{dom}((A \ B \ C \ D)) = \text{dom}(A) \cap \text{dom}(D)\). Then \([x] \in \text{dom}(S)\) implies that \(x \in \text{dom}(A)\) and \(u \in \text{dom}(B)\), and

\[
\begin{bmatrix} A \\ C \end{bmatrix} x + \begin{bmatrix} B \\ D \end{bmatrix} u = S \begin{bmatrix} 1 \\ 0 \\ x \\ u \end{bmatrix} = S \begin{bmatrix} x \\ u \end{bmatrix}.
\]

Thus \( S = [A \ B \ C \ D] \).

Proof of (iv): This is trivially true. \(\square\)

From time to time (especially in the study of passive impedance and transmission i/s/o systems) we shall also need i/s/o nodes of a more general type, that we simply call i/s/o nodes (mission i/s/o systems) we shall also need i/s/o nodes of a more general type, that need not be closed or have a dense domain.

By a multi-valued linear operator from \(X\) to \(Z\) we mean an operator whose graph is an arbitrary subspace of \([X \ Z]\). We denote the space of multivalued linear operators from \(X\) to \(Z\) by \(\mathcal{ML}(X; Z)\). For more details see Section A.4.

If we allow \( S \) to be multi-valued, then the basic equation (2.1.1) defining the dynamics of the system must be replaced by the relaxed equation

(2.1.10) \( \Sigma: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \text{dom}(S), \\ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} t \in I, \)

where the equality in (2.1.1) has been replaced by an inclusion.

2.1.5. Definition (cf. Definition 1.1.3).

(i) By an i/s/o node we mean a colligation \( \Sigma = (S; X, U, Y) \), where \( U, X \) and \( Y \) are H-spaces, and \( S \in \mathcal{ML}([X \ U]; [Y \ Y]) \) (i.e., \( S: [X \ U] \to [Y \ Y] \) is a multi-valued linear operator; see Section A.4). The spaces \( X, U, Y \) are called the state space, the input space, and the output space, respectively.

The operator \( S \) is called the (multi-valued) system operator of \( \Sigma \).

(ii) By a closed i/s/o node we mean an i/s/o node \( \Sigma = (S; X, U, Y) \) with a closed system operator \( S \).

(iii) By a single-valued i/s/o node we mean an i/s/o node \( \Sigma = (S; X, U, Y) \) with a single-valued system operator \( S \).

(iv) By a semi-regular i/s/o node we mean an i/s/o node \( \Sigma = (S; X, U, Y) \) with both single-valued and closed (i.e., \( S \) is single-valued and closed).

(v) By a classical trajectory generated by \( \Sigma \) on the interval \( I \) we mean a triple of functions \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \in \begin{bmatrix} C^1(I; X) \\ C(I; U) \\ C(I; Y) \end{bmatrix} \) satisfying (2.1.10) (for all \( t \in I \)).

Also in the case of a general (multi-valued) i/s/o node it is possible to define what we mean by its main, control, observation, and feedthrough operator, but out of these only the main operator will play a significant role.
2.1.6. Definition. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node.

(i) If $\Sigma$ is single-valued, then we define the main operator and the classical observation, control and feedthrough operators in the same way as in the case where $\Sigma$ is regular (see Definition 2.1.3).

(ii) In general we defined the main operator $A$ of $\Sigma$ (or of $S$) to be the multi-valued operator $A \in ML(X)$ defined by

$$\text{dom}(A) = \left\{ x \in X \mid \left[ x_0 \right] \in \text{dom}(S) \right\},$$

$$Ax := \left\{ z \in X \mid x \in \text{dom}(A) \text{ and } \left[ z \right] \in S[x_0] \text{ for some } y \in \mathcal{Y} \right\}.$$

2.1.7. Definition. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a (regular or non-regular) i/s/o node.

(i) A triple of functions $\left[ \begin{array}{c} x \hfill \\
 u \hfill \\
 y \end{array} \right] \in \left[ \begin{array}{c} C(I; X) \\
 L^1(I; U) \\
 L^1(I; Y) \end{array} \right]$ is a generalized trajectory generated by $\Sigma$ on a closed finite interval $I$ if there exists a sequence of classical trajectories $\left[ \begin{array}{c} x_n \\
 u_n \\
 y_n \end{array} \right]$ generated by $\Sigma$ on $I$ which converges to $\left[ \begin{array}{c} x \\
 u \\
 y \end{array} \right]$ in $\left[ \begin{array}{c} C(I; X) \\
 L^1(I; U) \\
 L^1(I; Y) \end{array} \right]$ as $n \to \infty$. Here the notion of a classical trajectory generated by $\Sigma$ on $I$ is defined as described in Definitions 2.1.1 and 2.1.5.

(ii) A triple of functions $\left[ \begin{array}{c} x \\
 u \\
 y \end{array} \right] \in \left[ \begin{array}{c} C(I; X) \\
 L^1_{\text{loc}}(I; U) \\
 L^1_{\text{loc}}(I; Y) \end{array} \right]$ is a generalized trajectory generated by $\Sigma$ on the general interval $I$ if the restriction of $\left[ \begin{array}{c} x \\
 u \\
 y \end{array} \right]$ to every finite closed subinterval $I'$ of $I$ is a generalized trajectory generated by $\Sigma$ on $I'$.

(iii) By the i/s/o (input/state/output) system induced by the i/s/o node $\Sigma$ we mean the node $\Sigma$ itself together with sets of classical and generalized trajectories generated by $\Sigma$. We use the same notation $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ for the system as for the i/s/o node, and alternatively write “trajectories of $\Sigma$” instead of “trajectories generated by the node $\Sigma$”. If the i/s/o node $\Sigma$ is closed, or semi-regular, or regular, or bounded, then the i/s/o system is called closed, or semi-regular, or regular, or bounded as well.

The three most important intervals $I$ in Definition 2.1.7 are $I = \mathbb{R}^+$, $I = \mathbb{R}^-$, and $I = \mathbb{R}$. We give the following special names to trajectories defined on these intervals.

2.1.8. Definition. By a future, past, or two-sided (classical or generalized) trajectory of an i/s/o system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a trajectory defined on $\mathbb{R}^+$, $\mathbb{R}^-$, or $\mathbb{R}$, respectively.

As will be shown later, classical and generalized trajectories of an i/s/o system have properties that are analogous to the properties of classical and generalized trajectories of a s/s system discussed in Chapter 1.

2.1.2. Bounded input/state/output systems. A specially simple class of i/s/o systems is the class of bounded i/s/o systems.

2.1.9. Definition. An i/s/o node or system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is bounded if $S \in B([\mathcal{X}; \mathcal{U}])$.

In the case of a bounded i/s/o node the system operator $S$ can be decomposed into four blocks $S = [\begin{array}{c|c}
A & B \\
\hline
C & D \end{array}]$, where the operators $A$, $B$, $C$, and $D$ are the same as in
2.1. INPUT/STATE/OUTPUT SYSTEMS (Jan 02, 2016) 69

Definition 2.1.3, i.e., \( A \) is the main operator, \( B \) is the (classical) control operator, \( C \) is the (classical) observation operator, and \( D \) is the (classical) feedthrough operator. Using these four operators equation (2.1.1) can be rewritten in the form

\[
\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} t \in I.
\]

Let us first look at the case where the operators \( B, C, \) and \( D \) are absent in (2.1.12) (which corresponds to the case where both \( U = \{0\} \) and \( Y = \{0\} \)). In this case the main operator \( A \) is the generator of a uniformly continuous group.

2.1.10. Definition. Let \( X \) be an \( H \)-space.

(i) A family \( \mathfrak{A}^t, t \in \mathbb{R}^+ \), of bounded linear operators \( \mathcal{X} \rightarrow \mathcal{X} \) is a semigroup in \( \mathcal{X} \) if \( \mathfrak{A}^0 = 1 \) and \( \mathfrak{A}^s \mathfrak{A}^t = \mathfrak{A}^{s+t} \) for all \( s, t \in \mathbb{R}^+ \).

(ii) A family \( \mathfrak{A}^t, t \in \mathbb{R}, \) of bounded linear operators \( \mathcal{X} \rightarrow \mathcal{X} \) is a group in \( \mathcal{X} \) if \( \mathfrak{A}^0 = 1 \) and \( \mathfrak{A}^s \mathfrak{A}^t = \mathfrak{A}^{s+t} \) for all \( s, t \in \mathbb{R} \).

(iii) The semigroup in (i) is strongly continuous (at zero) if \( \lim_{t \downarrow 0} \mathfrak{A}^t x = x \) for all \( x \in \mathcal{X} \). The group in (ii) is strongly continuous (at zero) if \( \lim_{t \rightarrow 0} \mathfrak{A}^t x = x \) for all \( x \in \mathcal{X} \).

(iv) We abbreviate ‘strongly continuous semigroup’ to ‘\( C_0 \) semigroup’ and ‘strongly continuous group’ to ‘\( C_0 \) group’;

(v) The semigroup in (i) and the group in (ii) is uniformly continuous if \( \mathfrak{A}^t \) is a continuous function of \( t \) in \( B(\mathcal{X}) \).

Note that every uniformly continuous group or semigroup of bounded linear operators is also a \( C_0 \) group or semigroup.

In the following theorem all the limits of operators in \( B(\mathcal{X}) \) and derivatives of functions with values in \( B(\mathcal{X}) \) are taken with respect to the standard norm topology of \( B(\mathcal{X}) \).

2.1.11. Theorem. Let \( \mathcal{X} \) be an \( H \)-space.

(i) Let \( A \in B(\mathcal{X}) \), and define

\[
\mathfrak{A}^t = e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k, \quad t \in \mathbb{R}.
\]

Then \( \mathfrak{A} \) is a uniformly continuous group in \( \mathcal{X} \), the function \( t \mapsto \mathfrak{A}^t \) is analytic on \( \mathbb{R} \) (and can be extended to an entire function on \( \mathbb{C} \)), and

\[
\frac{d^n}{dt^n} \mathfrak{A}^t = A^n \mathfrak{A}^t = \mathfrak{A}^t A^n, \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}^+.
\]

In particular,

\[
A = \lim_{t \to 0} \frac{1}{t} (\mathfrak{A}^t - 1_x).
\]

(ii) Let \( \mathfrak{A} \) be a uniformly continuous semigroup in \( \mathcal{X} \). Then there exists an operator \( A \in B(\mathcal{X}) \) such that \( \mathfrak{A}^t \) is given by (2.1.13) for all \( t \in \mathbb{R}^+ \), and (2.1.13) can be used to extend \( \mathfrak{A} \) to a uniformly continuous group on \( \mathcal{X} \).

Proof. Fix some admissible norm \( \|\cdot\|_\mathcal{X} \) in \( \mathcal{X} \), and denote the corresponding operator norm in \( B(\mathcal{X}) \) by \( \|\cdot\|_{B(\mathcal{X})} \).
(i) The series in (2.1.13) converges absolutely in $\mathcal{B}(\mathcal{X})$ since
\[
\sum_{n=0}^{\infty} \left( \frac{\|A\|^{n}}{n!} \right) \leq \sum_{n=0}^{\infty} \left( \frac{\|A\|^{n} \| \mathcal{X} \|}{n!} \right) = e^{\|A\| \| \mathcal{X} \|},
\]
The same estimate shows that
\[
(2.1.16) \quad \|e^{At}\|_{\mathcal{B}(\mathcal{X})} \leq e^{\|A\| \| \mathcal{X} \| t}, \quad t \in \mathbb{R}.
\]
Clearly $e^{0A} = 1$. Being a power series, the function $t \mapsto e^{At}$ is analytic, hence uniformly continuous for all $t$. By differentiating the power series (this is permitted since $e^{At}$ is analytic) we find that (2.1.15) holds. Thus, it only remains to verify the group property $e^{A(t+s)} = e^{As}e^{At}$, which is done as follows:
\[
e^{A(s+t)} = \sum_{n=0}^{\infty} \frac{A^{n}(s+t)^{n}}{n!} = \sum_{n=0}^{\infty} \frac{A^{n}}{n!} \sum_{k=0}^{n} \binom{n}{k} s^{k}t^{n-k}
\]
\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^{k}s^{k}}{k!} A^{n-k}t^{n-k} \frac{(n-k)!}{(n-k)!} = \sum_{k=0}^{\infty} \frac{A^{k}s^{k}}{k!} \sum_{n=k}^{\infty} \frac{A^{n-k}t^{n-k}}{(n-k)!}
\]
\[
= e^{As}e^{At}.
\]
(ii) For sufficiently small positive $h$, $\|1-1/h \int_{0}^{h} \mathfrak{A}^{s} \ ds\|_{\mathcal{B}(\mathcal{X})} < 1$, hence $1/h \int_{0}^{h} \mathfrak{A}^{s} \ ds$ is invertible, and so is $\int_{0}^{h} \mathfrak{A}^{s} \ ds$. By the semigroup property, for $0 < t < h$,
\[
\frac{1}{t} (\mathfrak{A}^{t} - 1) \int_{0}^{h} \mathfrak{A}^{s} \ ds = \frac{1}{t} \left( \int_{0}^{t} \mathfrak{A}^{s+t} \ ds - \int_{0}^{h} \mathfrak{A}^{s} \ ds \right)
\]
\[
= \frac{1}{t} \left( \int_{t}^{h} \mathfrak{A}^{s+t} \ ds - \int_{t}^{h} \mathfrak{A}^{s} \ ds \right)
\]
\[
= \frac{1}{t} \left( \int_{0}^{t} \mathfrak{A}^{s+t} \ ds - \int_{0}^{t} \mathfrak{A}^{s} \ ds \right).
\]
Multiply by $(\int_{0}^{h} \mathfrak{A}^{s} \ ds)^{-1}$ to the right and let $t \downarrow 0$ to get
\[
\lim_{t \downarrow 0} \frac{1}{t} (\mathfrak{A}^{t} - 1) = (\mathfrak{A}^{h} - 1) (\int_{0}^{h} \mathfrak{A}^{s} \ ds)^{-1}
\]
in the uniform operator norm.
Let $A = \lim_{t \downarrow 0} \frac{1}{t} (\mathfrak{A}^{t} - 1)$. Then $A \in \mathcal{B}(\mathcal{X})$, and we can define $e^{At}$ by the (2.1.13) for all $t \in \mathbb{R}$. Then
\[
\lim_{t \downarrow 0} \frac{1}{t} (\mathfrak{A}^{t} - 1) = A = \lim_{t \downarrow 0} \frac{1}{t} (e^{At} - 1).
\]
Since $\mathfrak{A}$ is a semigroup and and $t \mapsto e^{At}$ is a group we get for all $t \geq 0,$
\[
\lim_{h \downarrow 0} \frac{1}{h} (e^{-A(t+h)} \mathfrak{A}^{t+h} - e^{-At} \mathfrak{A}^{t})
\]
\[
= \lim_{h \downarrow 0} \frac{1}{h} (e^{-A(t+h)} (\mathfrak{A}^{t+h} - \mathfrak{A}^{t}) + (e^{-A(t+h)} - e^{-At}) \mathfrak{A}^{t})
\]
\[
= \lim_{h \downarrow 0} \frac{1}{h} (e^{-A(t+h)} (\mathfrak{A}^{h} - 1) \mathfrak{A}^{t} + e^{-At} \frac{1}{h} (e^{-Ah} - 1) \mathfrak{A}^{t})
\]
\[
= (e^{-At} A \mathfrak{A}^{t} - e^{-At} A \mathfrak{A}^{t}) = 0.
\]
This shows that the function \( t \mapsto e^{-At}x^t \) is continuously differentiable on \( \mathbb{R}^+ \) with a zero derivative, and hence \( e^{-At}x^t = e^{-At}x^0 = 1_X \) for all \( t \in \mathbb{R}^+ \). Multiplying this identity by \( e^{At} \) to the left and using the group property in the form \( e^{At}e^{-At} = 1_X \) we get \( x^t = e^{At} \) for all \( t \in \mathbb{R}^+ \). 

2.1.12. Definition.

(i) The operator \( A \) in (2.1.15) is called the generator of the uniformly continuous group \( \mathfrak{A} \) in part (i) of Theorem 2.1.11, and the uniformly continuous semigroup \( \mathfrak{A} \) in part (ii) of Theorem 2.1.11.

(ii) The uniformly continuous group \( \mathfrak{A} \) in part (i) of Theorem 2.1.11 is called the group generated by \( A \), and the uniformly continuous semigroup \( \mathfrak{A} \) in part (ii) of Theorem 2.1.11 is called the semigroup generated by \( A \).

See Definition 4.1.4 for an extension of this notion to the case where \( \mathfrak{A} \) is a \( C_0 \) semigroup.

2.1.13. Lemma. Let \( \mathcal{X} \) be an \( H \)-space, and let \( \mathfrak{A} \) be a uniformly continuous group on \( \mathcal{X} \) with generator \( A \in \mathcal{B}(\mathcal{X}) \). Then, for every \( x^0 \in \mathcal{X} \), every interval \( I \), and every \( t_0 \in I \), the equation

\[
2.1.17 \quad \dot{x}(t) = Ax(t), \quad t \in I,
\]

has a unique solution \( x \in C^1(I; \mathcal{X}) \) satisfying \( x(t_0) = x^0 \). This solution is given by

\[
2.1.18 \quad x(t) = \mathfrak{A}^{t-t_0}x^0, \quad t \in I.
\]

Proof. The right-hand side of (2.1.13) is an analytic \( \mathcal{X} \)-valued function of \( t \). Define \( x \) by (2.1.18). By differentiating \( x \) one finds that \( x \) is an analytic function which satisfies (2.1.17).

If \( x \in C^1(\mathbb{R}; I) \) is a solution of (2.1.17), then the function \( t \mapsto e^{-At}x(t) \) is continuously differentiable on \( I \), with derivative

\[
\frac{d}{dt} e^{-At}x(t) = -Ae^{-At}x(t) + e^{-At} \dot{x}(t) = -e^{-At}Ax(t) + e^{-At}Ax(t) = 0.
\]

Therefore this function is a constant on \( I \), so that \( e^{-At}x(t) = e^{-At_0}x(t_0) = e^{-At_0}x_0 \), \( t \in I \). Consequently

\[
x(t) = (e^{-At})^{-1}e^{-At_0}x_0 = e^{At}e^{-At_0}x_0 = e^{At(t-t_0)}x_0, \quad t \in I.
\]

By Lemma 2.1.13 if \( \Sigma = (A; \mathcal{X}, \{0\}, \{0\}) \), where \( A \in \mathcal{B}(\mathcal{X}) \), then for every \( x^0 \in \mathcal{X} \) and \( t_0 \in \mathbb{R} \) there exists a unique classical trajectory \( x \) of \( \Sigma \) satisfying \( x(t_0) = x^0 \), and it is given by (2.1.18). Moreover, every generalized trajectory of \( \Sigma \) on some interval \( I \) is automatically a classical trajectory of \( \Sigma \) on \( I \) (and it is even an analytic function of the time variable).

The case where the operators \( B, C, \) and \( D \) in (2.1.12) are non-zero is also well understood. In this case classical and generalized trajectories of \( \Sigma \) can be computed by means of the given initial state and input function as follows:

2.1.14. Theorem. Let \( \Sigma = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right) \) be a bounded i/s/o system. Then for every \( x^0 \in \mathcal{X} \), every interval \( I \), every \( t_0 \in I \), and every \( u \in L^1_{loc}(I; \mathcal{U}) \) the system \( \Sigma \) has a unique generalized trajectory \( \begin{bmatrix} x \\ u \end{bmatrix} \) on \( I \) (with input function \( u \)) satisfying
The state $x$ and the output $y$ of this trajectory is given by

$$
x(t) = A^{-t}x_0 + \int_{t_0}^{t} A^{t-s}Bu(s)\,ds, \quad t \in I,
$$

and

$$
y(t) = C A^{-t}x_0 + \int_{t_0}^{t} C A^{t-s}Bu(s)\,ds + Du(t),
$$

where $A$ is the uniformly continuous group defined in \((2.1.13)\). This trajectory is classical if and only if $u \in C(I;U)$.

**Proof.** We first consider the case where $u$ is continuous. Then the right-hand side of the first equation in \((2.1.19)\) is continuously differentiable, and by differentiating this equation with respect to $t$ we get the first equation in \((2.1.12)\). That also the second equation in \((2.1.12)\) holds follows from the second equation in \((2.1.19)\).

In the general case where $u$ only belongs to $L_{loc}^1(I;U)$ we can choose a sequence of continuous functions $u_n$ which converge to $u$ in $L_{loc}^1(I;U)$. We define $\begin{bmatrix} x_n^{} \end{bmatrix}$ by \((2.1.19)\) and $\begin{bmatrix} x_n^{} \end{bmatrix}$ by \((2.1.19)\) with $u$ replaced by $u_n$. Then $x_n \to x$ in $X$ uniformly on finite intervals and $y_n \to y$ in $L_{loc}^1(I;Y)$, and consequently $\begin{bmatrix} x_n^{} \end{bmatrix}$ is a generalized trajectory of $\Sigma$ on $I$.

We next prove uniqueness in the case where $u$ is continuous and $\begin{bmatrix} x \ y \end{bmatrix}$ is a classical trajectory of $\Sigma$ on $I$. Then it follows from the first equation in \((2.1.12)\) that the function $t \mapsto A^{-t}\varphi(t)$ is continuously differentiable on $I$, with derivative

$$
\frac{d}{dt}(A^{-t}\varphi(t)) = -A^{-t}x(t) + A^{-t}\varphi'(t) = A^{-t}Bu(t), \quad t \in I.
$$

By integrating this equation over $[t, t_0]$ (or $[t_0, t]$) and using the group property of $A$ we get the first equation in \((2.1.19)\). The second equation in \((2.1.19)\) then follows from the second equation in \((2.1.12)\).

Finally, if $\begin{bmatrix} x \ y \end{bmatrix}$ is a generalized trajectory of $\Sigma$ on $gI$, then there exists a sequence of classical trajectories $\begin{bmatrix} x_n^{} \ y_n^{} \end{bmatrix}$ of $\Sigma$ on $I$ such that $x_n \to x$ uniformly on bounded intervals, and $\begin{bmatrix} u_n^{} \ y_n^{} \end{bmatrix} \to \begin{bmatrix} u \ y \end{bmatrix}$ in $L_{loc}^1(I;U)$, where classical trajectories satisfy \((2.1.19)\) with $u$ replaced by $\begin{bmatrix} u_n^{} \ y_n^{} \end{bmatrix}$, also their limit $\begin{bmatrix} x \ y \end{bmatrix}$ must satisfy the same equation. \(\square\)

**2.1.15. Definition.** The uniformly continuous group $A$ defined in \((2.1.13)\) is called the evolution group of the i/s/o system $\Sigma = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}; X, U, Y)$.

**2.1.16. Remark.** In the case of a bounded i/s/o system $\Sigma$ the notion of a generalized trajectory generated by $\Sigma$ can also be defined in an alternative way, namely, a generalized trajectory $\begin{bmatrix} x \ y \end{bmatrix}$ of $\Sigma$ on $I$ is the unique solution of the equation \((2.1.12)\) in the following weaker sense: Instead of being continuously differentiable (as in the case of a classical trajectory) it is only true that $x$ is locally absolutely continuous, and the equations in \((2.1.12)\) holds only almost everywhere. If $\begin{bmatrix} x_n^{} \ y_n^{} \end{bmatrix}$ is a sequence of classical trajectories of $\Sigma$ which converges to $\begin{bmatrix} x \ y \end{bmatrix}$ in $\begin{bmatrix} C(I;X) \\ L_{loc}^1(I;U) \end{bmatrix}$, then $\begin{bmatrix} x \ y \end{bmatrix}$ is a generalized trajectory of $\Sigma$ in the above sense. The converse is also true:
generalized trajectory of $\Sigma$ in the above sense can be approximated by a sequence of classical trajectories of the above type. Thus, the class of generalized trajectories described above coincides with the class of generalized trajectories introduced in Definition 2.1.7. For details, see, e.g., Pazy [1983, Section 4.2] or Staffans [2005, Section 3.8].

2.1.3. Solvability, and the uniqueness and continuation properties.

In Chapter 1 we discussed basic properties of the sets of classical and generalized trajectories of s/s systems, and introduced several notions that were related to these sets of trajectories. Analogous notions can be introduced for i/s/o systems. Below we extend these notions to the class of i/s/o systems, namely solvability and the uniqueness and continuation properties. Additional notions and properties will be discussed in Section 2.4.

2.1.17. Definition (cf. Definitions 1.3.3 and 1.3.11). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o system.

(i) $\Sigma$ is forward solvable (or backward solvable, or two-sided solvable) if it is true that for every $[x^0_u y^0] \in \text{dom}(S)$ and every $[z^0_u y^0] \in S[x^0_u y^0]$ there is at least one classical trajectory $[x^u y]$ of $\Sigma$ on $\mathbb{R}^+$ (or on $\mathbb{R}^-$, or on $\mathbb{R}$, respectively) satisfying $[x(0) u(0)] = [x^0_u y^0]$ and $[\dot{x}(0) y(0)] = [z^0_u y^0]$.

(ii) $\Sigma$ has the forward (or backward, or two-sided) uniqueness property if for every finite closed interval $I$ with left end point zero (or with right end point zero, or containing the internal point zero, respectively), every $x^0 \in \mathcal{X}$, and every $[y] \in C(I; [\mathcal{Y}])$ there is at most one classical trajectory $[x^u_y]$ on $I$ (whose input and output are equal to the given function $u$ and $y$, respectively) satisfying $x(0) = x^0$.

(iii) $\Sigma$ is forward, backward, or two-sided uniquely solvable if it has both the respective properties in (i) and (ii).

(iv) $\Sigma$ has the forward (or backward, or two-sided) continuation property if it is true for every $T > 0$ that every generalized trajectory of $\Sigma$ on the interval $[0, T]$ can be continued to a generalized future trajectory of $\Sigma$.

2.1.18. Lemma. Every bounded i/s/o system is two-sided uniquely solvable and has the two-sided continuation property.

Proof. This follows from the representation formula (2.1.19) which gives existence and uniqueness of classical and generalized trajectories on any time interval.

□
2.2. Regular I/S/O Representations of Regular S/S Nodes (Jan 02, 2016)

2.2.1. Remark. In this and the next subsection, as well as in all other places where we discuss i/s/o representations of s/s nodes and systems, we shall usually denote the s/s node or system by Σ and the various i/s/o nodes and systems which are “i/s/o representations” of Σ by Σ_{i/s/o}.

2.2.1. The semi-regular s/s system induced by a semi-regular i/s/o system. Every semi-regular i/s/o node Σ_{i/s/o} = (S; X; U; Y) induces a semi-regular s/s node Σ in the following way:

2.2.2. Theorem. Let Σ_{i/s/o} = (S; X; U; Y) be a semi-regular i/s/o node.

(i) Let W be the product space \( W = [U; Y] \), and define

\[
V := \left\{ \begin{bmatrix} z \\ x \\ u \\ y \end{bmatrix} \in \begin{bmatrix} X \\ X' \end{bmatrix} \Bigg| \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} (S) \land \begin{bmatrix} z \\ y \end{bmatrix} = S \begin{bmatrix} x \\ u \end{bmatrix} \right\}
\]

(2.2.1)

\[
= \begin{bmatrix} 1_X & 0 & 0 & 0 \\ 0 & 0 & 1_X & 0 \\ 0 & [1_Y] & 0 & [1_U] \end{bmatrix} \text{gph}(S),
\]

where

(2.2.2) \( \text{gph}(S) = \left\{ \begin{bmatrix} z \\ Y \\ x \\ y \\ u \end{bmatrix} \in \begin{bmatrix} X \\ Y \\ X \end{bmatrix} \Bigg| \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} (S) \land \begin{bmatrix} z \\ y \end{bmatrix} = S \begin{bmatrix} x \\ u \end{bmatrix} \right\}, \)

and where the embedding operators \( U \to [U; 0] \subset W \) and \( Y \to [0; Y] \subset W \) have been denoted by \([1_U; 0]\) respectively \([0; 1_Y]\). Then Σ = (V; X; W) is a semi-regular s/s node.

(ii) If Σ_{i/s/o} is regular, then Σ is regular as well.

(iii) A triple \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical or generalized trajectory of Σ_{i/s/o} on some interval I if and only if \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical or generalized trajectory of Σ on I.

(iv) Σ is solvable or has the uniqueness property if and only if Σ_{i/s/o} is solvable or has the uniqueness property, respectively.

Proof. Proofs of (i) and (ii): That V defined by (2.2.1) is a closed subspace of the node space \( K = [X; W] \) follows from the representation (2.2.1) and the fact that S is closed. Condition (ii) in Definition 1.1.9 follows directly from (2.2.1). If, in addition, Σ_{i/s/o} is regular, i.e., if \( \text{dom} (S) \) is dense in \([X; U]\), then the projection \( X_0 \) of \( \text{dom} (S) \) onto \( X \) along \( U \) is dense in \( X \), which means that Σ is regular in this case.

Proofs of (iii) and (iv): The claim about the connection between the classical and generalized trajectories of Σ_{i/s/o} and Σ follows directly from (2.2.1) and Definitions 1.1.6 and (1.1.7). This correspondence of classical trajectories implies that Σ is solvable or has the uniqueness property if and only if Σ_{i/s/o} is solvable or has the uniqueness property.

□
2.2.3. Definition. The s/s node and system \( \Sigma = (V; \mathcal{X}, [\mathcal{X}]) \) in Theorem 2.2.2 is called the s/s node and system induced by \( \Sigma_{i/s/o} \).

2.2.4. Remark. The inverses of the two embedding operators \( \mathcal{U} \rightarrow [u_{(0)}] \) and \( \mathcal{Y} \rightarrow [y_{(0)}] \), appearing in (2.2.1), and denoted by \([u]\) respectively \([y]\) can be identified with the block matrix operator \([0 \ 1\mathcal{Y}]\) respectively \([0 \ 1\mathcal{Y}]\). Thus, (2.2.1) can be rewritten in the equivalent form

(2.2.3) \[
gph (S) = \begin{bmatrix}
1_{\mathcal{X}} & 0 & 0 \\
0 & 0 & 1_{\mathcal{Y}} \\
0 & 1_{\mathcal{X}} & 0 \\
0 & 0 & 1_{\mathcal{U}} \\
0 & 0 & 0
\end{bmatrix} V.
\]

2.2.5. Lemma. The regular i/s/o node \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{Y}) \) in Theorem 2.2.2 can be recovered from the generating subspace \( V \) of the s/s node \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) in the following way: \( \text{dom}(S) \) is given by

(2.2.4) \[
\text{dom}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{U} \right| \begin{bmatrix} z \\ x \\ y \end{bmatrix} \in V \text{ for some } \begin{bmatrix} z \\ y \end{bmatrix} \in \mathcal{Y} \right\},
\]

and \( S \begin{bmatrix} x \\ u \end{bmatrix} \) is equal to the unique vector \( \begin{bmatrix} z \\ y \end{bmatrix} \in \mathcal{Y} \) for which \( \begin{bmatrix} z \\ y \end{bmatrix} \) is dense in \( \mathcal{Y} \). \[\blacksquare\]

2.2.6. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be the semi-regular s/s node in Theorem 2.2.2 constructed from the given semi-regular i/s/o node \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \). Then the generating subspace \( V \) of \( \Sigma \) has the following property:

(i) If \( \begin{bmatrix} x \\ u \end{bmatrix} \in V \), then \( z = 0 \) and \( y = 0 \).

If, in addition, \( \Sigma_{i/s/o} \) is regular (and hence \( \Sigma \) is regular), then \( V \) also has the following property:

(ii) The set \( \text{dom}(S) \) is dense in \( \mathcal{Y} \).

Proof. Property (i) follows directly from (2.2.1), and property (ii) says that \( \text{dom}(S) \) is dense in \( [\mathcal{Y}] \). \[\blacksquare\]

2.2.2. Input/output representations of the signal space. Above we have showed that every regular i/s/o system \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) induces a s/s system \( \Sigma = (V; \mathcal{X}, [\mathcal{X}]) \). Below we shall look at the converse questions, and explain how one may construct (usually infinitely many) i/s/o systems which “represent” a given s/s system \( \Sigma \) in the sense that \( \Sigma \) can be recovered from its i/s/o representations in more or less the same way as was described above. The first step in this construction is to “identify” the given signal space \( \mathcal{W} \) with the product \( [\mathcal{U}] \), where \( \mathcal{W} = \mathcal{U} + \mathcal{Y} \) is some direct sum decomposition. Be start by taking a closer look at this identification.

Let \( \mathcal{W} \) be an \( H \)-space, and let \( \mathcal{W} = \mathcal{U} + \mathcal{Y} \) be a (topological) direct sum decomposition of \( \mathcal{W} \), i.e., both \( \mathcal{U} \) and \( \mathcal{Y} \) are closed subspaces of \( \mathcal{W} \), \( \mathcal{U} \cap \mathcal{Y} = \{0\} \), and \( \mathcal{U} + \mathcal{Y} = \mathcal{W} \). Thus, every \( w \in \mathcal{W} \) has a unique decomposition \( w = u + y \) where \( u \in \mathcal{U} \) and \( y \in \mathcal{Y} \). As explained above, we are thinking about the case where \( \mathcal{W} \) is the signal space of a s/s system \( \Sigma \) which we want to convert into an i/s/o system.
Thus, the operator 
\[ (2.2.7) \]
maps \( W \) one-to-one onto \( [\mathcal{U} \mathcal{Y}] \), and has the continuous inverse \n\[ (2.2.6) \]

Thus, for example, if \( R \) is an operator defined on \( [\mathcal{Y}] \) and \( w \in W \), then by \( Rw \) we mean \( R \mathcal{I} ([\mathcal{Y}], W) w \), and if \( P \) is an operator defined on \( W \) and \( [u] \in [\mathcal{Y}] \), then by \( P [u] \) we mean \( P \mathcal{I} ([u], W) [u] \). However, we do include these operators in those cases where confusion may arise (especially in the case where we at the same time consider two or more coordinate representations of the same space \( W \)).

Because of our interpretation of \( \mathcal{U} \) as the input space and \( \mathcal{Y} \) as the output space we shall refer to the above coordinate representation of \( W \) as an \textit{input/output representation}.

#### 2.2.7. Definition

Let \( W \) be an \( H \)-space.

(i) By an \textit{i/o representation} of \( W \) we mean the ordered pair \( (\mathcal{U}, \mathcal{Y}) \) of two closed subspaces \( \mathcal{U} \) and \( \mathcal{Y} \) of \( W \) such that \( W = \mathcal{U} + \mathcal{Y} \) is an ordered direct sum decomposition of \( W \). The spaces \( \mathcal{U} \) and \( \mathcal{Y} \) are called the \textit{input space} respectively \textit{output space} in this representation.

(ii) If \( (\mathcal{U}, \mathcal{Y}) \) is an i/o representation of \( W \), then the product space \( [\mathcal{U} \mathcal{Y}] \) is called a \textit{coordinate representation} of \( W \).

---

1In Chapters 10 and 11 we shall throughout interpret \( [\mathcal{Y}] \) as a Hilbert space with either the standard inner (orthogonal) product, or some other inner product induced by a signature operator in \( [\mathcal{Y}] \).
The operator $[I_u \ Y]_2$ that we encountered in (2.2.5) is a well-defined continuous operator from $[U \ Y]$ to $W$ for an arbitrary ordered pair $(U, Y)$ of closed subspaces of $W$ (i.e., even without the assumption that $W = U + Y$).

2.2.8. **Lemma.** Let $U$ and $Y$ be two closed subspaces of the $H$-space $W$, and denote the embedding operators $U \hookrightarrow W$ and $Y \hookrightarrow W$ by $[I_u \ Y]$ respectively $[I_y]$. Then $W = U + Y$ if and only if $[I_u \ Y]$ maps $[U \ Y]$ one-to-one onto $W$.

**Proof.** The easy proof of this lemma is left to the reader.

In the sequel we shall also encounter the situation where we want to determine if two closed subspaces $U_1$ and $Y_1$ of $W$ are direct complements to each other by using some known i/o representation $(U_2, Y_2)$ of $W$. In order to discuss this situation in more detail we first introduce the following notion.

2.2.9. **Definition.** Let $W$ be an $H$-space, $(U_1, Y_1)$ be an ordered pair of two closed subspaces of $W$, and let $(U_2, Y_2)$ be an i/o representation of $W$.

(i) By the **formal transition matrix** from $(U_1, Y_1)$ to $(U_2, Y_2)$ we mean the operator $\Theta \in \mathcal{B}(\{U_1; U_2\})$ defined by

$$\Theta := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} := \begin{bmatrix} P_{Y_2} & 0 \\ P_{Y_2} & 0 \end{bmatrix} \begin{bmatrix} I_{U_1} & I_{Y_1} \end{bmatrix} = \begin{bmatrix} P_{Y_2} & 0 \\ P_{Y_2} & 0 \end{bmatrix} \begin{bmatrix} I_{U_1} & I_{Y_1} \end{bmatrix}. $$

(ii) If, in addition, $(U_1, Y_1)$ is an i/o representation of $W$ (i.e., $W = U_1 + Y_1$), then we drop the word “formal” and call $\Theta$ the **transition matrix** from $(U_1, Y_1)$ to $(U_2, Y_2)$.

2.2.10. **Lemma.** Let $W$ be an $H$-space, $(U_1, Y_1)$ be an ordered pair of two closed subspaces of $W$, and let $(U_2, Y_2)$ be an i/o representation of $W$, and denote the formal transition matrix from $(U_1, Y_1)$ to $(U_2, Y_2)$ by $\Theta$ (see Definition 2.2.9).

(i) The following identity holds:

$$[I_{U_2} \ Y_2] \Theta = [I_{U_1} \ Y_1].$$

(ii) The transition matrix from $(U_1, Y_1)$ to $(U_2, Y_2)$ is $[\Theta_{11} \ 0, \Theta_{12} \ 1_{Y_2}]$, and the transition matrix from $(U_2, Y_1)$ to $(U_2, Y_2)$ is $[1_{U_2} \ 0, \Theta_{12} \ 1_{Y_2}]$.

(iii) Let $(U, Y)$ be another i/o representation of $W$, denote the formal transition matrix from $(U_1, Y_1)$ to $(U, Y)$ by $\Theta_1$, and denote the formal transition matrix from $(U_2, Y_2)$ to $(U, Y)$ by $\Theta_2$. Then $\Theta_1 = \Theta_2 \Theta$.

**Proof.** That (i) holds follows from (2.2.5) and Definition 2.2.9. Claim (ii) follows directly from Definition 2.2.9 and claim (iii) from (2.2.5) and Definition 2.2.9.

2.2.11. **Lemma.** Let $W$ be an $H$-space, $(U_1, Y_1)$ be an ordered pair of two closed subspaces of $W$, and let $(U_2, Y_2)$ be an i/o representation of $W$, and let $\Theta$ be the formal transition matrix from $(U_1, Y_1)$ to $(U_2, Y_2)$ (defined by (2.2.8)).

(i) $(U_1, Y_1)$ is an i/o representation of $W$ (i.e., $W = U_1 + Y_1$) if and only if $\Theta$ has an inverse $\Theta^{-1}$ in $\mathcal{B}(\{U_1; U_2\})$. This inverse $\Theta^{-1}$ is the transition matrix from $(U_1, Y_1)$ to $(U_2, Y_2)$, i.e.,

$$\bar{\Theta} := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} := \begin{bmatrix} P_{Y_1} \ 0 \\ P_{Y_1} \ 0 \end{bmatrix} \begin{bmatrix} I_{U_1} & I_{Y_1} \end{bmatrix} = \begin{bmatrix} P_{Y_1} \ 0 \\ P_{Y_1} \ 0 \end{bmatrix} \begin{bmatrix} I_{U_1} & I_{Y_1} \end{bmatrix}. $$
Moreover,

\[(2.2.11) \quad \left[ I_{U_2} \quad I_{Y_2} \right] \Theta \left[ P_{U_1}^{Y_1} \quad P_{Y_1}^{Y_2} \right] = I_W = \left[ I_{U_1} \quad I_{Y_1} \right] \tilde{\Theta} \left[ P_{U_1}^{Y_1} \quad P_{Y_1}^{Y_2} \right]. \]

(ii) \((U_1, Y_2)\) is an i/o representation of \(W\) (i.e., \(W = U_1 + Y_2\)) if and only if \(\Theta_{11} = P_{U_1}^{Y_1}\) has an inverse in \(\mathcal{B}(U_2; U_1)\). If \((U_1, Y_1)\) is an i/o representation of \(W\), then this is equivalent to the condition that \(\tilde{\Theta}_{22} = P_{U_2}^{Y_2}\) has an inverse in \(\mathcal{B}(Y_2; Y_1)\).

(iii) \((U_2, Y_1)\) is an i/o representation of \(W\) (i.e., \(W = U_2 + Y_1\)) if and only if \(\tilde{\Theta}_{22} = P_{U_2}^{Y_2}\) has an inverse in \(\mathcal{B}(Y_2; Y_1)\). If \((U_1, Y_1)\) is an i/o representation of \(W\), then this is equivalent to the condition that \(\Theta_{11} = P_{U_1}^{Y_1}\) has an inverse in \(\mathcal{B}(U_1; U_2)\).

**Proof.** Proof of (i): If \(\Theta\) maps \([X_1, Y_1]\) one-to-one onto \([X_2, Y_2]\), then it follows from Lemma 2.2.9 and Lemma 2.2.10 that \(W = U_1 + Y_2\), i.e., \((U_1, Y_1)\) is an i/o representation of \(\Sigma\). Conversely, suppose that \(W = U_1 + Y_2\) and denote the transition matrix from \((U_2, Y_2)\) to \((U_1, Y_1)\) by \(\tilde{\Theta}\). Then it follows from Lemma 2.2.10 with \((U, Y) = (U_1, Y_1)\) that \(\Theta\) and \(\tilde{\Theta}\) are inverses of each other. When we apply (2.2.5a) both to the representation \((U_1, Y_1)\) and to the representation \((U_2, Y_2)\), then we get

\[
\left[ I_{U_2} \quad I_{Y_2} \right] \left[ P_{Y_2}^{Y_1} \quad P_{Y_2}^{Y_2} \right] \left[ I_{U_1} \quad I_{Y_1} \right] = \left[ I_{U_2} \quad I_{Y_2} \right] \left[ P_{U_1}^{Y_1} \quad P_{U_1}^{Y_2} \right] = I_W.
\]

This proves the first half of (2.2.11). The second half is proved in the same way by interchanging \((U_1, Y_1)\) and \((U_2, Y_2)\).

**Proof of (ii):** The first claim in (ii) follows part (i) and Lemma 2.2.10. The second claim in (ii) follows in the same way from the fact that the transition matrix from \((U_1, Y_2)\) to \((U_1, Y_1)\) is \(\begin{bmatrix} 1_{U_2} & \tilde{\Theta}_{12} \\ 0 & \tilde{\Theta}_{22} \end{bmatrix}\).

**Proof of (iii):** The proof of (iii) is analogous to the proof of (ii). \(\square\)

2.2.12. **Lemma.** Let \(W\) be an \(H\)-space, and let \(U_1, U_2,\) and \(Y\) be three closed subspaces of \(W\). Then the following conditions are equivalent:

(i) Both \(U_1\) and \(U_2\) are direct complements to \(Y\), i.e., \(W = U_1 + Y = U_2 + Y\);

(ii) \(W = U_1 + Y\) and \(P_{U_2}^{Y_2}|_{U_2}\) maps \(U_2\) one-to-one onto \(U_1\) (and hence \(P_{Y_2}^{Y_1}\) has an inverse in \(\mathcal{B}(U_1; U_2)\));

(iii) \(W = U_1 + Y\) and there exists some \(D_1 \in \mathcal{B}(U_1; Y)\) such that

\[(2.2.12a) \quad \text{gph}(D_1) = \begin{bmatrix} P_{U_1}^{Y_1} \\ P_{Y_1}^{Y_2} \end{bmatrix} U_2,\]

or equivalently,

\[(2.2.12b) \quad U_2 = \begin{bmatrix} I_Y & I_{U_1} \end{bmatrix} \text{gph}(D_1).\]

(iv) \(W = U_2 + Y\) and \(P_{U_2}^{Y_2}|_{U_2}\) maps \(U_2\) one-to-one onto \(U_1\) (and hence \(P_{U_2}^{Y_2}\) has an inverse in \(\mathcal{B}(U_2; U_1)\));

(v) \(W = U_2 + Y\) and there exists some \(D_2 \in \mathcal{B}(U_2; Y)\) such that

\[(2.2.13a) \quad \text{gph}(D_2) = \begin{bmatrix} P_{U_2}^{Y_2} \\ P_{Y_2}^{Y_1} \end{bmatrix} U_1,\]
or equivalently,

\[ \mathcal{U}_1 = [I_y \quad I_{t_d}] \text{gph}(D_2). \]  

Suppose that these equivalent conditions hold, let \( \Theta \) be the transition matrix from \((\mathcal{U}_1, \mathcal{Y})\) to \((\mathcal{U}_2, \mathcal{Y})\), and let \( \tilde{\Theta} \) be the transition matrix from \((\mathcal{U}_2, \mathcal{Y})\) to \((\mathcal{U}_1, \mathcal{Y})\). Then

\[ D_1 = P_{\mathcal{Y}}^{t_1} (P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d})^{-1} = \tilde{\Theta}_{21} \Theta_{11}^{-1} = -\Theta_{21}, \]

\[ D_2 = P_{\mathcal{Y}}^{t_1} (P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d})^{-1} = \Theta_{21} \Theta_{11}^{-1} = -\tilde{\Theta}_{21}, \]

\[ \Theta = \begin{bmatrix} \Theta_{11} & 0 \\ \Theta_{21} & 1_y \end{bmatrix} = \begin{bmatrix} P_{\mathcal{U}_2}^{\mathcal{Y}} |_{\mathcal{U}_1} & 0 \\ P_{\mathcal{U}_2}^{\mathcal{Y}} |_{t_d} & 1_y \end{bmatrix} = \begin{bmatrix} \Theta_{11}^{-1} & 0 \\ -\Theta_{21} \Theta_{11}^{-1} & 1_y \end{bmatrix}, \]

\[ \tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & 0 \\ \tilde{\Theta}_{21} & 1_y \end{bmatrix} = \begin{bmatrix} P_{\mathcal{U}_1}^{\mathcal{Y}} |_{\mathcal{U}_1} & 0 \\ P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d} & 1_y \end{bmatrix} = \begin{bmatrix} \tilde{\Theta}_{11}^{-1} & 0 \\ -\Theta_{21} \Theta_{11}^{-1} & 1_y \end{bmatrix}. \]

**Proof.** We begin by observing that if \( \mathcal{W} = \mathcal{U}_2 + \mathcal{Y} \) then \( \Theta \) is well-defined (as a formal transition matrix) and it has the structure (2.2.14c), and that if \( \mathcal{W} = \mathcal{U}_1 + \mathcal{Y} \) then \( \tilde{\Theta} \) is well-defined (as a formal transition matrix) and it has the structure (2.2.14d) (at the moment we ignore the last equalities in (2.2.14c) and (2.2.14d)).

(ii) \( \iff \) (i) \( \iff \) (iv): This follows from Lemmas 2.2.10(ii) and 2.2.11(ii) with \( Y_1 = Y_2 = \mathcal{Y} \).

(ii) \( \iff \) (iii): If (ii) holds, then (iii) holds with \( D_1 = P_{\mathcal{Y}}^{t_1} (P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d}) \). Conversely, suppose that (iii) holds. Then \( P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d} \) maps \( \mathcal{U}_2 \) onto \( \mathcal{U}_1 \). To see that \( P_{\mathcal{U}_2}^{\mathcal{Y}} |_{t_d} \) is injective we let \( u_2 \in \mathcal{U}_2 \) and suppose that \( P_{\mathcal{U}_2}^{\mathcal{Y}} |_{t_d} u_2 = 0 \). Then (2.2.12a) implies that also \( P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d} u_2 = 0 \). Thus \( P_{\mathcal{U}_1}^{\mathcal{Y}} |_{t_d} \) is a bijection \( \mathcal{U}_2 \to \mathcal{U}_1 \).

(iv) \( \iff \) (v): This follows from the equivalence (ii) \( \iff \) (iii) if we interchange \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \).

We have now proved that (i)–(v) are equivalent. The remaining claims follow from the fact that according to Lemma 2.2.11(i) \( \Theta = \Theta^{-1} \). \( \Box \)

### 2.2.3. Semi-regular i/s/o representations of a semi-regular s/s node

We now turn to the problem of how to construct a semi-regular or regular i/s/o node \( \Sigma_{i/s/o} \) which “represents” the original regular s/s node in a natural way. The first step in this construction consists of a fixing some i/o decomposition \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) which satisfies condition (i) or both conditions (i) and (ii) and (ii) in Lemma 2.2.6. However, in the setting of Theorem 2.2.2 the signal space \( \mathcal{W} \) was by definition equal to the product space \( \left[ \mathcal{Y} \right] \). When we want to go in the opposite direction it is not longer quite true that \( \mathcal{W} \) is equal to \( \left[ \mathcal{Y} \right] \); instead \( \left[ \mathcal{Y} \right] \) is just a coordinate representation of \( \mathcal{W} \). To accommodate for this change it is convenient to make some minor changes in some of the earlier formulas. Formula (2.2.1) may be replaced by

\[ V := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \left[ X' \right] \begin{bmatrix} x \\ P_{t_d}^{\mathcal{Y}} w \end{bmatrix} \in \text{dom}(S) \text{ and } \begin{bmatrix} z \\ P_{t_d}^{\mathcal{Y}} w \end{bmatrix} = S \begin{bmatrix} x \\ P_{t_d}^{\mathcal{Y}} w \end{bmatrix} \right\} \]

\[ = \begin{bmatrix} 1_y & 0 & 0 \\ 0 & 1_x & 0 \\ 0 & I_y & I_{t_d} \end{bmatrix} \text{gph}(S), \]
where \( I_U \) and \( I_Y \) are the embedding operators \( I_U : U \hookrightarrow W \) and \( I_Y : Y \hookrightarrow W \), and formulas (2.2.3), and (2.2.4) may be replaced by

\[
\text{gph} (S) = \begin{bmatrix}
1_x & 0 & 0 & P^U_Y \\
0 & 0 & P^Y_U & 1_x \\
0 & 1_x & 0 & P^Y_U \\
0 & 0 & P^Y_U & 1_x \\
\end{bmatrix} V,
\]

respectively

\[
\text{dom} (S) = \begin{bmatrix}
0 & 1_x & 0 & P^Y_U \\
0 & 1_x & 0 & P^Y_U \\
\end{bmatrix} V.
\]

The last line in Lemma 2.2.5 may be replaced by “and \( S [ \hat{z} ] \) is equal to \( [ P^Y_U w ] \), where \( [ \hat{z} ] \) is the unique vector in \( V \) satisfying \( u = P^Y_U w \).” Finally, the appropriate version of condition (i) in Lemma 2.2.6 is the condition (2.2.18) in the following theorem.

2.2.13. THEOREM. Let \( \Sigma = (V; X, W) \) be a semi-regular s/s node.

(i) Let \( (U, Y) \) be an i/o representation of \( W \) which satisfies the condition

\[
V \cap \begin{bmatrix}
\hat{x} \\
0 \\
\end{bmatrix} = \{0\}.
\]

Then the right-hand side of (2.2.16) is the graph of a (unique) closed operator \( S : [ \hat{x} ] \rightarrow [ \hat{y} ] \). This operator is the system operator of a semi-regular i/s/o node \( \Sigma_{i/s/o} = (S; X, U, Y) \).

(ii) If \( \Sigma_{i/s/o} \) is regular, then \( \Sigma \) is regular as well.

(iii) \( \Sigma_{i/s/o} \) is regular (i.e., \( \text{dom} (S) \) is dense in \( [ \hat{x} ] \)) if and only if

\[
\begin{bmatrix}
0 & 1_x & 0 & P^Y_U \\
0 & 1_x & 0 & P^Y_U \\
\end{bmatrix} V \text{ is dense in } [ \hat{y} ].
\]

(iv) If \( \Sigma_{i/s/o} \) is regular, then \( \Sigma \) is regular as well.

(v) A pair \( [ \hat{z} ] \) is a classical or generalized trajectory of \( \Sigma \) on some closed interval \( I \) if and only if \( [ \hat{u} ] \) is a classical or generalized trajectory of \( \Sigma_{i/s/o} \) on \( I \), where \( u = P^Y_U w \) and \( y = P^Y_U w \).

(vi) The i/s/o system \( \Sigma_{i/s/o} \) is solvable or has the uniqueness property if and only if \( \Sigma \) is solvable or has the uniqueness property.

PROOF. Condition (2.2.18) is equivalent to the condition that \( V \) has a graph representation of the form (2.2.15) for some closed operator \( S : [ \hat{x} ] \rightarrow [ \hat{y} ] \). This operator is unique determined by its graph, and hence by \( V \), and its domain is given by (2.2.17). The remainder of the proof is analogous to the last part of the proof of Theorem 2.2.2. \( \Box \)

2.2.14. DEFINITION. Let \( \Sigma = (V; X, W) \) be a regular s/s node.

(i) An i/o representation \( (U, Y) \) of \( W \) is called semi-i/s/o-admissible for \( \Sigma \) if condition (2.2.18) holds.

(ii) An i/o representation \( (U, Y) \) of \( W \) is called i/s/o-admissible for \( \Sigma \) if conditions (2.2.18) and (2.2.19) hold.
(iii) By the semi-regular or regular i/s/o representation of $\Sigma$ corresponding to a semi-i/s/o-admissible or i/s/o-admissible i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ we mean the semi-regular or regular i/s/o node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ constructed in Theorem 2.2.13.

(iv) An i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ is called boundedly i/s/o-admissible for $\Sigma_{i/s/o}$ if the corresponding i/o node is bounded (i.e., the system operator of $\Sigma_{i/s/o}$ is bounded).

(v) We also apply the same terminology with the word “node” replaced by the word “system”.

2.2.15. Proposition. If $\Sigma = (V; \mathcal{X}, \begin{bmatrix} u_0 \\ y \end{bmatrix})$ is the semi-regular s/s node induced by the semi-regular i/s/o node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, then the i/o representation $\begin{bmatrix} [u_0] \\ [y] \end{bmatrix}$ of $\Sigma$ is semi-i/s/o-admissible for $\Sigma$, and the corresponding i/s/o representation of $\Sigma$ is $\begin{bmatrix} 1_x \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$-similar to $\Sigma_{i/s/o}$. Conversely, if $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a semi-regular i/s/o representation of the semi-regular s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$, then the semi-regular s/s node induced by $\Sigma_{i/s/o}$ is $\begin{bmatrix} 1_x \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix}$-similar to $\Sigma$.

Proof. This follows from Definitions 1.2.11, 2.2.3, 2.2.14, and 2.3.11. □

2.2.16. Theorem. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a semi-regular s/s system. Define $\mathcal{W}_0$ by (1.1.7). Then an i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ is semi-i/s/o-admissible for $\Sigma$ if and only if $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$.

Proof. Suppose first that $(\mathcal{U}, \mathcal{Y})$ is semi-i/s/o-admissible for $\Sigma$, i.e., suppose that (2.2.18) holds. Let $y \in \mathcal{Y} \cap \mathcal{W}_0$. Since $y \in \mathcal{W}_0$ there exists some $z \in \mathcal{X}$ such that $\begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathcal{V}$, and therefore (2.2.18) implies that $y = 0$. Thus $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$. Conversely, suppose that $\mathcal{Y} \cap \mathcal{W}_0 = \{0\}$. Let $\begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathcal{V}$ where $y \in \mathcal{Y}$. Then $y \in \mathcal{W}_0$, and hence $y = 0$. By assumption $\Sigma$ is semi-regular, and it therefore follows from condition (ii) in Definition 1.1.9 that also $z = 0$. Consequently (2.2.18) holds. □

2.2.17. Lemma. Let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-regular i/s/o representation of the semi-regular s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a semi-regular. Define $\mathcal{W}_0$ and $\mathcal{V}_0$ by (1.1.7) and (1.1.6), and denote the classical control and feedthrough operators of $S$ by $B$ respectively $D$. Then

$$W_0 = \begin{bmatrix} I_Y & I_U \end{bmatrix} \text{ gph } (D), \quad \text{ gph } (D) = \begin{bmatrix} P_{Y}^{U} \\ P_{U}^{Y} \end{bmatrix} W_0,$$

$$V_0 = \begin{bmatrix} 1_x \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \begin{bmatrix} 0 \\ Y \end{bmatrix} \end{bmatrix} \text{ gph } \begin{bmatrix} B \end{bmatrix}, \quad \text{ gph } \begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} 1_x \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ P_{Y}^{U} \end{bmatrix} V_0,$$

where $I_U$ and $I_Y$ are the embedding operators $U \hookrightarrow \mathcal{V}$ respectively $\mathcal{Y} \hookrightarrow \mathcal{W}$.

Proof. This follows from Definition 2.1.3, 2.2.15, 2.2.16, 1.1.7, and 1.1.6. □

2.2.18. Theorem. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a regular s/s system. Define $\mathcal{W}_0$ by (1.1.7). Let $\mathcal{U} = \overline{\mathcal{U}_0}$ and let $\mathcal{Y}$ be an arbitrary direct complement to $\mathcal{U}$ (such a complement always exists, since $\mathcal{W}$ is supposed to be an $H$-space).
(i) The i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ is i/s/o-admissible for $\Sigma$ if and only if $\{0\} \subset \text{dom}(S)$, where $S$ is the system operator of the corresponding regular i/s/o representation $\Sigma_{i/s/o} = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ of $\Sigma$.

(ii) If $\{0\} \neq \mathcal{U} \neq \mathcal{W}$, then there exists infinitely many i/s/o-admissible i/o representations of $\mathcal{W}$ for $\Sigma$.

(iii) If $\mathcal{W}_0 = \mathcal{W}$, then $\Sigma$ has exactly one regular i/s/o representation $\Sigma_{i/s/o} = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, namely the one where $\mathcal{U} = \mathcal{W}$ and $\mathcal{Y} = \{0\}$.

Proof. Proof of (i): It follows from Theorem 2.2.16 that the i/o representation $(\mathcal{U}, \mathcal{Y})$ is semi-i/s/o-admissible for $\Sigma$, so it only remains to show that dom $(S)$ is dense in $[X]_U$.

Let $\left[ \begin{array}{c} x_0 \\ u_0 \end{array} \right] \in [X]_U$, let $\mathcal{O}_{x^0}$ be a neighborhood of $x^0$ in $\mathcal{X}$, and let $\mathcal{O}_{w^0}$ be a neighborhood of $w^0$ in $\mathcal{U}$. By condition (iii) in Definition 1.1.9, there exists some $\left[ \begin{array}{c} \bar{x}_1 \\ \bar{u}_1 \end{array} \right] \in \mathcal{V}$ such that $x_1 \in \mathcal{O}_{x^0}$. Let $u_1 = P_{\mathcal{Y}_1}^{\mathcal{V}} w_1$, and define $\mathcal{O}_{w^0 - u_1} = \{ u \in \mathcal{U} \mid u + u_1 \in \mathcal{O}_{w^0} \}$. Then $\mathcal{O}_{w^0 - u_1}$ is a neighborhood of $w^0 - u_1$ in $\mathcal{U}$, and since $\mathcal{U} = \mathcal{W}_0$, there exists some $w_2 \in \mathcal{O}_{w^0 - u_1}$. By (1.1.7), this means that there exists some $\bar{x}_2 \in \mathcal{X}$ such that $\left[ \begin{array}{c} \bar{x}_2 \\ 0 \end{array} \right] \in \mathcal{V}$. Define $\left[ \begin{array}{c} x \\ w \end{array} \right] = \left[ \begin{array}{c} \bar{x}_1 \\ \bar{u}_1 \end{array} \right] + \left[ \begin{array}{c} \bar{x}_2 \\ 0 \end{array} \right]$. Then $\left[ \begin{array}{c} x \\ w \end{array} \right] \in \mathcal{V}$, $x \in \mathcal{O}_{x^0}$, and $P_{\mathcal{Y}_1}^{\mathcal{V}} w = u_1 + w_2 \in \mathcal{O}_{w^0}$. This shows that also condition (2.2.19) holds. By Theorem 2.2.13, $(\mathcal{U}, \mathcal{Y})$ is an i/s/o-admissible representation for $\Sigma$. That $\{0\} \subset \text{dom}(S)$ follows from (2.2.17) and (1.1.7).

Proof of (ii): That there exists infinitely many regular i/s/o representations for $\Sigma$ if neither $\mathcal{U} = \{0\}$ nor $\mathcal{U} = \mathcal{W}$ follows from the fact that in this case $\mathcal{U}$ has infinitely many direct complements $\mathcal{Y}$ in $\mathcal{W}$, and each i/o representation $(\mathcal{U}, \mathcal{Y})$ is i/s/o admissible for $\Sigma$.

Proof of (iii): If $\mathcal{W}_0 = \mathcal{W}$, then it follows from Theorem 2.2.16 that $\mathcal{Y} = \{0\}$, and hence $\mathcal{U} = \mathcal{W}$. $\square$

2.2.19. Remark. As we shall see in Chapters 3 and 4, for the classes of bounded s/s systems and semi-bounded s/s systems the subspace $\mathcal{W}_0$ of $\mathcal{W}$ defined in (1.1.7) an important role in the parameterization of all bounded or semi-bounded i/s/o representations. In these cases $\mathcal{W}_0$ is closed. However, as the following examples show, $\mathcal{W}_0$ need not always be closed or nontrivial, and for more general classes of s/s system the space $\mathcal{W}_0$ is not that significant.

2.2.20. Example. Let $\Sigma = (\mathcal{V}; L^1(\mathbb{R}^+), \mathbb{C})$ be either the s/s node in Example 1.4.3 or the s/s node in Example 1.4.6. These two examples are time reflections of each other, so they have the same i/s/o-admissible i/o representations of the signal space $\mathcal{W} = \mathbb{C}$. In both these examples the subspace $\mathcal{W}_0$ in (1.1.7) is $\mathcal{W}_0 = \{0\}$. However, it follows from our earlier analysis of Examples 1.4.3 and 1.4.6 that, in addition to the i/s/o-admissible i/o representation $(\{0\}, \mathbb{C})$ found in Theorem 2.2.18, also the i/o decomposition $(\mathbb{C}, \{0\})$ is i/s/o-admissible. Thus, the condition $\mathcal{W}_0 = \{0\}$ does not imply uniqueness of an i/s/o-admissible i/o decomposition. The same examples shows that it is even possible that every possible i/o decomposition of the signal space may be i/s/o-admissible. Out of the two possible i/o representations $(\{0\}, \mathbb{C})$ and the $(\mathbb{C}, \{0\})$ of the signal space $\mathbb{C}$ the former one is more natural in Example 1.4.3 and the latter is more natural in Example 1.4.6 in the sense that in...
order for the corresponding i/s/o representation to be well-posed\footnote{See Chapter 8} (in the forward time direction) we must choose the i/o representation $\{0\}, \mathbb{C}$ in Example 1.4.5 and the i/o representation $(\mathbb{C}, \{0\})$ in Example 1.4.6. With the opposite choice the corresponding i/s/o representations are backward well-posed but not forward well-posed.

Example 2.2.21 can is a special case of the following example.

2.2.21. Example. Let $\mathcal{X}$ and $\mathcal{W}$ be $H$-spaces, and let $L: \mathcal{X} \to \mathcal{X}$ and $\Gamma: \mathcal{X} \to \mathcal{W}$ be two linear operators with $\text{dom}(L) = \text{dom}(\Gamma)$, and define the subspace $V$ of $[\mathcal{X}, \mathcal{W}]$ by

$$(2.2.22) \quad V := \left\{ \begin{bmatrix} Lx \\ x \\ \Gamma x \end{bmatrix} \mid x \in \text{dom}(L) = \text{dom}(\Gamma) \right\}.$$  

In addition, suppose that $L$ and $\Gamma$ has the following properties:

(i) $\left[ \frac{L}{\Gamma} \right]: \mathcal{X} \to \left[ \frac{\mathcal{X}}{\mathcal{W}} \right]$ is closed (with $\text{dom}(\left[ \frac{L}{\Gamma} \right]) = \text{dom}(L) = \text{dom}(\Gamma)$);

(ii) $\text{dom}(L) \cap \ker(\Gamma)$ is dense in $\mathcal{X}$.

Then $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is a regular s/s node, the subspace $\mathcal{W}_0$ in (1.1.7) is given by $\mathcal{W}_0 = \{0\}$, and every i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ is i/s/o-admissible for $\Sigma$.

A s/s node of the type described in Example 2.2.21 is called a boundary control s/s node, and it appears, e.g., in the theory of conservative boundary triplets. One particular example of such a boundary control s/s node is Example 1.4.5, where $\mathcal{X} = \mathcal{L}^2(\mathbb{R}^+)$, $\mathcal{W} = \mathbb{C}$, $L\varphi = \varphi'$, $\Gamma \varphi = \varphi(0)$, and $\text{dom}(L) = \text{dom}(\Gamma) = \mathcal{W}^{1,2}(\mathbb{R}^+)$. Note that Example 1.4.5 satisfies not only conditions (i) and (ii) above, but in addition, in that example $L$ is closed and $\Gamma$ is surjective (these two conditions are often added to conditions (i) and (ii) above). See, e.g., Arov et al. \cite{2012a, 2012b} for details.

**Proof of Example 2.2.21** Since $V$ is a reordered version the graph of the closed operator $\left[ \frac{L}{\Gamma} \right]$, $V$ is closed. It follows from (ii) that $\text{dom}(L)$ is dense in $\mathcal{X}$. This implies that $\Sigma$ is a regular s/s node. Clearly $\mathcal{W}_0 = \{0\}$ in this case.

Define the operator $S: \left[ \frac{\mathcal{X}}{\mathcal{U}} \right] \to \left[ \frac{\mathcal{X}}{\mathcal{Y}} \right]$ by

$$\text{dom}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \left[ \frac{\mathcal{X}}{\mathcal{U}} \right] \mid x \in \text{dom}(L) = \text{dom}(\Gamma) \quad \text{and} \quad P \mathcal{U} x = u \right\},$$

$$S \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} L \\ P \mathcal{U} \Gamma \end{bmatrix} x, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom}(S).$$

Then $V$ has the representation (2.2.15). It follows from (ii) that $\text{dom}(S)$ is dense in $\left[ \frac{\mathcal{X}}{\mathcal{U}} \right]$ (this domain contains $\text{dom}(L) \cap \ker(\Gamma)$). Since $\left[ \frac{L}{\Gamma} \right]$ is closed, also $V$ is closed, and it then follows from (2.2.15) that $S$ is closed (the graph of $S$ coincides with $V$ after a permutation of the components). Thus, $\Sigma_{i/s/o} = (S; \mathcal{U}, \mathcal{U}, \mathcal{Y})$ is a regular i/s/o representation of $\Sigma$, i.e., the i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ is i/s/o-admissible for $\Sigma$. □
2.2.22. Example. Let $X$ and $W$ be $H$-spaces, let $B: W \to X$ be a closed unbounded operator with dense domain, and let $\Sigma = (V; X, X')$ be the s/s node with generating subspace

\[
V := \left\{ \begin{bmatrix} Bw \\ x \\ w \end{bmatrix} \in \begin{bmatrix} X \\ \operatorname{dom}(B) \end{bmatrix} \right. \right|\left. \begin{bmatrix} x \\ w \end{bmatrix} \in X \cap \operatorname{dom}(B) \right. \right\}.
\]

Let $Y$ be an arbitrary closed subspace of $W$ which satisfies $Y \cap \operatorname{dom}(B) = \{0\}$, and let $U$ be an arbitrary direct complement to $Y$ in $W$. Then the i/o decomposition $(U, Y)$ is i/s/o-admissible for $\Sigma$. In particular, the i/o decomposition $(W, \{0\})$ is i/s/o-admissible for $\Sigma$ (take $Y = 0$ above). In this example the subspace $W_0$ in (1.1.7) is given by $W_0 = \operatorname{dom}(B)$, and hence $W_0$ is dense in $W$. This system is two-sided uniquely solvable.

Proof. Since $B$ is closed, also $V$ is closed. It follows from (1.1.7) that $W_0 = \operatorname{dom}(B)$. (By Theorem 2.2.18, the i/o decomposition $(W, \{0\})$ is i/s/o-admissible for $\Sigma$, which takes care of the special case where $Y = \{0\}$).

Let $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$, i.e., $x \in X$, $w \in \operatorname{dom}(B)$, and $z = Bw$. Define $u = P^W_Y w$ and $y = P^W_Y w$. If $u = 0$, then $y = w$, and hence $y \in Y \cap \operatorname{dom}(B) = \{0\}$. This implies that $V$ has a graph representation of the type (2.2.15), where $S$ is the closed unbounded operator which for each $\begin{bmatrix} x \\ w \end{bmatrix} \in V$ maps $\begin{bmatrix} x \\ P^W_Y w \end{bmatrix}$ into $\begin{bmatrix} Bw \\ P^W_Y w \end{bmatrix}$ (that this operator is closed follows from (2.2.15) and the fact that $V$ is closed). Clearly $\operatorname{dom}(S) = \begin{bmatrix} x \\ P^W_Y \operatorname{dom}(B) \end{bmatrix}$, which is dense in $\begin{bmatrix} X \\ Y \end{bmatrix}$. Thus, the i/o representation $(U, Y)$ is i/s/o-admissible for $\Sigma$.

It is easy to see that this example is uniquely solvable, since the classical two-sided trajectories $\begin{bmatrix} x \\ w \end{bmatrix}$ of $\Sigma$ are of the form

\[
x(t) = x(0) + \int_0^t w(s) \, ds, \quad t \in \mathbb{R}.
\]

2.2.23. Remark. By Theorem 2.2.18, every regular s/s system $\Sigma$ has a regular i/s/o representation. However, the explicit construction in Theorem 2.2.18 of such an i/s/o representation is not very interesting from a physical point of view, due to the fact that it does not guarantee that these specific i/s/o representations have any additional properties apart from regularity. Later we shall impose additional conditions on $\Sigma$ which makes it possible to obtain i/s/o representations with more interesting properties (which do not necessarily include the notion of regularity). See, in particular Theorem 2.2.29 below which applies if $\Sigma$ is bounded, the discussion in Chapter 5 of i/s/o representations with nonempty resolvent sets, the discussion in Chapter 9 of well-posed i/s/o representations, and the discussion of the various i/s/o representations of passive s/s systems in Chapter 11. See also and Examples 2.2.20, 2.2.21 and 2.2.22 above. Out of these three examples the first two are significant from a physical point of view, but the last one is primarily of academic interest, since, as we shall see in Example 5.3.5 below, the resolvent set of this s/s system is empty.

2.2.5. Parametrization of regular i/s/o representations. We next turn the question to what extent it is possible to parameterize all regular i/s/o representations of a s/s node $\Sigma$ with the help of one fixed regular i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$. We begin with the following lemma.
2.2.24. Lemma. Let \( \Sigma_{i/s/o} = (S_1; X, U_1, Y_1) \) be a regular i/s/o representation of a regular s/s node \( \Sigma = (V; X, W) \). Let \( \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \) is the transition matrix from \( (U_1, Y_1) \) to \( (U_2, Y_2) \), and let \( \tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} \) be the transition matrix from \( (U_2, Y_2) \) to \( (U_1, Y_1) \) (see (2.2.8) and (2.2.10)).

(i) The two \( 4 \times 4 \) block matrices in the formula below are invertible, and the are the inverses of each other:

\[
\begin{bmatrix}
1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11}
\end{bmatrix}^{-1} = \begin{bmatrix}
1_X & 0 & 0 & 0 \\
0 & \tilde{\Theta}_{22} & 0 & \tilde{\Theta}_{21} \\
0 & 0 & 1_X & 0 \\
0 & \tilde{\Theta}_{12} & 0 & \tilde{\Theta}_{11}
\end{bmatrix}.
\]

(ii) Suppose that \( (U_2, Y_2) \) is i/s/o-admissible for \( \Sigma \), and denote the corresponding i/s/o representation of \( \Sigma \) by \( \Sigma_{i/s/o} = (S_2; X, U_2, Y_2) \). Then

(a) the graphs of \( S_2 \) is related to the graph os \( S_1 \) as follows:

\[
\text{gph} (S_2) = \begin{bmatrix}
1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11}
\end{bmatrix} \text{gph} (S_1).
\]

(b) the operator

\[
M := \begin{bmatrix}
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11} 
\end{bmatrix} \begin{bmatrix}
S_1 \\
1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11}
\end{bmatrix} \bigg|_{\text{dom}(S_1)}
\]

maps \( \text{dom}(S_1) \) one-to-one onto \( \text{dom}(S_2) \), and its inverse \( \tilde{M} := M^{-1} \) is given by

\[
\text{dom}(S_2)
\]

\[
M := \begin{bmatrix}
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11} 
\end{bmatrix} \begin{bmatrix}
S_2 \\
1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11}
\end{bmatrix} \bigg|_{\text{dom}(S_2)}
\]

(iii) Conversely, if the right-hand side of (2.2.25) is the graph of a single-valued operator \( S_2 \) with dense domain, then \( (U_2, Y_2) \) is i/s/o-admissible for \( \Sigma \), and \( S_2 \) is the generating operator of the corresponding regular i/s/o representation of \( \Sigma \).

Proof. Proof of (i): That (2.2.24) holds follows from the fact that \( \Theta \) and \( \tilde{\Theta} \) are inverses of each other (see Lemma 2.2.11).
Proof of (ii)(a): Suppose that (2.2.25) holds if \((U_2, Y_2)\) is i/s/o-admissible follows from (2.2.15) and (2.2.16) which give
\[
gph(S_2) = \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{U_2}^{Y_2} \\ 0 & 1_X & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & 1_X \\ 0 & I_{Y_1} & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{U_2}^{Y_2} \\ 0 & 0 & 0 \end{bmatrix} gph(S_1)
\]
\[
= \begin{bmatrix} 1_X & 0 & 0 & 0 \\ 0 & P_{Y_2}^{U_2} & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 1_X & 0 \\ 0 & P_{U_2}^{Y_2} & 0 & P_{U_2}^{Y_2} \end{bmatrix} gph(S_1)
\]
\[
= \begin{bmatrix} 1_X & 0 & 0 & 0 \\ 0 & \Theta_{22} & 0 & \Theta_{21} \\ 0 & 0 & 1_X & 0 \\ 0 & \Theta_{12} & 0 & \Theta_{11} \end{bmatrix} gph(S_1).
\]

Proof of (ii)(b): Define \(\text{dom}(F)\) by (1.1.4). It follows from (2.2.8) that the operator \(M\) can be factored into
\[
M = \begin{bmatrix} 1_X & 0 \\ 0 & P_{U_2}^{Y_2} \end{bmatrix} \begin{bmatrix} 0 & 1_X \\ I_{Y_1} & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 \\ 0 & I_{U_1} \end{bmatrix} |_{\text{dom}(S_1)} = \begin{bmatrix} 0 & 1_X \\ 0 & 0 \end{bmatrix} P_{U_2}^{Y_2}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P_{U_2}^{Y_2}^{-1}.
\]
All the three operators above are invertible: The last operator maps \(\text{dom}(S_1)\) one-to-one onto \(\text{gph}(S_1)\), by (2.2.15) the next operator maps \(\text{gph}(S_1)\) one-to-one onto \(\text{dom}(F)\), and by (2.2.16) the final operator maps \(\text{dom}(F)\) one-to-one onto \(\text{dom}(S_2)\). Moreover, for \(i = 1, 2\) we have
\[
\left( \begin{bmatrix} 0 & 1_X \\ I_{Y_1} & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 \\ 0 & I_{U_1} \end{bmatrix} |_{\text{dom}(S_1)} \right)^{-1} = \begin{bmatrix} 0 & 1_X \\ 0 & 0 \end{bmatrix} P_{U_2}^{Y_2}^{-1}.
\]
From this follows that \(M\) maps \(\text{dom}(S_1)\) one-to-one onto \(\text{dom}(S_2)\), and that \(M^{-1} = M\).

Proof of (iii): If (2.2.25) holds, then the same computation that we carried out above to prove (ii)(a) combined with (2.2.15) gives
\[
gph(S_2) = \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{U_2}^{Y_2} \\ 0 & 1_X & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & 1_X \\ 0 & I_{Y_1} & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 0 \end{bmatrix} gph(S_1)
\]
\[
= \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{U_2}^{Y_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & 1_X \\ 0 & I_{Y_1} & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & 1_X \\ 0 & I_{U_1} & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{U_2}^{Y_2} \\ 0 & 0 & 0 \end{bmatrix} gph(S_1)
\]
\[
= \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{U_2}^{Y_2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_{Y_2}^{U_2} \\ 0 & 0 & 0 \end{bmatrix} gph(S_1).
\]
By Definition (2.2.14) this means that \(\Sigma_{i/s/o}^{2} = (S_2; X, U_2, Y_2)\) is a regular i/s/o representation of \(\Sigma\). □

The following theorem gives a parameterization of all regular i/s/o representations of a s/s node \(\Sigma\) with the help of one fixed regular i/s/o representation \(\Sigma_{i/s/o}^{2}\) of \(\Sigma\).
2.2.25. **Theorem.** Let \( \Sigma_{i/s/o} = (S_1; X, U_1, Y_1) \) be a semi-regular i/o representation of a semi-regular s/s node \( \Sigma = (V; X, W) \). Let \((U_2, \mathcal{V}_2)\) be an i/o representation of \( \mathcal{W} \), let \( \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \) be the transition matrix from \((U_1, \mathcal{V}_1)\) to \((U_2, \mathcal{V}_2)\) (see (2.2.8)), and define the operators \( K: [X_{u_1}] \to [X_{u_2}] \) and \( M: [X_{u_1}] \to [X_{u_2}] \) by

\[
K := \begin{bmatrix} 0 & 0 \\ 0 & \Theta_{21} \end{bmatrix} + \begin{bmatrix} 1X \\ 0 \end{bmatrix} \Theta_{11} S_1, \quad \text{dom}(K) = \text{dom}(S_1),
\]

\[
M := \begin{bmatrix} 1X & 0 \\ 0 & \Theta_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Theta_{12} \end{bmatrix} S_1, \quad \text{dom}(M) = \text{dom}(S_1).
\]

(i) The i/o representation \((U_2, \mathcal{V}_2)\) is semi-i/s/o-admissible for \( \Sigma \) if and only if \( M \) is injective.

(ii) The i/o representation \((U_2, \mathcal{V}_2)\) is i/s/o-admissible for \( \Sigma \) if and only if \( M \) is injective and \( \text{rng}(M) \) is dense in \([X_{u_2}]\).

(iii) The i/o representation \((U_2, \mathcal{V}_2)\) of \( \mathcal{W} \) is boundedly i/s/o-admissible for \( \Sigma \) if and only if \( M \) maps \( \text{dom}(S_1) \) one-to-one onto \([X_{u_2}]\).

(iv) Suppose that the i/o representation \((U_2, \mathcal{V}_2)\) of \( \mathcal{W} \) is semi-i/s/o-admissible for \( \Sigma \), denote the corresponding i/s/o representation of \( \Sigma \) by \( \Sigma_{i/s/o} = (S_2; X, U_2, Y_2) \), let \( \bar{\Theta} = \Theta^{-1} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \) be the transition matrix from \((U_2, \mathcal{V}_2)\) to \((U_2, \mathcal{V}_2)\) (see (2.2.10)), and define \( \bar{K} \) and \( \bar{M} \) by

\[
\bar{K} := \begin{bmatrix} 0 & 0 \\ 0 & \bar{\Theta}_{21} \end{bmatrix} + \begin{bmatrix} 1X \\ 0 \end{bmatrix} \bar{\Theta}_{11} S_2, \quad \text{dom}(K) = \text{dom}(S_2),
\]

\[
\bar{M} := \begin{bmatrix} 1X & 0 \\ 0 & \bar{\Theta}_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{\Theta}_{12} \end{bmatrix} S_2, \quad \text{dom}(M) = \text{dom}(S_2).
\]

Then \( M \) maps \( \text{dom}(S_1) \) one to one onto \( \text{dom}(S_2) \), \( M^{-1} = \bar{M} \), and

\[
S_2 = KM^{-1} = \bar{K}M, \quad S_1 = \bar{K}M^{-1} = \bar{K}M.
\]

**Proof.** Proofs of (i) and (ii): That (i) and (ii) hold follows from Lemma (2.2.24).

Proof of (iii): Claim (iii) follows from (i) and the closed graph theorem (\( S_2 \) is bounded if and only if \( \text{dom}(S_2) = [X_{u_2}] \)).

Proof of (iv): That \( M \) maps \( \text{dom}(S_1) \) one-to-one onto \( \text{dom}(S_2) \) and that \( M^{-1} = \bar{M} \) follows from Lemma (2.2.24). Since \( M \) and \( \bar{M} \) are invertible it follows from (2.2.25) that \( S_2 = KM^{-1} \) and \( S_1 = \bar{K}M^{-1} \).

2.2.26. **I/s/o representations of bounded s/s systems.** Next we shall look at the question of the existence of bounded i/s/o representations of a s/s node. We begin with the following preliminary lemma.

2.2.26. **Lemma.** Let \( \Sigma = (V; X, W) \) be a s/s node, let \((U, \mathcal{V})\) be an i/o representation of \( W \). Then the following conditions are equivalent:

(i) \((U, \mathcal{V})\) is boundedly i/s/o-admissible for \( \Sigma \);
(ii) V has the following three equivalent representations:

\[ V = \left[ \begin{array}{cccc} 1_X & 0 & 0 & 0 \\ 0 & 0 & 1_X & 0 \\ 0 & I_Y & 0 & I_U \end{array} \right] \text{gph} \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right), \]

\[ V = \text{rng} \left( \begin{bmatrix} A & B \\ 1_X & 0 \\ I_Y & 0 \end{bmatrix} \right), \]

\[ V = \ker \left( \begin{bmatrix} -1_X & A & BP_Y^U \\ 0 & C & DP_Y^U - P_Y^U \end{bmatrix} \right) \]

for some block matrix operator \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in B(\mathbb{X};\mathbb{Y}) \). Here the block operator matrix in (2.2.33b) maps \( \mathbb{X} \) into \( \mathbb{R} \), and the block operator matrix in (2.2.33c) maps \( \mathbb{R} \) into \( \mathbb{Y} \).

If these equivalent conditions hold, then the system operator of the bounded i/o representation of \( \Sigma \) corresponding to the i/o representation \( (U, Y) \) of \( W \) is the operator \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) in (ii). Moreover, \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is determined uniquely by \( V \) and the i/o representation \( (U, Y) \) of \( W \), and \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) can be recovered from \( V \) as follows:

\[ \text{gph} \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{bmatrix} 1_X & 0 & 0 \\ 0 & 0 & P_Y^U \\ 0 & 1_X & 0 \\ 0 & 0 & P_Y^U \end{bmatrix} V. \]

**Proof.** This follows from Definition 2.2.14, (2.2.15), and (2.2.16). \( \square \)

See Theorem 2.2.29 below for a list of additional conditions which are equivalent to conditions (i) and (ii) in Lemma 2.2.26.

**2.2.27. Theorem.** Let \( \Sigma = (V; \mathbb{X}, \mathbb{W}) \) be a bounded s/s node with node space \( \mathbb{R} \), and define \( W_0 \) and \( V_0 \) as in (1.1.7) and (1.1.6). Then the following conditions are equivalent:

(i) \( \Sigma \) is a bounded (and hence regular);
(ii) conditions (i)–(iii) in Definition 1.1.15 hold and \( W_0 \) is closed in \( \mathbb{W} \);
(iii) \( \Sigma \) has a bounded i/s/o representation.

If these equivalent conditions hold, then

(iv) For every direct complement \( \mathbb{Y} \) to \( W_0 \) the i/o representation \( (W_0, \mathbb{Y}) \) is boundedly i/s/o-admissible.
(v) If \( \Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} ; \mathbb{X}, \mathbb{U}, \mathbb{Y} \right) \) is an arbitrary bounded i/s/o representation of \( \Sigma \), then (2.2.20) and (2.2.21) hold.

**Proof.** (i) \( \Rightarrow \) (ii): By condition (iv) in Definition 1.1.15 the set \( \begin{bmatrix} 0 & 1_X & 0 \\ 0 & 0 & 1_W \end{bmatrix} \mathcal{V} \) is closed in \( \begin{bmatrix} \mathbb{X} \\ \mathcal{W} \end{bmatrix} \), and consequently also

\[ \begin{bmatrix} \{0\} \\ W_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_W \end{bmatrix} \left( \mathcal{V} \cap \begin{bmatrix} \mathbb{X} \\ \{0\} \end{bmatrix} \right) = \begin{bmatrix} 0 & 1_X & 0 \\ 0 & 0 & 1_W \end{bmatrix} \mathcal{V} \cap \begin{bmatrix} \mathbb{X} \\ \{0\} \end{bmatrix} \]

is closed. Thus \( W_0 \) is closed.
(ii) ⇒ (iv) ⇒ (iii): Suppose that (ii) holds, and let \( \mathcal{Y} \) be an arbitrary complement to \( W_0 \) in \( W \). By Theorem 2.2.18 the i/o representation \( (W_0, \mathcal{Y}) \) is i/s/o-admissible for \( \Sigma \) and \( [0,0,0,0] \) \( \in \text{dom}(S) \), where \( \Sigma_{i/s/o} = (\mathcal{X}, W_0, \mathcal{Y}) \) is the corresponding i/s/o representation. The operator \( S \) is closed since \( V \) is closed, and \( \text{dom}(S) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \nu_{W_0} \end{bmatrix} \right\} V \). By condition (iii) in Definition 1.1.15 the projection of \( \text{dom}(S) \) onto its first component is equal to \( \mathcal{X} \). Thus, for every \( x^0 \in \mathcal{X} \) there exists some \( u^0 \in W_0 \) such that \( [\begin{bmatrix} x^0 \\ u^0 \end{bmatrix}] \in \text{dom}(S) \). Since also \( [\begin{bmatrix} 0 \\ u^0 \end{bmatrix}] \in \text{dom}(S) \), the difference \( [\begin{bmatrix} x^0 \\ u^0 \end{bmatrix}] - [\begin{bmatrix} 0 \\ u^0 \end{bmatrix}] \in \text{dom}(S) \). Thus, \( \text{dom}(S) = \left\{ \begin{bmatrix} x^0 \\ 0 \end{bmatrix} \right\} \). Since \( S \) is closed, this implies that \( S \in \mathcal{B}(\left\{ \begin{bmatrix} x^0 \\ 0 \end{bmatrix} \right\} ; \left\{ \begin{bmatrix} x^0 \\ 0 \end{bmatrix} \right\}) \). Thus (ii) ⇒ (iv). Trivially (iv) ⇒ (iii).

(iii) ⇒ (i): Let \( \Sigma = (\begin{bmatrix} A & D \\ C & D \end{bmatrix} : \mathcal{X}, U, \mathcal{Y}) \) be a bounded i/s/o representation of \( \Sigma \). Then \( V \) is closed since \( [A \ B] \) is bounded and hence closed (this is condition (i) in Definition 1.1.15). Condition (ii) in Definition 1.1.15 follows from (2.2.33c), and condition (iii) in Definition 1.1.15 follows from (2.2.33b). It follows from (2.2.33a) that \( [\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}] V = \ker(\Sigma + DP_{U|W_0}^Y - P_{U|W_0}^Y) \), and hence this set is closed. Thus conditions (i)–(iv) in Definition 1.1.15 are satisfied, and hence \( V \) is bounded.

We have now proved that (i)–(iv) are equivalent. That these conditions imply (v) follow from Lemma 2.2.17.

2.2.28. COROLLARY. Every bounded s/s system is two-sided uniquely solvable and has the two-sided continuation property.

PROOF. This follows from Lemma 2.1.18 and Theorems 2.2.2, 2.2.13, and 2.2.27.

2.2.29. THEOREM. Let \( \Sigma = (V; \mathcal{X}, W) \) be a bounded s/s node, let \( (U, \mathcal{Y}) \) be an i/o representation of \( W \), and define \( W_0 \) as in (1.1.7). Then \( W_0 \) is closed, and the following conditions are equivalent:

(i) the i/o representation \( (U, \mathcal{Y}) \) of \( W \) is boundedly i/s/o-admissible for \( \Sigma \);
(ii) \( \mathcal{Y} \) is a direct complement to \( W_0 \) (and \( U \) is a direct complement to \( \mathcal{Y} \));
(iii) \( P_{U|W_0}^\mathcal{Y} \) maps \( W_0 \) one-to-one onto \( U \) (and hence it has an inverse in \( \mathcal{B}(U; W_0) \));
(iv) \( \text{rng} \left( \begin{bmatrix} P_{U|W_0}^\mathcal{Y} \\ P_{U|W_0}^\mathcal{Y} \end{bmatrix} \right) \) is the graph of an operator \( D \in \mathcal{B}(U, \mathcal{Y}) \).

Suppose that the equivalent conditions (i)–(iv) above hold. The following additional claims are true:

(v) The operator \( D \) in (iv) is equal to the feedthrough operator \( D \) of the bounded i/s/o representation \( \Sigma_{i/s/o} = (\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{X}, U, \mathcal{Y}) \) of \( \Sigma \) corresponding to the i/o representation \( (U, \mathcal{Y}) \) of \( W \).

(vi) Let \( \Sigma_{i/s/o} = (\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} : \mathcal{X}, W_0, \mathcal{Y}) \) be the bounded i/s/o representation corresponding to the i/o representation \( (W_0, \mathcal{Y}) \) of \( \Sigma \). Then \( D_0 = 0 \) and the system operator of the i/s/o representation \( \Sigma_{i/s/o} \) in (v) is given by

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_0 & B_0(P_{U|W_0}^\mathcal{Y})^{-1} \\ C_0 & P_{U|W_0}^\mathcal{Y}(P_{U|W_0}^\mathcal{Y})^{-1} \end{bmatrix}.
\]

In particular, \( A \) and \( C \) do not depend on the choice of \( U \), and \( D = P_{U|W_0}^\mathcal{Y}(P_{U|W_0}^\mathcal{Y})^{-1} \) depends only on \( W_0 \) (which depends on \( V \)), \( U \), and \( \mathcal{Y} \). Moreover, \( D = 0 \) if and only if \( U = W_0 \).
Proof. That \( \mathcal{W}_0 \) is closed follows from Theorem 2.2.29 (ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv): This follows from Lemma 2.2.12 (i) \( \Rightarrow \) (i): Let \( \mathcal{Y} \) be a direct complement to \( \mathcal{W}_0 \), and let \( \mathcal{U} \) be a direct complement to \( \mathcal{Y} \). By Theorem 2.2.27 the i/o representation \((\mathcal{W}_0, \mathcal{Y})\) is a boundedly i/s/o admissible i/o representation for \( \Sigma \). Denote the corresponding i/s/o representation of \( \Sigma \) by \( \Sigma_{i/s/o}^0 = ([A_0 B_0; C_0 D_0] : \mathcal{X}, \mathcal{W}_0, \mathcal{Y}) \). Let \( \Theta \) be the transition matrix from \((\mathcal{W}_0, \mathcal{Y})\) to \((\mathcal{U}, \mathcal{Y})\) (see Definition 2.2.9). Then

\[
\Theta = \begin{bmatrix}
\Theta_{11} & 0 \\
\Theta_{21} & 1_{\mathcal{Y}}
\end{bmatrix} = \begin{bmatrix}
P_0^\mathcal{Y}|_{\mathcal{W}_0} & 0 \\
P_1^\mathcal{Y}|_{\mathcal{W}_0} & 0
\end{bmatrix},
\]

and by Lemma 2.2.12 \( \Theta_{11} \) maps \( \mathcal{W}_0 \) one-to-one onto \( \mathcal{U} \). Define \( K \) and \( M \) by (2.2.28) and (2.2.29) with \( S_1 \) replaced by \([C_0 D_0] \). Then

\[
K = \begin{bmatrix}
0 & 0 \\
0 & \Theta_{21}
\end{bmatrix} + \begin{bmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{bmatrix} = \begin{bmatrix}
A_0 & B_0 \\
C_0 & \Theta_{21} + D_0
\end{bmatrix}, \quad M = \begin{bmatrix}
1_{\mathcal{X}} & 0 \\
0 & \Theta_{11}
\end{bmatrix},
\]

and hence \( M \) maps \([X_{\mathcal{W}_0}] \) one-to-one onto \([\mathcal{Y}] \). By Theorem 2.2.27 the decomposition \((\mathcal{U}, \mathcal{Y})\) is boundedly i/s/o admissible for \( \Sigma \), and the system operator \([C \ D]\) of the corresponding i/s/o representation \( \Sigma_{i/s/o} = ([C \ D] : \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is given by

(2.2.36)

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
A_0 & B_0\Theta_{11}^{-1} \\
C_0 & (\Theta_{21} + D_0)\Theta_{11}^{-1}
\end{bmatrix}.
\]

This agrees with (2.2.35), apart from the fact that we have not yet proved that \( D_0 = 0 \).

(i) \( \Rightarrow \) (iv): This follows from Theorem 2.2.27 (v).

We have now proved that (i)–(iv) are equivalent. That (v) holds follows from Theorem 2.2.27 (v). It follows from (iv) and (v) that \( D = P_0^\mathcal{Y}(P_1^\mathcal{Y}|_{\mathcal{W}_0})^{-1} \). In particular, taking \( \mathcal{U} = \mathcal{W}_0 \) we get \( D_0 = 0 \). Since \( D_0 = 0 \) (2.2.35) follows from (2.2.36). Finally, \( D = 0 \) if an only if \( P_0^\mathcal{Y} = 0 \). This is true if and only if \( \mathcal{W}_0 \subset \mathcal{U} \). But this inclusion cannot be strict since \( \mathcal{W}_0 + \mathcal{Y} = \mathcal{U} + \mathcal{Y} = \mathcal{W} \). Thus \( D = 0 \) if and only if \( \mathcal{U} = \mathcal{W}_0 \).

Motivated by Theorem 2.2.27 we introduce the following terminology:

2.2.30. Definition. By the canonical input space of a bounded i/s/o system we mean the (closed) subspace \( \mathcal{W}_0 \) defined in (1.1.7).

2.2.7. Parametrization of bounded i/s/o representations. Our next theorem gives a parameterization of all bounded i/s/o representations of a bounded s/s node \( \Sigma \) with the help of one fixed bounded i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \).

2.2.31. Theorem. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a bounded s/s node with the bounded i/s/o representation \( \Sigma_{i/s/o} = ([A_1 \ B_1; C_1 \ D_1] : \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1) \). Let \( (\mathcal{U}_2, \mathcal{Y}_2) \) be an i/o representation of \( \mathcal{W} \), and let \( \Theta \) and \( \tilde{\Theta} \) be the transition matrices from \((\mathcal{U}_1, \mathcal{Y}_1)\) to \((\mathcal{U}_2, \mathcal{Y}_2)\) respectively from \((\mathcal{U}_2, \mathcal{Y}_2)\) to \((\mathcal{U}_1, \mathcal{Y}_1)\) (these are given by (2.2.8) and (2.2.10)).

(i) The following conditions are equivalent:

(a) the i/o representation \((\mathcal{U}_2, \mathcal{Y}_2)\) of \( \mathcal{W} \) is boundedly i/s/o/-admissible for \( \Sigma \);

(b) \( \mathcal{Y}_2 \) is a direct complement to \( \mathcal{W}_0 \), where \( \mathcal{W}_0 \) is the canonical input space defined in (1.1.7).
(c) the operator \( \Theta_{11} + \Theta_{12} D_1 \) maps \( \mathcal{U}_1 \) one-to-one onto \( \mathcal{U}_2 \) (and hence it has an inverse in \( \mathcal{B}(\mathcal{U}_2; \mathcal{U}_1) \));

(d) the operator \( \Theta_{22} - D_1 \Theta_{12} \) maps \( \mathcal{V}_2 \) one-to-one onto \( \mathcal{V}_1 \) (and hence it has an inverse in \( \mathcal{B}(\mathcal{V}_2; \mathcal{V}_1) \)).

(ii) Suppose that the equivalent conditions in (i) hold, and denote the i/s/o representation of \( \Sigma \) corresponding to the i/o decomposition \((\mathcal{U}_i, \mathcal{Y}_i)\) by \( \Sigma_{i/s/o} = ([A_i, B_i] : \mathcal{X}, \mathcal{U}_i, \mathcal{Y}_i), i = 1, 2. \) Then

(a) The operators listed below are boundedly invertible, with the following inverses:

\[
(\Theta_{11} + \Theta_{12} D_1)^{-1} = \tilde{\Theta}_{11} + \tilde{\Theta}_{12} D_2, \\
(\Theta_{22} - D_1 \Theta_{12})^{-1} = \Theta_{22} - D_2 \Theta_{12},
\]

\[
\begin{bmatrix}
\Theta_{12} C_1 \\
\Theta_{11} + \Theta_{12} D_1
\end{bmatrix}^{-1} = \begin{bmatrix}
1X \\
\Theta_{12} C_2 - \Theta_{11} D_2
\end{bmatrix},
\]

\[
\begin{bmatrix}
1X \\
-B_1 \tilde{\Theta}_{12}
\end{bmatrix}^{-1} = \begin{bmatrix}
1X \\
0
\end{bmatrix} - B_2 \Theta_{12}.
\]

(b) The system operator \([A_2, B_2]_{C_2, D_2}\) of \( \Sigma_{i/s/o} \) can be obtained from the system operator \([A_1, B_1]_{C_1, D_1}\) of \( \Sigma_{i/s/o} \) in the following way:

\[
\begin{bmatrix}
A_2 \\
B_2
\end{bmatrix} = \begin{bmatrix}
A_1 \\
B_1
\end{bmatrix} \begin{bmatrix}
\Theta_{22} C_1 + \Theta_{22} D_1 \\
\Theta_{12} C_1 + \Theta_{11} D_1
\end{bmatrix}^{-1},
\]

or equivalently,

\[
A_2 = A_1 - B_1 (\Theta_{11} + \Theta_{12} D_1)^{-1} \Theta_{12} C_1,
\]

\[
B_2 = B_1 (\Theta_{11} + \Theta_{12} D_1)^{-1}.
\]

(c) The system operator \([A_2, B_2]_{C_2, D_2}\) of \( \Sigma_{i/s/o} \) can also be obtained from the system operator \([A_1, B_1]_{C_1, D_1}\) of \( \Sigma_{i/s/o} \) in the following alternative way:

\[
\begin{bmatrix}
A_2 \\
B_2
\end{bmatrix} = \begin{bmatrix}
1X \\
0
\end{bmatrix} - B_1 \tilde{\Theta}_{12} \begin{bmatrix}
A_1 \\
B_1 \tilde{\Theta}_{11}
\end{bmatrix},
\]

or equivalently,

\[
A_2 = A_1 + B_1 \tilde{\Theta}_{12} (\Theta_{22} - D_1 \tilde{\Theta}_{12})^{-1} C_1,
\]

\[
B_2 = B_1 \tilde{\Theta}_{11} + B_1 \tilde{\Theta}_{12} (\Theta_{22} - D_1 \tilde{\Theta}_{12})^{-1} (-\tilde{\Theta}_{21} + D_1 \tilde{\Theta}_{11}),
\]

\[
C_2 = (\tilde{\Theta}_{22} - D_1 \tilde{\Theta}_{12})^{-1} C_1,
\]

\[
D_2 = (\tilde{\Theta}_{22} - D_1 \tilde{\Theta}_{12})^{-1} (-\tilde{\Theta}_{21} + D_1 \tilde{\Theta}_{11}).
\]

**Proof.** (i)(a) \( \Leftrightarrow \) (i)(b): This equivalence follows from Theorem 2.2.29

(i)(b) \( \Leftrightarrow \) (i)(c): By 2.2.29 the \( \mathcal{Y}_1 \) is a direct complement to \( \mathcal{W}_0 \), and \( \mathcal{Y}_1 \) is a boundedly i/s/o-admissible decomposition for \( \Sigma \). By Lemma 2.2.12 and Theorem 2.2.29(v) the transition matrix \( \Theta_1 \) from \( (\mathcal{W}_0, \mathcal{Y}_1) \) to \( (\mathcal{U}_1, \mathcal{Y}_1) \) is given by

\[
\Theta_1 = \begin{bmatrix}
P_{\mathcal{Y}_1 | \mathcal{W}_0}^1 & 0 \\
P_{\mathcal{Y}_1 | \mathcal{W}_0}^1 & 1_{\mathcal{Y}_1}
\end{bmatrix} = \begin{bmatrix}
P_{\mathcal{Y}_1 | \mathcal{W}_0}^1 & 0 \\
P_{\mathcal{Y}_1 | \mathcal{W}_0}^1 & 1_{\mathcal{Y}_1}
\end{bmatrix}.
\]


where $P_{U_1}^{Y_1}|_{W_0}$ maps $W_0$ one-to-one onto $U_1$. Therefore by Lemma \[2.2.10\] iii) the transition matrix $\Theta_2$ from $(W_0, Y_1)$ to $(U_2, Y_2)$ is given by

$$
\Theta_2 = \Theta \Theta_1 = \begin{bmatrix}
(\Theta_{11} + \Theta_{12} D_1) P_{U_1}^{Y_1}|_{W_0} & \Theta_{12}

(\Theta_{21} + \Theta_{22} D_1) P_{U_1}^{Y_1}|_{W_0} & \Theta_{22}
\end{bmatrix},
$$

and by Lemma \[2.2.10\] ii) the formal transition matrix $\Theta_3$ from $(W_0, Y_2)$ to $(U_2, Y_2)$ is given by

$$
\Theta_3 = \begin{bmatrix}
(\Theta_{11} + \Theta_{12} D_1) P_{U_1}^{Y_1}|_{W_0} & 0

(\Theta_{21} + \Theta_{22} D_1) P_{U_1}^{Y_1}|_{W_0} & 1_{Y_2}
\end{bmatrix}.
$$

By Lemma \[2.2.12\] $W_0$ is a direct complement to $Y_2$ if and only if $(\Theta_{11} + \Theta_{12} D_1) P_{U_1}^{Y_1}|_{W_0}$ maps $W_0$ one-to-one onto $U_2$, or equivalently, $\Theta_{11} + \Theta_{12} D_1$ maps $U_1$ one-to-one onto $U_2$. Thus (i)(b) $\Leftrightarrow$ (i)(c).

(i)(b) $\Leftrightarrow$ (i)(d): The transition matrix $\tilde{\Theta}_1$ from $(U_1, Y_1)$ to $(W_0, Y_1)$ is given by

$$
\tilde{\Theta}_1 = \Theta_1^{-1} = \begin{bmatrix}
(P_{U_1}^{Y_1}|_{W_0})^{-1} & 0

-D_1 & 1_{Y_1}
\end{bmatrix},
$$

and hence the transition matrix $\tilde{\Theta}_2$ from $(U_2, Y_2)$ to $(W_0, Y_1)$ is given by

$$
\tilde{\Theta}_2 = \tilde{\Theta}_1 \Theta = \begin{bmatrix}
(P_{U_1}^{Y_1}|_{W_0})^{-1} \tilde{\Theta}_{11} & (P_{U_1}^{Y_1}|_{W_0})^{-1} \tilde{\Theta}_{12}

\Theta_{21} - D_1 \tilde{\Theta}_{11} & \Theta_{22} - D_1 \tilde{\Theta}_{12}
\end{bmatrix}.
$$

By Lemma \[2.2.10\] ii) the formal transition matrix $\tilde{\Theta}_3$ from $(W_0, Y_2)$ to $(W_0, Y_1)$ is given by

$$
(2.2.42)
\tilde{\Theta}_3 = \begin{bmatrix} 1_{W_0} & (P_{U_1}^{Y_1}|_{W_0})^{-1} \tilde{\Theta}_{12} \\
0 & \Theta_{22} - D_1 \tilde{\Theta}_{12} \end{bmatrix}.
$$

By Lemma \[2.2.12\] $W_0$ is a direct complement to $Y_2$ if and only if $\tilde{\Theta}_{22} - D_1 \tilde{\Theta}_{12}$ maps $Y_2$ one-to-one onto $Y_1$. Thus (i)(b) $\Leftrightarrow$ (i)(d).

We have now proved (i), and continue with the proof of (ii).

Proof of (ii)(b), \[2.2.37a\], \[2.2.37b\], and \[2.2.37c\]: That (ii)(b) and \[2.2.37c\] hold follows from Theorem \[2.2.25\] and clearly \[2.2.37c\] implies \[2.2.37a\].

Proofs of (ii)(c), \[2.2.37b\], and \[2.2.37d\]: By using a kernel characterization of $gph\left(\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\right)$ we get from \[2.2.25\]

$$
gph\left(\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -1_X & 0 & A_1 & B_1 \\
0 & -1_{Y_1} & C_1 & D_1 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11} \end{bmatrix}^{-1}\right)
= \ker\left(\begin{bmatrix} -1_X & 0 & A_1 & B_1 \\
0 & -1_{Y_1} & C_1 & D_1 \end{bmatrix} \begin{bmatrix} 1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11} \end{bmatrix}^{-1}\right)
= \ker\left(\begin{bmatrix} -1_X & 0 & B_1 \tilde{\Theta}_{12} - A_1 D_1 \Theta_{11} \\
0 & -\tilde{\Theta}_{22} + D_1 \tilde{\Theta}_{12} \end{bmatrix} \begin{bmatrix} A_1 & B_1 \tilde{\Theta}_{11} \\
C_1 & \tilde{\Theta}_{21} \end{bmatrix}^{-1}\right).
$$

The kernel of the last $2 \times 4$ block matrix operator does not change if it is multiplied by the operator $\begin{bmatrix} 1_X & 0 & -B_1 \tilde{\Theta}_{12} \\
0 & \Theta_{22} & 0 & \Theta_{21} \end{bmatrix}^{-1}$ to the left. This converts the first two columns of that block matrix into $\begin{bmatrix} -1_X & 0 & 0 \\
0 & -1_{Y_2} \end{bmatrix}$, and the resulting kernel description of $gph\left(\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}\right)$ gives us \[2.2.40\].
Therefore

\[
\begin{bmatrix}
1_X & -B_1 \Theta_{12} \\
0 & \Theta_{22} - D_1 \Theta_{12}
\end{bmatrix}
\begin{bmatrix}
-B_2 \Theta_{12} \\
\Theta_{22} - D_2 \Theta_{12}
\end{bmatrix}
= \begin{bmatrix}
1_X & -B_1 \Theta_{12} \\
0 & \Theta_{22} - D_1 \Theta_{12}
\end{bmatrix}
\begin{bmatrix}
1_X & -B_1 \Theta_{12} \\
0 & \Theta_{22} - D_1 \Theta_{12}
\end{bmatrix}
\begin{bmatrix}
B_2 \\
D_2
\end{bmatrix}
\Theta_{12}
= \begin{bmatrix}
-\Theta_{22} - D_1 \Theta_{12} \\
\Theta_{22} - D_1 \Theta_{12}
\end{bmatrix}
\begin{bmatrix}
B_2 \\
D_2
\end{bmatrix}
\Theta_{12}
= \begin{bmatrix}
1_X \\
0
\end{bmatrix}
(\Theta_{12} \Theta_{22} + \Theta_{11} \Theta_{12}) = \begin{bmatrix}
0 \\
1_Y
\end{bmatrix},
\]

where we in the last equality use the facts that \(\Theta_{22} \Theta_{22} + \Theta_{11} \Theta_{12} = 1_{1_Y}\) and \(\Theta_{12} \Theta_{22} + \Theta_{11} \Theta_{12} = 0\) since \(\Theta\) and \(\Theta\) are inverses of each other. This implies that

\[
\begin{bmatrix}
1_X & -B_1 \Theta_{12} \\
0 & \Theta_{22} - D_1 \Theta_{12}
\end{bmatrix}
\begin{bmatrix}
1_X & -B_1 \Theta_{12} \\
0 & \Theta_{22} - D_2 \Theta_{12}
\end{bmatrix}
= \begin{bmatrix}
1_X & 0 \\
0 & 1_Y
\end{bmatrix}.
\]

Thus \([1_X - B_1 \Theta_{12}]_{\Theta_{22} - D_1 \Theta_{12}}\) is a right inverse of \([1_X - B_1 \Theta_{12}]_{\Theta_{22} - D_1 \Theta_{12}}\). Since we know that \(\Theta_{22} - D_1 \Theta_{12}\) is invertible also the operator \([1_X - B_1 \Theta_{12}]_{\Theta_{22} - D_1 \Theta_{12}}\) is invertible, and therefore this right inverse is also a left inverse. This implies (2.2.37b) and (2.2.37d). □

It follows from Theorem 2.2.29 that if both \((U_1, Y_1)\) and \((U_2, Y_2)\) are boundedly i/s/o-admissible i/o representations of \(W\) for \(\Sigma\), then also \((W_0, Y_1)\) and \((W_0, Y_2)\) are boundedly i/s/o-admissible i/o representations for \(\Sigma\) where \(W_0\) is defined by (1.17). It is therefore possible to pass from the i/o representation \((U_1, Y_1)\) of \(W\) to the i/o representation \((U_2, Y_2)\) in three steps: first we replace \((U_1, Y_1)\) by \((W_0, Y_1)\), then replace \((W_0, Y_1)\) by \((W_0, Y_2)\), and finally replace \((W_0, Y_2)\) by \((U_2, Y_2)\). In the first and last of these transformations the input space changes but the output space remains the same, and in the middle transformation the output space changes, but the input space remains the same. These two different transformations turn out to be of a quite different nature. The first and last transformations (where the output space stays the same) can be interpreted as similarity transformations with an extra modification of a feedthrough term, but the second transformation (where the input space stays the same) is more complicated: it can be interpreted as an output feedback connection combined with a similarity transform. See Examples 2.3.15 and 2.3.26 for details.
is of the type \( \Theta = \begin{bmatrix} \bar{\Theta}_{11} & 0 \\ \bar{\Theta}_{21} & 1_y \end{bmatrix} \), and

\[
\begin{bmatrix}
    A_2 & B_2 \\
    C_2 & D_2
\end{bmatrix} =
\begin{bmatrix}
    A_1 & B_1 \Theta_{11}^{-1} \\
    C_1 & (\Theta_{21} + D_1) \Theta_{11}^{-1}
\end{bmatrix}
\]

(2.2.43)

\[
= \begin{bmatrix}
    A_1 & -B_1 \bar{\Theta}_{11} \\
    C_1 & (-\Theta_{21} + D_1) \Theta_{11}^{-1}
\end{bmatrix}.
\]

In particular, \( \Sigma_1 \) and \( \Sigma_2 \) have the same main and observation operators.

(ii) If \( \mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U} \), then the transition matrix \( \Theta \) from \((\mathcal{U}, \mathcal{Y}_1)\) to \((\mathcal{U}, \mathcal{Y}_2)\) is of the type \( \Theta = \begin{bmatrix} 1_u & \Theta_{12} \\ 0 & \Theta_{22} \end{bmatrix} \), the transition matrix \( \bar{\Theta} \) from \((\mathcal{U}, \mathcal{Y}_1)\) to \((\mathcal{U}, \mathcal{Y}_1)\) is of the type \( \bar{\Theta} = \begin{bmatrix} 1_u & 0 \\ 0 & \Theta_{22} \end{bmatrix} \), and

\[
\begin{bmatrix}
    A_2 & B_2 \\
    C_2 & D_2
\end{bmatrix} =
\begin{bmatrix}
    A_1 & B_1 \\
    \Theta_{22} C_1 & \Theta_{22} D_1
\end{bmatrix}
\begin{bmatrix}
    1_x \\
    \Theta_{12} C_1 & 1_u + \Theta_{12} D_1
\end{bmatrix}^{-1}
\]

(2.2.44)

\[
= \begin{bmatrix}
    1_x & -B_1 \bar{\Theta}_{12} \\
    0 & \bar{\Theta}_{22} - D_1 \bar{\Theta}_{12}
\end{bmatrix}^{-1}
\begin{bmatrix}
    A_1 & B_1 \\
    C_1 & D_1
\end{bmatrix}.
\]

2.2.8. I/s/o representations of general s/s nodes (Jan 02, 2016).

Above we have discussed regular i/s/o representations of regular s/s nodes. We next look at the case where the s/s node and the i/s/o representation are closed, but not necessarily semi-regular.

The main difficulty in our earlier proofs of the existence of i/s/o representations of a s/s system was the need to assure that the candidate for the system operator \( S \) of an i/s/o representation was single-valued, and in the case of a regular representation that \( S \) has a dense domain. If we permit the system operator to be multi-valued with a non-dense domain, then these difficulties disappear, and the theory of i/s/o representations becomes very simple.

2.2.33. THEOREM. Let \( \Sigma_{i/s,o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o node.

(i) Let \( \mathcal{W} \) be the product space \( \mathcal{W} = [\begin{bmatrix} \mathcal{Y} \end{bmatrix}] \), and define \( V \) by

\[
V := \left\{ \begin{bmatrix} z \\ x \\ [\begin{bmatrix} y \end{bmatrix}] \end{bmatrix} \subset \mathcal{X} \mid \begin{bmatrix} x \\ [\begin{bmatrix} y \end{bmatrix}] \end{bmatrix} \in \text{dom} \,(S) \text{ and } \begin{bmatrix} z \\ [\begin{bmatrix} y \end{bmatrix}] \end{bmatrix} \in S \begin{bmatrix} x \\ [\begin{bmatrix} y \end{bmatrix}] \end{bmatrix} \right\}
\]

(2.2.45)

\[
= \begin{bmatrix}
    1_x & 0 & 0 \\
    0 & 0 & 1_x \\
    0 & 0 & 0
\end{bmatrix} \text{gph}(S),
\]

where the embedding operators \( \mathcal{U} \to \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \subset \mathcal{W} \) and \( \mathcal{Y} \to \begin{bmatrix} \mathcal{Y} \\ 0 \end{bmatrix} \subset \mathcal{W} \) have been denoted by \( \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \) respectively \( \begin{bmatrix} \mathcal{Y} \\ 0 \end{bmatrix} \). Then \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) is a s/s node. If \( \Sigma_{i/s,o} \) is closed, or semi-regular, or regular, or bounded, then \( \Sigma \) is closed, or regular, or bounded, respectively.

(ii) A triple \( \begin{bmatrix} x \\ [\begin{bmatrix} y \end{bmatrix}] \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma_{i/s,o} \) on some interval \( I \) if and only if \( \begin{bmatrix} z \\ [\begin{bmatrix} y \end{bmatrix}] \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma \) on \( I \).

(iii) The s/s system \( \Sigma \) is solvable or has the uniqueness property if and only if \( \Sigma_{i/s,o} \) is solvable or has the uniqueness property.
PROOF. The proof of this theorem is contained in the proof of Theorem 2.2.2 (simply omit the proofs of the claims that \( V \) has properties (i)–(iii) in Definition 1.1.9). \( \square \)

To go in the opposite direction we need to make a small adjustment to (2.2.15) (i.e., we replace an equality sign by an inclusion) and write

\[
\begin{align*}
V := \bigg\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix} & \bigg| \begin{bmatrix} x \\ P_U y \end{bmatrix} \in \text{dom} (S) \text{ and } \begin{bmatrix} z \\ P_U y \end{bmatrix} \in S \begin{bmatrix} x \\ P_U y \end{bmatrix} \bigg\} \\
= \begin{bmatrix} 1 \alpha & 0 & 0 & 0 \\ 0 & 0 & 1 \alpha & 0 \\ 0 & I_{XY} & 0 & I_{UX} \end{bmatrix} \text{gph}(S),
\end{align*}
\]

but formulas (2.2.16) and (2.2.17) still remain valid.

2.2.34. THEOREM. Let \( \Sigma = (V; X, W) \) be a s/s node.

(i) Let \((U, Y)\) be an arbitrary i/o representation of \( W \), and let \( S \in \mathcal{ML}([X]; [Y]) \) be the operator whose graph is given by (2.2.16). Then \( S \) is the system operator of a i/s/o node \( \Sigma_{i/s/o} = (S; X, U, Y) \). If \( \Sigma \) is closed, then \( \Sigma_{i/s/o} \) is closed, and if \( \Sigma \) is semi-regular or regular and the i/o decomposition \((U, Y)\) is semi-i/s/o-admissible or i/s/o-admissible for \( \Sigma \), then \( \Sigma_{i/s/o} \) is semi-regular respectively regular.

(ii) A pair \([\frac{z}{x}]\) is a classical or generalized trajectory of \( \Sigma \) on some closed interval \( I \) if and only if \([\frac{x}{u}]\) is a classical or generalized trajectory of \( \Sigma_{i/s/o} \) on \( I \), where \( u = P_Y w \) and \( y = P_Y w \).

(iii) The i/s/o system \( \Sigma_{i/s/o} \) is solvable or has the uniqueness property if and only if \( \Sigma \) is solvable or has the uniqueness property.

PROOF. The proof of this theorem is a simplified version of the proof of Theorem 2.2.13. \( \square \)

2.2.35. DEFINITION.

(i) If \( \Sigma_{i/s/o} = (S; X, U, Y) \) is an i/s/o node, then the s/s node \( \Sigma = (V; X, W) \) constructed in Theorem 2.2.33 is called the s/s node induced by \( \Sigma_{i/s/o} \).

(ii) If \( \Sigma = (V; X, W) \) is a s/s node, then by the i/s/o representation of \( \Sigma \) corresponding to the i/o representation \((U, Y)\) of \( W \) we mean the i/s/o node \( \Sigma_{i/s/o} = (S; X, U, Y) \) constructed in Theorem 2.2.34.

(iii) We also use the same terminology with the word “node” replaced by the word “system”.

2.2.36. LEMMA. Let \( \Sigma = (V; X, W) \) be a s/s node, and let \( \Sigma_{i/s/o} = (S; X, U, Y) \) be an i/s/o representation of \( \Sigma \).

(i) \( \Sigma \) is closed if and only if \( \Sigma_{i/s/o} \) is closed.

(ii) \( \Sigma \) satisfies condition (ii) in Definition 1.1.9 if and only if \( \text{mul} (S) \cap [\begin{bmatrix} X \\ \hat{0} \end{bmatrix}] = \{0\} \).

(iii) \( \Sigma \) satisfies condition (iii) in Definition 1.1.9 if and only if \([1 \alpha \ 0] \text{dom} (S) \) is dense in \( \mathcal{X} \).

(iv) \( \Sigma \) is regular if and only if \( \text{mul} (S) \cap [\begin{bmatrix} X \\ \hat{0} \end{bmatrix}] = \{0\} \) and \([1 \alpha \ 0] \text{dom} (S) \) is dense in \( \mathcal{X} \).
In particular, if $\Sigma$ has a regular i/s/o representation, then $\Sigma$ is regular.

**Proof.** This follows from Definition 1.1.9 and the representation (2.2.46) of $V$. □

2.2.37. **Remark.** The converse to the above claim is not true: If $\Sigma = (V; X, W)$ is a regular s/s node, then to every i/o representation $(U, Y)$ of $W$ there corresponds some i/o representation $\Sigma_{i/o}$ of $\Sigma$. In most cases not all of these representations will be regular (except when $W = \{0\}$). This is, in particular, true for finite-dimensional systems. To see this, suppose that $\Sigma = (V; X, W)$ is a finite-dimensional regular s/s system with the property that every i/o representation of $\Sigma$ is i/o-admissible. Then, in particular, both $(0, W)$ and $(W, 0)$ are i/o-admissible i/o representations for $\Sigma$. The i/o-admissibility of the representation $(0, W)$ implies that $\text{dim } V = \text{dim } X$, and the i/o-admissibility of the representation $(W, 0)$ implies that $\text{dim } V = \text{dim } X + \text{dim } W$. This is possible if and only if $W = \{0\}$. Thus, every finite-dimensional s/s system with a nonzero signal space $W$ has at least one non-regular i/o representation. (The infinite-dimensional case is different, as we saw in Example 2.2.21.)

2.2.38. **Lemma.** Let $\Sigma = (V; X, W)$ be a s/s node, and let $\Sigma_{i/o} = (S; X, U, Y)$ be an i/s/o representation of $\Sigma$. Then the characteristic node bundle $\hat{E}$ of $\Sigma$ is given by

\[
\hat{E}(\lambda) = \begin{bmatrix}
-1_X & 0 & \lambda & 0 \\
0 & 0 & 1_X & 0 \\
0 & I_Y & 0 & I_U \\
0 & 0 & 1_X & 0 \\
0 & I_Y & 0 & I_U \\
\end{bmatrix} \text{gph} (S) \\
= \begin{bmatrix}
-1_X & 0 & 0 & 0 \\
0 & 0 & 1_X & 0 \\
0 & I_Y & 0 & I_U \\
\end{bmatrix} \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right), \quad \lambda \in \mathbb{C}.
\]

**Proof.** This follows from Lemma 1.6.2 and the representation (2.2.46) of $V$. □

2.2.39. **Lemma.** Let $\Sigma_i = (V; X, U, Y)$ and $\Sigma_{i/o}^1 = (S_1; X, U, Y_1)$ be two i/o representations of a s/s node $\Sigma = (V; X, W)$, and define $\Theta$ by (2.2.8). Then

\[
\text{gph} (S_1) = \begin{bmatrix}
1_X & 0 & 0 & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
0 & 0 & 1_X & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11} \\
\end{bmatrix} \text{gph} (S).
\]

In particular, $\text{dom} (S_1) = \text{rng}(M)$ and $\text{rng} (S_1) = \text{rng} (K)$ where $K \in \mathcal{ML}([X; U]; [X])$ and $M \in \mathcal{ML}([X; U]; [X])$ operators defined in (2.2.28) and (2.2.29).
Proof. By (2.2.16) and (2.2.46),
\[
\text{gph} \left( S_1 \right) = \begin{bmatrix}
1_X & 0 & 0 \\
0 & 0 & P_{\bar{Y}_{\bar{U}_1}}^I \\
0 & 1_X & 0 \\
0 & 0 & P_{\bar{Y}_{\bar{U}_1}}^I \\
\end{bmatrix} V = \begin{bmatrix}
1_X & 0 & 0 \\
0 & 0 & P_{\bar{Y}_{\bar{U}_1}}^I \\
0 & 1_X & 0 \\
0 & 0 & P_{\bar{Y}_{\bar{U}_1}}^I \\
\end{bmatrix} \begin{bmatrix}
1_X & 0 & 0 & 0 \\
0 & 0 & 1_X & 0 \\
0 & 0 & 1_X & 0 \\
0 & 0 & 1_X & 0 \\
\end{bmatrix} \text{gph}(S)
\]
\[
= \begin{bmatrix}
1_X & 0 & 0 & 0 \\
0 & 0 & 1_X & 0 \\
0 & 0 & 1_X & 0 \\
0 & 0 & 1_X & 0 \\
\end{bmatrix} \text{gph}(S).
\]
This together with (2.2.48) gives (2.2.48). □

2.2.40. Remark. It is clear from Theorem 2.2.34 the notion of an i/s/o representation of a s/s node is “too general” to be useful without any further additional conditions. The same comment applies to the notions of a (regular or non-regular) s/s or i/s/o node or system. In the next chapters we shall impose some additional conditions on s/s and i/s/o nodes and systems which lead to a more meaningful theory.
2.3. Transforms of I/S/O Nodes (Jan 02, 2016)

In this section we take a closer look at a number of transforms that can be applied to i/s/o nodes. Most of these are i/s/o counter part of the transforms of s/s systems discussed in Section 1.2, but some of them make specific use of the i/s/o structure of the systems, namely the parallel, the difference, the cascade connection, and two types of output feedback connections.

2.3.1. Time reflection of an i/s/o node. The idea behind the definition of the “time reflection” of a i/s/o system is the same as in the case of a s/s system, i.e., to reverse the direction of time in equation 2.1.1, i.e., to replace $x(t)$ by $x(-t)$ and $w(t)$ by $w(-t)$.

2.3.1. Definition (cf. Definition 1.2.1). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node or system. By the time reflection $\Sigma_\mathcal{R}$ of $\Sigma$ we mean the i/s/o node or system $\Sigma_\mathcal{R} = (V_\mathcal{R}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with system operator

\[
S_\mathcal{R} = \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix} S.
\]

2.3.2. Lemma (cf. Lemma 1.2.2). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node.

(i) The time reflection $\Sigma_\mathcal{R}$ of $\Sigma$ is closed, or regular, or semi-bounded, or bounded if and only if $\Sigma$ is closed, or regular, or bounded, respectively.

(ii) $\Sigma = (\Sigma_\mathcal{R})_\mathcal{R}$, i.e., $\Sigma$ is equal to the time reflection of the time reflection of itself.

Proof. This follows directly from the relevant definitions.

2.3.3. Lemma (cf. Lemma 1.2.4). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o system, and let $\Sigma_\mathcal{R} = (S_\mathcal{R}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be the time reflection of $\Sigma$. Then $\left[ \begin{array}{c} x \\ u \\ y \\ \end{array} \right]$ is a classical or generalized trajectory of $\Sigma$ on the interval $I$ if and only if $\left[ \begin{array}{c} x_\mathcal{R} \\ u_\mathcal{R} \\ y_\mathcal{R} \end{array} \right]$ is a classical or generalized trajectory of $\Sigma_\mathcal{R}$ on $I_\mathcal{R} = \{ -t | t \in I \}$.

Proof. The proof is analogous to the proof of Lemma 1.2.2.

2.3.4. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node or system and let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of $\Sigma$. Then the time reflection $\Sigma_{i/s/o}^R$ of $\Sigma_{i/s/o}$ is the i/s/o representation of the time reflection $\Sigma^R$ of $\Sigma$ corresponding to the decomposition $W = U + \mathcal{Y}$ of $\mathcal{W}$.

Proof. This follows from the relevant definitions.

2.3.2. Time rescaling of an i/s/o node. Time rescaling works in the same way in the i/s/o setting as in the s/s setting.

2.3.5. Definition (cf. Definition 1.2.5). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node or system and let $\gamma > 0$. By the time $\gamma$-rescaling of $\Sigma$ we mean the i/s/o node or system $\Sigma_{\gamma} = (S_{\gamma}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with system operator

\[
S_{\gamma} := \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} S.
\]

2.3.6. Lemma (cf. Lemma 1.2.6). Let $\Sigma$ be an i/s/o, and let $\gamma > 0$. 

Proof. This follows from the relevant definitions.
(i) The time $\gamma$-rescaling $\Sigma$ is closed, or semi-regular, or regular, or bounded if and only if $\Sigma$ is closed, or semi-regular, or regular, or bounded, respectively.

(ii) The time $\gamma_1$-rescaling of the time $\gamma_2$-rescaling of $\Sigma$ is equal to the time $\gamma_1\gamma_2$-rescaling of $\Sigma$, where $\gamma = \gamma_1\gamma_2$.

(iii) Time reflection and time $\gamma$-rescaling commute, i.e., the time $\gamma$-rescaling of the time reflection of $\Sigma$ is equal to the time reflection of the time $\gamma$-rescaling of $\Sigma$.

**Proof.** This follows directly from the relevant definitions. \(\square\)

2.3.7. **Lemma** (cf. Lemma 1.2.7). Let $\Sigma = (S; X, U, Y)$ be an i/s/o system, let $\gamma > 0$, and let $\Sigma_{\gamma} = (S_{\gamma}; X, U, Y)$ be the time $\gamma$-rescaling of $\Sigma$. Then $\begin{bmatrix} x_{\gamma}(t) \\ u_{\gamma}(t) \\ y_{\gamma}(t) \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma$ on the interval $I$ if and only if $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma_{\gamma}$ on the interval $I_{\gamma}$, where $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_{\gamma}(t) \\ u_{\gamma}(t) \\ y_{\gamma}(t) \end{bmatrix}$, $t \in I_{\gamma} := \{ \gamma t \mid t \in I \}$.

**Proof.** The proof is analogous to the proof of Lemma 1.2.4. \(\square\)

2.3.3. **Exponentially weighted i/s/o nodes.** The exponential weighting of an i/s/o node is the i/s/o counterpart of the exponential weighting of an s/o node.

2.3.8. **Definition** (cf. Lemma 1.2.8). Let $\Sigma = (S; X, U, Y)$ be an i/s/o node or system, and let $\alpha \in \mathbb{C}$. By the exponential weighting by $\alpha$ of $\Sigma$ we mean the i/s/o node or system $\Sigma_{\alpha} = (S_{\alpha}; X, U, Y)$ with generating subspace

\[
S_{\alpha} := S + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}.
\]

2.3.9. **Lemma** (cf. Lemma 1.2.9). Let $\Sigma$ be an i/s/o node.

(i) The exponential $\alpha$-weighting of $\Sigma$ is closed, or semi-regular, or regular, or bounded if and only if $\Sigma$ is closed, or semi-regular, or regular, or bounded, respectively.

(ii) If $\Sigma_{\alpha}$ is the exponential $\alpha$-weighting of $\Sigma$, then the exponential $\beta$-weighting of $\Sigma_{\alpha}$ is the exponential $(\alpha + \beta)$-weighting of $\Sigma$.

(iii) $\Sigma_{\alpha}$ is the exponential $\alpha$-weighting of $\Sigma$ if and only if $\Sigma$ is the exponential $(-\alpha)$-weighting of $\Sigma$.

(iv) The exponential $\alpha$-weighting of the time reflection of $\Sigma$ is the time reflection of the $(-\alpha)$-weighting of $\Sigma$.

(v) The exponential $\alpha$-weighting of the time $\gamma$-scaling of $\Sigma$ is the time $\gamma$-scaling of $\Sigma$ the exponential $\alpha/\gamma$-weighting of $\Sigma$.

**Proof.** This follows directly from the relevant definitions. \(\square\)

2.3.10. **Lemma** (cf. Lemma 1.2.10). Let $\Sigma = (S; X, U, Y)$ be an i/s/o system, let $\alpha \in \mathbb{C}$, and let $\Sigma_{\alpha} = (S_{\alpha}; X, U, Y)$ be the exponential $\alpha$-weighting of $\Sigma$. Then $\begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma$ on some time interval $I$ if and only if $\begin{bmatrix} x_{\alpha}(t) \\ u_{\alpha}(t) \\ y_{\alpha}(t) \end{bmatrix}$ defined by

\[
\begin{bmatrix} x_{\alpha}(t) \\ u_{\alpha}(t) \\ y_{\alpha}(t) \end{bmatrix} := e^{\alpha t} \begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix}, \quad t \in I,
\]
is a classical or generalized trajectory of $\Sigma_\alpha$ on $I$.

**Proof.** The proof is analogous to the proof of Lemma 1.2.4. □

**2.3.4. Similarity of i/s/o nodes.** The notion of similarity of two i/s/o nodes is also analogous to the notion of similarity of two s/s nodes.

**2.3.11. Definition** (cf. Definition 1.2.11). Let $\Sigma = (S; X, U, Y)$ and $\Sigma_1 = (S_1; X_1, U_1, Y_1)$ be two i/s/o nodes or systems.

(i) We say that $\Sigma_1$ is $(P, Q, R)$-similar to $\Sigma$ if $P \in B(X; X_1)$, and $Q \in B(U; Y_1)$, and $R \in B(Y; Y_1)$ are bicontinuous bijections and

\[
S_1 = \begin{bmatrix} P & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Q^{-1} \end{bmatrix} S \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}.
\]

In this case we also call $\Sigma_1$ the $(P, Q, R)$-similarity transformation of $\Sigma$. The operators $P$, $Q$, and $R$ in (i) are called the state similarity operator, the input similarity operator, and the output similarity operator, respectively.

(ii) $\Sigma$ and $\Sigma_1$ are similar if $\Sigma_1$ is $(P, Q, R)$-similar to $\Sigma$ for some $P$, $Q$, and $R$.

(iii) We say that $\Sigma_1$ is $P$-similar to $\Sigma$ if $U_1 = U$ and $Y_1 = Y$ and $\Sigma_1$ is $(P, 1_U, 1_Y)$-similar to $\Sigma$. In this case we also call $\Sigma_1$ the $P$-similarity transformation of $\Sigma$.

Note that (2.3.4) is equivalent to the condition

\[
gph(S_1) = \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q \end{bmatrix} gph(S).
\]

Also note that if $\Sigma = ([A \ B]; X, U, Y)$ is a bounded i/s/o node, then $\Sigma_1 = ([A_1 \ B_1]; X_1, U_1, Y_1)$ is bounded and

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} P & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}.
\]

**2.3.12. Lemma** (cf. Lemma 1.2.12). Let $\Sigma = (S; X, U, W)$ and $\Sigma_1 = (S_1; X_1, U_1, Y)$ be two i/s/o nodes or systems.

(i) If $\Sigma$ and $\Sigma_1$ are similar, then $\Sigma_1$ is closed, or semi-regular, or regular, or bounded if and only if $\Sigma$ is closed, or semi-regular, or regular, or bounded, respectively.

(ii) $\Sigma_1$ is $(P, Q, R)$-similar to $\Sigma$ if and only if $\Sigma$ is $(P^{-1}, Q^{-1}, R^{-1})$-similar to $\Sigma_1$. Thus, similarity of two i/s/o nodes or systems is an equivalence relation.

(iii) The time reflection of $\Sigma_1$ is $(P, Q, R)$-similar to the time reflection of $\Sigma$.

(iv) If $\Sigma_1$ is $(P, Q, R)$-similar to $\Sigma$, then for each $\alpha \in \mathbb{C}$ the exponential $\alpha$-weighting of the time reflection of $\Sigma_1$ is $(P, Q, R)$-similar to the exponential $\alpha$-weighting of of $\Sigma$.

**Proof.** This follows immediately from the relevant definitions. □
2.3.13. LEMMA (cf. Lemma 1.2.13). Let $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1)$ be an i/s/o system which is $(P, Q, R)$-similar to $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. Then $P_\Sigma = \begin{bmatrix} x & u \\ y & \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma$ on the interval $I$ if and only if $P_\Sigma = \begin{bmatrix} x & u \\ y & \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma_1$ on $I$.

PROOF. This follows immediately from the definitions of these notions. □

2.3.14. LEMMA.

(i) If the i/s/o node or system $\Sigma_{i/s/o}^1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1)$ is $(P, Q, R)$-similar to the i/s/o node or system $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, then the s/s node or system $\Sigma_1 = (V_1; \mathcal{X}_1, \begin{bmatrix} U_1 \\ Y_1 \end{bmatrix})$ induced by $\Sigma_{i/s/o}^1$ is $(P, \begin{bmatrix} Q \\ R \end{bmatrix})$-similar to the s/s node or system $\Sigma = (V; \mathcal{X}, \begin{bmatrix} U \\ Y \end{bmatrix})$ induced by $\Sigma_{i/s/o}$ (see Definition 2.2.35).

(ii) Suppose that the s/s node or system $\Sigma^1 = (V_1; \mathcal{X}_1, \mathcal{W}_1)$ is $(P, Q)$-similar to the node or system $\Sigma = (V; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, and let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of $\Sigma$. Define $Q_1 = Q_{|U}, R_1 = Q_{|Y}, U_1 = \text{rng} (Q_1),$ and $Y_1 = \text{rng} (R_1).$ Then $(U_1, Y_1)$ is an i/o representation of $\mathcal{W}_1$, and the corresponding i/s/o representation $\Sigma_{i/s/o}^1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1)$ of $\Sigma_1$ is $(P_1, Q_1, R_1)$-similar to $\Sigma_{i/s/o}$.

PROOF. (i) This follows directly from Definitions 1.2.11, 2.3.11, and 2.2.35.

(ii) It is easy to see that $\mathcal{W}_1 = \mathcal{U}_1 + \mathcal{Y}_1$ (since $Q$ is a continuous bijection), and $Q$ has the block matrix decomposition \( \begin{bmatrix} Q_1 & 0 \\ 0 & R_1 \end{bmatrix} \) with respect to the decompositions $\mathcal{W} = \mathcal{U} + \mathcal{Y}$ of $\mathcal{W}$ and $\mathcal{W}_1 = \mathcal{U}_1 + \mathcal{Y}_1$ of $\mathcal{W}_1$. □

2.3.15. EXAMPLE. The system operator $\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ in (2.2.43) can be obtained from $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ by first adding a feedthrough term $\Theta_{21}$ to $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ and then applying a $(1_X, \Theta_{11}, 1_Y)$-similarity transformation (or alternatively, first we apply a $(1_X, \Theta_{11}, 1_Y)$-similarity transformation) and then add the feedthrough term $\Theta_{21}\Theta_{11}^1$. Addition of feedthrough terms will be discussed in Definition 2.3.27 below.

2.3.5. The $(P, Q, R)$-image of an i/s/o node. By relaxing the conditions that we put on the operators $P$, $Q$, and $R$ in Definition 1.2.11 we arrive at the notion of a $(P, Q, R)$-image of an i/s/o node.

2.3.16. DEFINITION (cf. Definition 1.2.14). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o node or system, let $\mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1$ be two $H$-spaces, and let $P: \mathcal{X} \to \mathcal{X}_1, Q: \mathcal{U} \to \mathcal{U}_1,$ and $R: \mathcal{Y} \to \mathcal{Y}_1$ be continuous linear operators with closed domains $\text{dom} (P) \subset \mathcal{X}, \text{dom} (Q) \subset \mathcal{U},$ and $\text{dom} (R) \subset \mathcal{Y}$.

(i) By the $(P, Q, R)$-image of $\Sigma$ we mean the i/s/o node or system $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1)$ where

$$
\text{gph} (S_1) = \begin{bmatrix}
P & 0 & 0 & 0 \\
0 & R & 0 & 0 \\
0 & 0 & P & 0 \\
0 & 0 & 0 & Q
\end{bmatrix} \left( \text{gph} (S) \cap \begin{bmatrix} \text{dom} (P) \\ \text{dom} (R) \\ \text{dom} (P) \\ \text{dom} (Q) \end{bmatrix} \right).
$$

(ii) If (2.3.7) holds and $\text{dom} (P) = \mathcal{X}, \text{dom} (Q) = \mathcal{U},$ and $\text{dom} (R) = \mathcal{Y}$, then we call $\Sigma_1$ a $(P, Q, R)$-image of $\Sigma$ with full domain.
(iii) If (2.3.7) holds and \( P, Q, \) and \( R \) are injective, then we call \( \Sigma_1 \) a **injective** \((P, Q, R)-\text{image of } \Sigma\).

(iv) If (2.3.7) holds and \( P, Q, \) and \( R \) are surjective, then we call \( \Sigma_1 \) a **surjective** \((P, Q, R)-\text{image of } \Sigma\).

Most (if not all) of our \((P, Q, R)\)-images will be surjective, some will be injective, and some will have full domain.

2.3.17. **Lemma** (cf. Lemma 2.2.15). Let \( \Sigma = (S; \mathcal{X}, \mathcal{Y}) \) and \( \Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) be two i/s/o nodes or systems.

(i) \( \Sigma_1 \) is \((P, Q, R)\)-similar to \( \Sigma \) if and only if \( \Sigma_1 \) is an injective and surjective \((P, Q, R)-\text{image of } \Sigma\) with full domain.

(ii) If \( \Sigma_1 \) is the \((P, Q, R)-\text{image of } \Sigma\), then the time reflection of \( \Sigma_1 \) is the \((P, Q, R)-\text{image of } \Sigma\) to the time reflection of \( \Sigma \).

**Proof.** This follows immediately from the relevant definitions. \( \square \)

Note that we do not claim that \( \Sigma_1 \) is closed or semi-regular or regular or bounded whenever \( \Sigma \) is closed or semi-regular or regular or bounded.

2.3.18. **Lemma.** If the i/s/o node or system \( \Sigma_{i/o} = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) is the \((P, Q, R)-\text{image of } \Sigma\) of the i/s/o node or system \( \Sigma_{i/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), then the s/s node or system \( \Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{T}_{\mathcal{X}_1}) \) induced by \( \Sigma_{i/o} \) is the \((P, [Q R])-\text{image of } \Sigma\) of the s/s node or system \( \Sigma = (V; \mathcal{X}, [Q R]) \) induced by \( \Sigma_{i/o} \) (see Definition 2.2.35).

**Proof.** This follows from Definitions 2.2.14 and 2.3.16 \( \square \)

2.3.19. **Lemma.** Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o node or system, and let \( \Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) be the \((P, Q, R)-\text{image of } \Sigma\) of \( \Sigma \), where \( P: \mathcal{X} \to \mathcal{X}_1 \), \( Q: \mathcal{U} \to \mathcal{U}_1 \) and \( R: \mathcal{Y} \to \mathcal{Y}_1 \) are continuous linear operators with closed domains \( \text{dom}(P) \subset \mathcal{X} \), \( \text{dom}(Q) \subset \mathcal{U} \), and \( \text{dom}(R) \subset \mathcal{Y} \).

(i) If \( [x_0]_w \) is a classical or generalized trajectories of \( \Sigma \) on some interval \( I \) satisfying \( x(t) \in \text{dom}(P) \) and \( w(t) \in \text{dom}(Q) \) for (almost) all \( t \in I \), then \( \left[P^t x\right]_w \) is a classical respectively generalized trajectory of \( \Sigma \) on \( I \).

(ii) If both \( P \) and \( Q \) are injective and surjective (i.e., \( \Sigma_1 \) is an injective and surjective \((P, Q)-\text{image of } \Sigma\) ), then the converse is also true, i.e., to any \( [x_0]_w \) is a classical or generalized trajectories of \( \Sigma_1 \) on some interval \( I \) there exists a (unique) trajectory \( [x_0]_w \) of \( \Sigma \) on \( I \) such that \( x(t) \in \text{dom}(P) \) and \( w(t) \in \text{dom}(Q) \) for (almost) all \( t \in I \) and \( \left[P^t x\right]_w = \left[R^t w\right]_w \).

**Proof.** This follows from Theorem 2.2.34 and Lemma 2.3.19 \( \square \)

2.3.6. **Parts and projections of an i/s/o node.** Parts and projections of i/s/o nodes are defined in the same way as parts and projections of s/s nodes.

2.3.20. **Definition** (cf. Definition 1.2.17). Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) and \( \Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) be two s/s nodes or systems.

(i) We call \( \Sigma_1 \) the **part of \( \Sigma \)** in \( \left[x_0\right]_{\mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1} \) if \( \mathcal{X}_1, \mathcal{U}_1 \), and \( \mathcal{Y}_1 \) are closed subspaces of \( \mathcal{X}, \mathcal{U}, \) and \( \mathcal{Y} \), respectively, \( \Sigma_1 \) is the \((1, \mathcal{X} | \mathcal{X}_1, 1 \mathcal{U} | \mathcal{U}_1, 1 \mathcal{Y} | \mathcal{Y}_1)-\text{image of } \Sigma\),
i.e.,

\[(2.3.8) \quad \text{gph} (S_1) = \text{gph} (S) \cap \begin{bmatrix} X_1 \\ Y_1 \\ X_1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ X_1 \end{bmatrix} = \text{gph} (S) \cap \begin{bmatrix} X_1 \\ Y_1 \\ X_1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ X_1 \end{bmatrix} \]

(ii) We call \( \Sigma_1 \) the static projection of \( \Sigma \) onto \( \begin{bmatrix} X_1 \\ U_1 \\ Y_1 \end{bmatrix} \) along \( \begin{bmatrix} X_2 \\ U_2 \\ Y_2 \end{bmatrix} \) if \( X = X_1 + X_2, U = U_1 + U_2, \) and \( Y = Y_1 + Y_2, \) and \( \Sigma_1 \) is the \( (P_{X_1}X_1, P_{U_1}U_1, P_{Y_1}Y_1) \)-image of \( \Sigma, \) i.e.,

\[(2.3.9) \quad \text{gph} (S_1) = \begin{bmatrix} P_{X_1}X_1 & 0 & 0 & 0 \\ 0 & P_{Y_1}Y_1 & 0 & 0 \\ 0 & 0 & P_{X_1}X_1 & 0 \\ 0 & 0 & 0 & P_{W_1}W_1 \end{bmatrix} \text{gph} (S). \]

Note, in particular, that the \( (1_X|X_1, 1_U|U_1, 1_Y|Y_1) \)-image in part (i) of Definition 2.3.20 is both injective and surjective, and that the \( (P_{X_1}X_1, P_{U_1}U_1, P_{Y_1}Y_1) \)-image in part (ii) has full domain.

2.3.21. Lemma (cf. Lemma 1.2.19). Let \( \Sigma = (S; X, U, Y) \) be an i/s/o node, let \( X = X_1 + X_2, U = U_1 + U_2, \) and \( Y = Y_1 + Y_2, \)

(i) The part \( \Sigma_{\text{part}} = (S_{\text{part}}; X_1; U_1; Y_1) \) of \( \Sigma \) in \( \begin{bmatrix} X_1 \\ U_1 \\ Y_1 \end{bmatrix} \) has the following properties:

(a) If \( \Sigma \) is closed then \( \Sigma_{\text{part}} \) is closed.
(b) If \( S \) is single-valued, then \( S_{\text{part}} \) is single-valued.

(ii) The static projection \( \Sigma_{\text{proj}} = (S_{\text{proj}}; X_1; U_1; Y_1) \) of \( \Sigma \) onto \( \begin{bmatrix} X_1 \\ U_1 \\ Y_1 \end{bmatrix} \) along \( \begin{bmatrix} X_2 \\ U_2 \\ Y_2 \end{bmatrix} \) has the following properties:

(a) If \( \text{dom} (S) \) is dense in \( \begin{bmatrix} X \\ U \end{bmatrix} \), then \( \text{dom} (S_{\text{part}}) \) is dense in \( \begin{bmatrix} X_1 \\ U_1 \\ Y_1 \end{bmatrix} \).

Proof. This follows directly from Definition 1.2.17.

2.3.22. Lemma. Lemma 1.2.20 is also true in the i/s/o setting.

Proof. This follows from Lemma 2.3.18.

2.3.7. Static output feedback. The idea behind the notion of a static output feedback is to change the properties of a given i/s/o system \( \Sigma = (S; X, U, Y) \) by adding a bounded multiple \( Ky \) of the output to the input, as drawn in Figure ??.

2.3.23. Definition. Let \( \Sigma = (S; X, U, Y) \) be an i/s/o node or system, and let \( K \in B(Y; U) \). By the static output feedback connection of \( \Sigma \) with feedback operator \( K \) we mean the i/s/o node or system \( \Sigma_1 = (S_1; X_1; U_1; Y_1) \) where

\[(2.3.10) \quad \text{gph} (S_1) = \begin{bmatrix} 1_X & 0 & 0 & 0 \\ 0 & 1_Y & 0 & 0 \\ 0 & 0 & 1_X & 0 \\ 0 & -K & 0 & 1_U \end{bmatrix} \text{gph} (S). \]

2.3.24. Lemma. Let \( \Sigma = (S; X, U, Y) \) and \( \Sigma_1 = (S_1; X_1; U_1; Y_1) \) be two i/s/o nodes or systems (with the same input and output spaces).
(i) $\Sigma_1$ is the static output feedback connection of $\Sigma$ with feedback operator $K$ if and only if $\Sigma$ is the static output feedback connection of $\Sigma_1$ with feedback operator $-K$.

(ii) If $\Sigma_1$ is the static output feedback connection of $\Sigma$ with feedback operator $K$, then the time reflection of $\Sigma_1$ is the static output feedback connection of $\Sigma$ with feedback operator $K$ of the time reflection of $\Sigma$.

(iii) If $\Sigma$ is bounded with generating operator $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $\Sigma_1$ is bounded if and only if $1_{\mathcal{U}} - KD$ has an inverse in $\mathcal{B}(\mathcal{U})$, or equivalently, if and only if $1_{\mathcal{Y}} - DK$ has an inverse in $\mathcal{B}(\mathcal{Y})$. If this is the case, then the generating operator $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ of $\Sigma_1$ is given by

$$
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ -KC & 1_{\mathcal{U}} - KD \end{bmatrix}^{-1} \\
= \begin{bmatrix} 1_{\mathcal{X}} & -BK \\ 0 & 1_{\mathcal{Y}} - DK \end{bmatrix}^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix},
$$

or more explicitly,

$$
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
A + B(1_{\mathcal{U}} - KD)^{-1}KC & B(1_{\mathcal{U}} - KD)^{-1}K \\
1_{\mathcal{Y}} + D(1_{\mathcal{U}} - KD)^{-1}K & D(1_{\mathcal{U}} - KD)^{-1}
\end{bmatrix}
= \begin{bmatrix}
A + BK(1_{\mathcal{Y}} - DK)^{-1}C & B[1_{\mathcal{U}} + K(1_{\mathcal{Y}} - DK)^{-1}D] \\
(1_{\mathcal{Y}} - DK)^{-1}C & (1_{\mathcal{Y}} - DK)^{-1}D
\end{bmatrix}.
$$

PROOF. (i) This follows from \eqref{2.3.11} and \eqref{2.3.12}.

(ii) See Definitions \ref{2.3.1} and \ref{2.3.10}.

(iii) Recall that by, e.g., \cite{Staffans2005, Lemma A.4.1}, $1_{\mathcal{U}} - KD$ has an inverse in $\mathcal{B}(\mathcal{U})$ if and only if $1_{\mathcal{Y}} - DK$ has an inverse in $\mathcal{B}(\mathcal{Y})$, and that $(1_{\mathcal{Y}} - DK)^{-1} = 1_{\mathcal{Y}} + D(1_{\mathcal{U}} - KD)^{-1}K$.

By \eqref{2.3.10}, if $\Sigma$ is bounded with generating operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$
gph (S_1) = \text{rng} \begin{bmatrix}
1_{\mathcal{X}} & 0 & 0 & 0 \\
0 & 1_{\mathcal{Y}} & 0 & 0 \\
0 & 0 & 1_{\mathcal{X}} & 0 \\
0 & -K & 0 & 1_{\mathcal{U}}
\end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}
$$

$$
= \text{rng} \begin{bmatrix} A & B \\ C & D \\ 1_{\mathcal{X}} & 0 \\ -KC & 1_{\mathcal{U}} - KD
\end{bmatrix}.
$$

If $1_{\mathcal{U}} - KD$ has an inverse in $\mathcal{B}(\mathcal{U})$, then by multiplying the block matrix operator in the right-hand side of \eqref{2.3.13} by $\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{X}} \\ 0 & -K & 0 & 1_{\mathcal{U}}
\end{bmatrix}^{-1}$ to the right we find that $\Sigma_1$ is bounded and that \eqref{2.3.11} holds. Conversely, suppose that $\Sigma_1$ is bounded. Then it follows from \eqref{2.3.13} that $1_{\mathcal{U}} - KD$ is surjective (since dom $(S_1) = \begin{pmatrix} \mathcal{X} \\ \mathcal{U} \end{pmatrix}$).

The single-valuedness of $S_1$ implies that $Du = 0$ whenever $(1 - KD)u = 0$, and hence $u = 0$. Thus, $1_{\mathcal{U}} - KD$ is both injective and surjective, so it has an inverse in $\mathcal{B}(\mathcal{U})$.

\begin{lemma}
Let $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1)$ be the static output feedback connection of $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with feedback operator $K$. Then $\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}$ is a classical or
generalized trajectory of \( \Sigma \) on the interval \( I \) if and only if \( \begin{bmatrix} u \cdot K \end{bmatrix} y \) is a classical or generalized trajectory of \( \Sigma_1 \) on \( I \).

**Proof.** This follows immediately from the definitions of these notions. \( \square \)

2.3.26. **Example.** The system operator \( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \) in (2.2.44) can be obtained from \( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \) by first performing an output feedback with feedback operator \(-\Theta_{12}\) and then applying a \((1_X, 1_U, \Theta_{22})\)-similarity transformation.

2.3.8. **Adding bounded inputs and output to an i/s/o node.** In Section 1.2 we described how to add bounded inputs and outputs to a s/s node. This can, of course, also be done in the i/s/o setting. Moreover, in the i/s/o setting it is easy to permit some additional feedthrough terms.

2.3.27. **Definition** (cf. Definition 1.2.21). Let \( \Sigma = (S; X, U, Y) \) be an i/s/o node or system, let \( U_0 \) and \( Y_1 \) be \( H \)-spaces, and let \( B_1 \in \mathcal{B}(U_1; X), \ C_1 \in \mathcal{B}(X; Y_1), \) and \( \begin{bmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{bmatrix} \in \mathcal{B}(U_0; U_1, Y_1) \).

(i) The i/s/o node or system \( \Sigma_1 = \left( S_1; X, \begin{bmatrix} U \\ U_1 \end{bmatrix}, \begin{bmatrix} Y \\ Y_1 \end{bmatrix} \right) \) where
\[
\text{dom} (S_1) = \left[ \text{dom}(S) \right] \begin{bmatrix} \text{dom}(S) \end{bmatrix} U_1,
\]
(2.3.14)
\[
S_1 \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} D_{00} \\ C_1 \end{bmatrix} x + D_{10} u + D_{11} u_1 \end{bmatrix} u_1, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} (S_1),
\]

is called the (bounded) i/o extension of \( \Sigma \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( \begin{bmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{bmatrix} \).

(ii) The i/s/o node or system \( \Sigma_2 = \left( S_2; X, \begin{bmatrix} U \\ U_1 \end{bmatrix}, Y \right) \) where
\[
\text{dom} (S_2) = \text{dom} (S_1),
\]
(2.3.15)
\[
S_2 \begin{bmatrix} x \\ u \end{bmatrix} = S \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} D_{00} \end{bmatrix} u + \begin{bmatrix} B_1 \\ D_{01} \end{bmatrix} u_1, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} (S_2),
\]

is called the (bounded) input extension of \( \Sigma \) with control operator \( B_1 \) and feedthrough operator \( \begin{bmatrix} D_{00} & D_{01} \end{bmatrix} \).

(iii) The i/s/o node or system \( \Sigma_2 = \left( S_3; X, \begin{bmatrix} U \\ Y_1 \end{bmatrix} \right) \) where
\[
S_3 \begin{bmatrix} x \\ u \end{bmatrix} = S \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} D_{00} \end{bmatrix} u + \begin{bmatrix} C_1 \end{bmatrix} x + D_{10} u, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom} (S_3) = \text{dom} (S),
\]
(2.3.16)

is called the (bounded) output extension of \( \Sigma \) with observation operator \( C_1 \) and feedthrough operator \( \begin{bmatrix} D_{00} & D_{10} \end{bmatrix} \).

Note that (2.3.14) can be alternatively written in the form
\[
gph (S_1) = \begin{bmatrix} 1_X & 0 & 0 & 0 \\ 0 & 1_Y & 0 & D_{00} \\ 0 & 0 & C_1 & D_{10} \\ 0 & 0 & 0 & 1_{U_1} \\ & & & \end{bmatrix} \begin{bmatrix} B_1 \\ D_{01} \\ D_{11} \\ U_1 \end{bmatrix},
\]
(2.3.17)
and also (2.3.15) and (2.3.16) can be written analogously (simply drop the third row or the last column). Clearly input and output extensions can be regarded as special
cases of i/o extensions (take either \( Y_1 = \{0\} \) or \( U_1 = \{0\} \)). Also note that the i/o extension can be obtained by first performing an input extension of \( \Sigma \) with control operator \( B_1 \) and feedthrough operator \( \begin{bmatrix} D_{00} & D_{01} \end{bmatrix} \), and then performing an output extension of the resulting system with observation operator \( C_1 \) and feedthrough operator \( \begin{bmatrix} D_{10} & D_{11} \end{bmatrix} \). An alternative interpretation is also possible, where one first does an output extension and then an input extension.

2.3.28. **Lemma** (cf. Lemma 2.2.14). Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o node or system, and let \( \Sigma_1 = (S; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) be a bounded i/o extension of \( \Sigma \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( \begin{bmatrix} D_{00} & D_{01} \end{bmatrix} \).

(i) \( \Sigma_1 \) is closed, or semi-regular, or regular, or bounded if and only if \( \Sigma \) is closed, or semi-regular, or regular, or bounded, respectively.

(ii) If \( \Sigma \) and \( \Sigma_1 \) are bounded, then the system operator \( S_1 \) of \( \Sigma_1 \) is given by

\[
S_1 = \begin{bmatrix}
A & B & B_1 \\
C & D + D_{00} & D_{01} \\
C_1 & D_{01} & D_{11}
\end{bmatrix},
\]

where \( S = \begin{bmatrix} \frac{A}{C} & B \end{bmatrix} \) is the system operator of \( \Sigma \).

(iii) The time reflection of \( \Sigma_1 \) is the bounded i/o extension of the time reflection of \( \Sigma \) with the same control operator, observation operator, and feedthrough operator.

**Proof.** The easy proof is left to the reader. \( \square \)

2.3.29. **Lemma.**

(i) Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o node or system, and let \( \Sigma_{i/s/o}^1 = (S; \mathcal{X}_1, \begin{bmatrix} U_1 \end{bmatrix}, \begin{bmatrix} Y_1 \end{bmatrix}) \) be a bounded i/o extension of \( \Sigma_{i/s/o} \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( \begin{bmatrix} D_{00} & D_{01} \end{bmatrix} \).

Then the s/s node or system induced by \( \Sigma_{i/s/o} \) is the bounded i/o extension of the s/s node or system induced by \( \Sigma \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( D_{11} \).

(ii) Conversely, let \( \Sigma = (V; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o node or system, and let \( \Sigma_1 = (V; \mathcal{X}_1, \begin{bmatrix} U_1 \end{bmatrix}, \begin{bmatrix} Y_1 \end{bmatrix}) \) be a bounded i/o extension of \( \Sigma \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( D_{11} \). Let \( (\mathcal{U}, \mathcal{Y}) \) be an i/o representation of \( \Sigma \) by \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), and denote the i/s/o representation of \( \Sigma_1 \) corresponding to the i/o representation \( \begin{bmatrix} U_1 \end{bmatrix}, \begin{bmatrix} Y_1 \end{bmatrix} \) of \( \Sigma \) by \( \Sigma_{i/s/o}^1 = (S; \mathcal{X}_1, \begin{bmatrix} U_1 \end{bmatrix}, \begin{bmatrix} Y_1 \end{bmatrix}) \). Then \( \Sigma_{i/s/o}^1 \) is the bounded i/o extension of \( \Sigma_{i/s/o} \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( \begin{bmatrix} D_{00} & D_{01} \end{bmatrix} \).

(iii) The i/o representation \( (\mathcal{U}, \mathcal{Y}) \) of \( \Sigma \) is i/s/o-admissible or boundedly i/s/o-admissible for \( \Sigma \) if and only if the i/o representation \( \begin{bmatrix} U_1 \end{bmatrix}, \begin{bmatrix} Y_1 \end{bmatrix} \) is i/s/o-admissible respectively boundedly i/s/o-admissible for \( \Sigma_1 \) (cf. Definition 2.2.17).

**Proof.** That (i) and (ii) hold follows from (1.2.15) and (2.3.17). Claim (iii) follows from Definition 2.2.17 and Lemma 2.3.28. \( \square \)
2.3.30. **Lemma** (cf. Lemma 1.2.23). Let $\Sigma = (S; X, U, Y)$ be an i/s/o system, let $C \in B(X; Y_1)$, and let $\Sigma_1 = (S_1; X, U_1, Y_1)$ be the output extension of $\Sigma$ with observation operator $C_1$ and zero feedthrough operator. Then the following claims are true:

(i) $\Sigma$ is the static projection of $\Sigma_1$ onto $[\Sigma]$ along $\{0\}$; 
(ii) $\begin{bmatrix} x \\ u \\ y \\ z \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma$ on $I$ if and only if $\begin{bmatrix} x \\ u \\ y \\ z \end{bmatrix}$ is a classical respectively generalized trajectory of $\Sigma_1$ on $I$.

**Proof.** The proof is analogous to the proof of Lemma 1.2.23. $\square$

2.3.31. **Lemma** (cf. Lemma 1.2.24). Let $\Sigma = (S; X, U, Y)$ be an i/s/o system, let $B_1 \in B(U_1; X)$, and let $\Sigma_1 = (S_1; X, U_1, Y_1)$ be the input extension of $\Sigma$ with control operator $B_1$ and zero feedthrough operator. Then the following claims are true:

(i) $\Sigma$ is the part of $\Sigma_1$ in $[\Sigma]$; 
(ii) $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma$ on $I$ if and only if $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ is a classical respectively generalized trajectory of $\Sigma_1$ on $I$. 
(iii) If $S$ and $S_1$ are single-valued, if $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix}$ are classical trajectories of $\Sigma$ respectively $\Sigma_1$ on some interval $I$, and if $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ u_1(t) \end{bmatrix}$ at some point $t \in I$, then $\hat{x}_1(t) = \hat{x}(t) + B_1 u_1(t)$ and $y_1(t) = y(t)$.

**Proof.** The proof is analogous to the proof of Lemma 1.2.23. $\square$

2.3.9. **The cross product of two i/s/o nodes.** The cross product of two i/s/o systems simply consists of the two given system, interpreted as one system with two independent parts, as drawn in Figure ??.

2.3.32. **Definition.** Let $\Sigma_i = (S_i; X_i, U_i, Y_i)$, $i = 1, 2$, be two i/s/o nodes or systems. By the **cross product** of $\Sigma_1$ and $\Sigma_2$ we mean the i/s/o node or system $\Sigma = (S; X, U, Y)$ with state space $X = [X_1, X_2]$, input space $U = [U_1, U_2]$, and output space $Y = [Y_1, Y_2]$ with system operator $S$ whose graph is given by 

(2.3.19) 

$$
gph(S) = \left\{ \begin{bmatrix} z \\ y \\ x \\ u \\ y \\ z \\ z_2 \\ z_1 \\ y_1 \\ x_1 \\ x_2 \\ u_1 \\ u_2 \\ y_2 \\ y_2 \\ y_1 \end{bmatrix} \in gph(S_i), \ i = 1, 2 \right\}.
$$

We denote the cross product of $\Sigma_1$ and $\Sigma_2$ by $\Sigma_1 \times \Sigma_2$.

2.3.33. **Example.** Let $\Sigma_i = ([A_i B_i]_i X_i, U_i, Y_i)$, $i = 1, 2$, be two bounded i/s/o nodes. Then $\Sigma_{\times} := \Sigma_1 \times \Sigma_2$ is the bounded i/s/o node $\Sigma_{\times} = \left( [A_{\times} B_{\times}]_i X_{\times}, U_{\times}, Y_{\times} \right)$. [Note: The exact forms of $A_{\times}$, $B_{\times}$, $X_{\times}$, $U_{\times}$, and $Y_{\times}$ are not provided in the text and would need to be derived based on the definitions given.]
with system operator

\[
\begin{bmatrix}
A_x & B_x \\
C_x & D_x
\end{bmatrix} = \begin{bmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{bmatrix}
\]

(2.3.20)

2.3.34. Lemma. Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i) \), \( i = 1, 2 \), be two i/s/o systems, and let \( \Sigma = \Sigma_1 \times \Sigma_2 \) be the cross product of \( \Sigma_1 \) and \( \Sigma_2 \). Then \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma \) on some interval \( I \) if and only if \( x = \begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix} \) and \( y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \), where \( \begin{bmatrix} x_1 & u_1 & y_1 \end{bmatrix} \) is a classical respectively generalized trajectory of \( \Sigma_i \) in \( I \), \( i = 1, 2 \).

Proof. This follows directly from Definitions 2.1.1, 2.1.5, 2.1.7, and 2.3.32. □

Our main use of the cross product of two i/s/o nodes will be as a preliminary connection from which many other connections can be derived (such as the parallel and difference connections).

2.3.10. \((P, Q, R)\)-interconnections of i/s/o nodes. The \((P, Q)\)-interconnections in Definition 1.2.27 have natural i/s/o counterparts.

2.3.35. Definition. Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i) \), \( i = 1, 2 \), be two i/s/o nodes or systems, let \( \mathcal{X}, \mathcal{U}, \) and \( \mathcal{Y} \) be \( H \)-spaces, and let \( P: \mathcal{X}_1 \to \mathcal{X}, \ Q: \mathcal{U}_1 \to \mathcal{U}, \) and \( R: \mathcal{Y}_1 \to \mathcal{Y} \) be continuous linear surjective operators with closed domains \( \text{dom}(P) \subset \mathcal{X}_1 \), \( \text{dom}(Q) \subset \mathcal{U}_1 \), and \( \text{dom}(R) \subset \mathcal{Y}_1 \). By the \((P, Q, R)\)-interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) we mean the \((P, Q, R)\)-image \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) of the cross product \( \Sigma_1 \times \Sigma_2 = (S_x; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) of \( \Sigma_1 \) and \( \Sigma_2 \), i.e., \( S \) is given by

\[
\text{gph}(S) = \begin{bmatrix}
P & 0 & 0 & 0 \\
0 & R & 0 & 0 \\
0 & 0 & P & 0 \\
0 & 0 & 0 & Q
\end{bmatrix} \left( \text{gph}(S_x) \cap \begin{bmatrix}
\text{dom}(P) \\
\text{dom}(Q)
\end{bmatrix} \right).
\]

(2.3.21)

2.3.36. Lemma (cf. Lemma 1.2.28). Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i) \), \( i = 1, 2 \), be two i/s/o nodes or systems and let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be the \((P, Q, R)\)-interconnection of \( \Sigma_1 \) and \( \Sigma_2 \), where \( P: \mathcal{X}_1 \to \mathcal{X}, \ Q: \mathcal{U}_1 \to \mathcal{U}, \) and \( R: \mathcal{Y}_1 \to \mathcal{Y} \) are continuous linear surjective operators with closed domains \( \text{dom}(P) \subset \mathcal{X}_1 \), \( \text{dom}(Q) \subset \mathcal{U}_1 \), and \( \text{dom}(R) \subset \mathcal{Y}_1 \). If \( \begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix}, \ i = 1, 2 \) are classical or generalized trajectories of \( \Sigma_i \), \( i = 1, 2 \) on some interval \( I \) satisfying \( \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \text{dom}(P), \ \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in \text{dom}(Q) \), and \( \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \text{dom}(R) \) for (almost) all \( t \in I \), then \( \begin{bmatrix} x \\ u \\ y \end{bmatrix}, \) where \( x = P \begin{bmatrix} x_1 \end{bmatrix}, \) and \( u = Q \begin{bmatrix} u_1 \end{bmatrix}, \) and \( y = R \begin{bmatrix} y_1 \end{bmatrix} \) is a classical respectively generalized trajectory of \( \Sigma \) on \( I \).

Proof. This follows from Lemmas 2.3.19 and 2.3.34. □

Also the “short circuit” connection in Definition 1.2.29 has a natural i/s/o counterpart.
2.3.37. **Definition** (cf. Definition 2.2.29). Let \( \Sigma_i = (S_i; X_i, U, Y) \), \( i = 1, 2 \), be two i/o nodes or systems (with the same input and output spaces), and let \( X_0 \) be a closed subspace of \( \left[ \frac{X_1}{X_2} \right] \). By the \( X_0 \)-short circuit \( \Sigma = (S; X_0, U, Y) \) of \( \Sigma_1 \) and \( \Sigma_2 \) we mean the \((P, Q, R)\)-connection of \( \Sigma_1 \) and \( \Sigma_2 \) where \( P = 1_{\left[ \frac{X_1}{X_2} \right]} |_{X_0} \), \( \text{dom} (Q) = \{ [u] \mid u \in U \} \), \( Q [u] = u \), \( [u] \in \text{dom} (Q) \), \( \text{dom} (R) = \{ [y] \mid y \in Y \} \), and \( R [y] = y \), \([y] \in \text{dom} (Y)\), i.e.,

\[
\text{gph} (S) = \left\{ \begin{bmatrix} z_i \\ y \\ x_i \\ u \end{bmatrix} \in \left[ \begin{bmatrix} X_0 \\ Y \\ X_1 \\ U \end{bmatrix} \right] \mid \begin{bmatrix} x_i \\ y \\ y_i \\ y_i \end{bmatrix} \in \text{gph} (S_i), \ i = 1, 2 \right\}.
\]

This notion will be particularly useful in the study of intertwined i/o systems (see Definition 2.5.22).

2.3.11. **Parallel and difference connections.** If \( \Sigma_1 \) and \( \Sigma_2 \) are two i/o systems with the same input and output spaces, then we may define the parallel and difference connections of \( \Sigma_1 \) and \( \Sigma_2 \). In both cases the input signals of the two systems are taken to be the same, and the output signals are added to respectively subtracted from each other, as drawn in Figure ??.

2.3.38. **Definition.** Let \( \Sigma_i = (S_i; X_i, U, Y) \), \( i = 1, 2 \), be two i/o nodes or systems (with the same input and output spaces).

(i) By the **parallel connection** of \( \Sigma_1 \) and \( \Sigma_2 \) we mean the i/o node or system \( \Sigma = (S; X, U, Y) \), where \( X = \left[ \frac{X_1}{X_2} \right] \) and \( S \) is given by

\[
\text{gph} (S) = \left\{ \begin{bmatrix} \frac{z_1}{z_2} \\ y_1 + y_2 \\ \frac{x_1}{x_2} \\ x_i \\ u \end{bmatrix} \in \left[ \begin{bmatrix} X \\ Y \\ X_1 \\ X_2 \\ U \end{bmatrix} \right] \mid \begin{bmatrix} x_i \\ y_i \\ y_i \\ y_i \end{bmatrix} \in \text{gph} (S_i), \ i = 1, 2 \right\}.
\]

We denote the parallel connection of \( \Sigma_1 \) and \( \Sigma_2 \) by \( \Sigma_1 \parallel \Sigma_2 \).

(ii) The **difference connection** of \( \Sigma_1 \) and \( \Sigma_2 \) is defined in the same way, with “\( y_1 + y_2 \)” replaced by “\( y_1 - y_2 \)”. We denote the difference connection of \( \Sigma_1 \) and \( \Sigma_2 \) by \( \Sigma_1 \ominus \Sigma_2 \).

Note that if \( \dim U = \dim Y = 0 \), then the parallel connection \( \Sigma_1 \parallel \Sigma_2 \) of \( \Sigma_1 \) and \( \Sigma_2 \) can be identified with the cross product \( \Sigma_1 \times \Sigma_2 \) of \( \Sigma_1 \) and \( \Sigma_2 \), and if \( \dim X = 0 \), then \( \Sigma_1 \ominus \Sigma_2 \) can be identified with the sum \( S_1 + S_2 \). Observe, in particular, that \( \Sigma_1 \parallel \Sigma_2 \) need not be closed, even if both \( \Sigma_1 \) and \( \Sigma_2 \) is closed (the sum of two closed operators need not be closed).

2.3.39. **Example.** Let \( \Sigma_i = \left( \left[ \begin{bmatrix} A_i \\ B_i \end{bmatrix} \right]; X_i, U, Y \right) \), \( i = 1, 2 \) be two bounded i/o nodes (with the same input and output spaces). Then \( \Sigma_1 \parallel \Sigma_2 \) is the bounded i/o node \( \Sigma_{\parallel} = \left( \left[ \begin{bmatrix} A_{\parallel} \\ B_{\parallel} \end{bmatrix} \right]; X, U, Y \right) \) with system operator

\[
\begin{bmatrix} A_{\parallel} \\ C_{\parallel} \end{bmatrix} \parallel \begin{bmatrix} B_{\parallel} \\ D_{\parallel} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D_1 + D_2 \end{bmatrix}.
\]
and $\Sigma_1 \parallel \Sigma_2$ is the bounded i/s/o node $\Sigma_1 = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} ; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ with system operator

(2.3.25) $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} C_1 & D_1 \\ -C_2 & D_2 \end{bmatrix}$.

In the following lemma we use a special “continuity condition” of the following type:

2.3.40. DEFINITION. We say that an i/s/o node or system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ has a continuously determined output if there exists a continuous linear operator $H$ such that

(2.3.26) $\text{dom} (H) = \begin{cases} \begin{bmatrix} z \\ x \\ u \end{bmatrix} \in \mathcal{X} & \text{and} \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \text{gph} (S) \text{ for some } y \in \mathcal{Y} \end{cases}$, \hspace{1cm} \text{gph} (S) = \begin{cases} \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \mathcal{X} & \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \text{dom} (H) \text{, } y = H \begin{bmatrix} z \\ x \\ u \end{bmatrix} \end{cases}$.

Observe that if $\Sigma$ has a continuously determined output, then $\Sigma$ is closed if and only if $S$ is closed (since $H$ is closed if and only if $\text{dom} (H)$ is closed). The operator $H$ in Definition 2.3.40 is unique, since an operator is uniquely determined by its graph. Also observe that every bounded i/s/o node has a continuously determined output. A sufficient condition for a node $\Sigma$ to have a continuously determined output will be given in Lemma 5.2.14 (every i/s/o node with a nonempty resolvent set has a continuously determined output).

2.3.41. LEMMA. Let $\Sigma_i = (S; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$ be two i/s/o systems (with the same input and output spaces), and let $\Sigma \parallel = \Sigma_1 \parallel \Sigma_2$.

(i) If $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ is a classical or generalized trajectory of $\Sigma_i$ in some interval $I$, $i = 1, 2$, then $\begin{bmatrix} x \\ y \end{bmatrix}$ is a classical respectively generalized trajectory of $\Sigma \parallel$ in $I$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = y_1 + y_2$.
(ii) If, in addition, both $\Sigma_1$ and $\Sigma_2$ (are closed and) have continuously determined outputs, then to each classical trajectory $\begin{bmatrix} x \\ y \end{bmatrix}$ of $\Sigma \parallel$ on some interval $I$ there corresponds unique classical trajectories $\begin{bmatrix} x \\ y \end{bmatrix}$ of $\Sigma_i$ on $I$, $i = 1, 2$, such that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = y_1 + y_2$.

Claims (i) and (ii) are also true if we replace the parallel connection $\Sigma \parallel = \Sigma_1 \parallel \Sigma_2$ by the difference connection $\Sigma_\ominus = \Sigma_1 \ominus \Sigma_2$ and throughout replace “$y_1 + y_2$” by “$y_1 - y_2$”.

PROOF. (i) Claim (i) follows directly from Definitions 2.1.1, 2.1.5, 2.1.7 and 2.3.32.
(ii) Denote the functions $H$ in Definition 2.3.40 by $H_i$, $i = 1, 2$. Then it follows from (2.3.23) that $\text{gph} \ (S)$ has the representation

$$\text{gph} \ (S) = \left\{ \left[ \begin{array}{c} z_i \\ y_i + y_2 \\ x_i \\ u \end{array} \right] \mid y_i = H_i \left[ \begin{array}{c} z_i \\ x_i \\ u \end{array} \right], \ i = 1, 2 \right\}.$$  

This implies that also $\Sigma$ has a continuously determined output, with the function $H$ in Definition 2.3.40 corresponding to $S$ given by

$$H \left[ \begin{array}{c} z_1 \\ z_2 \\ x_1 \\ u \end{array} \right] = H_1 \left[ \begin{array}{c} z_1 \\ x_1 \\ u \end{array} \right] + H_2 \left[ \begin{array}{c} z_2 \\ x_2 \\ u \end{array} \right], \quad \left[ \begin{array}{c} z_i \\ x_i \\ u \end{array} \right] \in \text{dom} \ (H_i), \quad i = 1, 2.$$  

Let $\left[ \begin{array}{c} u \\ y \end{array} \right]$ be a classical trajectory of $\Sigma$ on $I$. Then $x$ is continuously differentiable on $I$, and $u$ is continuous. Split $x$ into $x = [x_1, x_2]$ in accordance with the decomposition $X = [X_1, X_2]$. Then $x_i$ is continuously differentiable in $X_i, i = 1, 2$. Denote the functions $H$ in Definition 2.3.40 by $H_i$, $i = 1, 2$, and define $y_i = H_i \left[ \begin{array}{c} x_i \\ x_i \\ u \end{array} \right]$. Then $\left[ \begin{array}{c} x_i \\ u \end{array} \right]$ is a classical trajectory of $\Sigma_i$ on $I$, $i = 1, 2$, and $y = y_1 + y_2$.

The proof for the difference connection is analogous. \hfill \Box

2.3.42. Lemma. Let $\Sigma_1 = (S_1; X_1, Y_1, \mathcal{U}), i = 1, 2$, be two i/s/o nodes (with the same input and output spaces), and let $\Sigma_i = (S_i; X, Y, \mathcal{U})$ and $\Sigma_\| = (S_\|; X, Y, \mathcal{U})$ be the parallel and difference connections of $\Sigma_1$ and $\Sigma_2$ with $X = [X_1, X_2]$. Then $\Sigma_\|$ is the $(1_X, Q_\|, R_\|)$-interconnection of $\Sigma_1$ and $\Sigma_2$ and $\Sigma_\| = (1_X, Q_\|, R_\|)$-interconnection of $\Sigma_1$ and $\Sigma_2$ where $Q_\|, R_\|, Q_\parallel$, and $R_\parallel$ are defined by

$$\text{dom} \ (Q_\|) = \text{dom} \ (Q_\parallel) = \{ [u] \mid u \in \mathcal{U} \},$$

$$Q_\| ([u]) = Q_\parallel ([u]) = u, \quad [u] \in \text{dom} \ (Q),$$

$$R_\| [y_1, y_2] = y_1 + y_2, \quad R_\parallel [y_1, y_2] = y_1 - y_2, \quad [y_1, y_2] \in \left[ \begin{array}{c} Y_1 \\ Y_2 \end{array} \right].$$

Thus,

$$\text{gph} \ (S_\|) = \left[ \begin{array}{cccc} 1_X & 0 & 0 & 0 \\ 0 & 1_Y & 1_Y & 0 \\ 0 & 0 & 1_X & 0 \\ 0 & 0 & 0 & Q_\parallel \end{array} \right] \left( \text{gph} \ (S_X) \cap \text{dom} \ (Q_\|) \right),$$

$$\text{gph} \ (S_\parallel) = \left[ \begin{array}{cccc} 1_X & 0 & 0 & 0 \\ 0 & 1_Y & -1_Y & 0 \\ 0 & 0 & 1_X & 0 \\ 0 & 0 & 0 & Q_\parallel \end{array} \right] \left( \text{gph} \ (S_X) \cap \text{dom} \ (Q_\parallel) \right),$$

where $S_X$ is the system operator of the cross product of $\Sigma_1$ and $\Sigma_2$.

Proof. This follows from Definitions 2.3.16 and 2.3.38. \hfill \Box

2.3.12. Cascade connections. If $\Sigma_1$ and $\Sigma_2$ are two i/s/o systems and the input space of $\Sigma_1$ coincides with the output space of $\Sigma_2$, then we may define the cascade connection of $\Sigma_1$ and $\Sigma_2$. We simply take the input of $\Sigma_1$ to be the output ??.
2.3.43. Definition. Let \( \Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{V}) \) and \( \Sigma_2 = (S_1; \mathcal{X}_2, \mathcal{V}, \mathcal{Y}) \) be two i/s/o nodes or systems. By the cascade connection of \( \Sigma_2 \) and \( \Sigma_1 \) we mean the i/s/o node or system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), where \( \mathcal{X} = [\mathcal{X}_1 \mathcal{X}_2] \) and \( S \) is given by (2.3.29)

\[
\text{gph}(S) = \left\{ \begin{bmatrix} z_1 \\ z_2 \\ x_2 \\ u \\ v \\ y \\ x_1 \\ v \\ u \end{bmatrix} \in \text{gph}(S_2), \begin{bmatrix} z_1 \\ y \\ x_1 \\ v \end{bmatrix} \in \text{gph}(S_1) \text{ for some } v \in \mathcal{V} \right\}.
\]

We denote the cascade connection of \( \Sigma_2 \) and \( \Sigma_1 \) by \( \Sigma_1 \circ \Sigma_2 \).

2.3.44. Example. Let \( \Sigma_1 = ([A_1 B_1; C_1 D_1]; \mathcal{X}_1, \mathcal{V}, \mathcal{Y}) \) and \( \Sigma_2 = ([A_2 B_2; C_2 D_2]; \mathcal{X}_2, \mathcal{U}, \mathcal{V}) \) be two bounded i/s/o systems. Then \( \Sigma_1 \circ \Sigma_2 \) is the bounded i/s/o node \( \Sigma_0 = ([A_0 B_0; C_0 D_0]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) with system operator (2.3.30)

\[
\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ C_1 & D_1 C_2 & D_1 D_2 \end{bmatrix}.
\]

2.3.45. Lemma. Let \( \Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{V}, \mathcal{Y}) \) and \( \Sigma_2 = (S_1; \mathcal{X}_2, \mathcal{U}, \mathcal{V}) \) be two i/s/o nodes, and let \( \Sigma_0 = \Sigma_1 \circ \Sigma_2 \).

(i) If \( \begin{bmatrix} x_1 \\ x_2 \\ u \\ v \\ y \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma_2 \) and \( \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma_1 \) in some interval \( I \), i = 1, 2,\) then \( \begin{bmatrix} x_1 \\ x_2 \\ u \\ v \\ y \end{bmatrix} \) is a classical respectively generalized trajectory of \( \Sigma_0 \) in \( I \) where \( x = [x_1 x_2] \).

(ii) If, in addition, both \( \Sigma_1 \) and \( \Sigma_2 \) (are closed and) have continuously determined outputs, then to each classical trajectory \( \begin{bmatrix} x_1 \\ x_2 \\ x_1 \\ y \end{bmatrix} \) of \( \Sigma_0 \) on some interval \( I \) there corresponds a unique classical trajectory \( \begin{bmatrix} x_2 \\ x_2 \\ u \\ v \\ y \end{bmatrix} \) of \( \Sigma_2 \) and a unique classical trajectory \( \begin{bmatrix} x_1 \\ x_1 \\ x_1 \\ y \end{bmatrix} \) of \( \Sigma_1 \) on \( I \), i = 1, 2, such that \( x = [x_1 x_2] \).

Proof. The proof is analogous to the proof of Lemma 2.3.41.

As opposed to the parallel and difference connections of two i/s/o systems the cascade connection cannot be interpreted as a \((P, Q, R)\)-connection (due to the fact that “the output of the first system is connected to the input of the second”, which leads to a non-diagonal term in counterpart of (2.3.28)). However, it can be interpreted as a \((P, Q)\)-connection of the induced s/s systems.

2.3.46. Lemma. Let \( \Sigma_1^{i/s/o} = (S_1; \mathcal{X}_1, \mathcal{V}, \mathcal{Y}) \) and \( \Sigma_2^{i/s/o} = (S_1; \mathcal{X}_2, \mathcal{U}, \mathcal{V}) \) be two i/s/o nodes, and let \( \Sigma_0^{i/s/o} = (S_0; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) the cascade connection of \( \Sigma_2 \) and \( \Sigma_1 \) with \( \mathcal{X} = [\mathcal{X}_1 \mathcal{X}_2] \). Denote the s/s nodes induced by \( \Sigma_1^{i/s/o}, \Sigma_2^{i/s/o}, \) and \( \Sigma_0^{i/s/o} \) by \( \Sigma_1, \Sigma_2, \) and \( \Sigma_0 \), respectively. Then \( \Sigma_0 \) is the \((1, \mathcal{X}, \mathcal{Q}_o)\)-interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) where \( \mathcal{Q}_o \) is defined by

\[
\text{dom}(\mathcal{Q}_o) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ u \\ v \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \\ y \end{bmatrix} \right\},
\]

\[
\mathcal{Q}_o \begin{bmatrix} x_1 \\ x_2 \\ u \\ v \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \\ y \end{bmatrix} \in \text{dom}(\mathcal{Q}_o).
\]

Proof. This follows from Definitions 1.2.14 and 2.3.43.
2.3.13. **Dynamic feedback.** Dynamic output feedback can be seen as an extension of static output feedback. Instead of adding a multiple of the output of one i/s/o system to the input of the same system we now have two i/s/o systems \( \Sigma_1 = (S_1, X_1, U, Y) \) and \( \Sigma_2 = (S_2, X_2, Y, U) \) (with opposite input and output spaces), and we add the output of \( \Sigma_1 \) to the input of \( \Sigma_2 \), and also add the output of \( \Sigma_2 \) to the input of \( \Sigma_1 \), as drawn in Figure ??.

2.3.47. **Definition.** Let \( \Sigma_i = (S_i; X_i, U_i, Y_i), \ i = 1, 2, \) be two i/s/o nodes or systems, and let \( K_1 \in B(Y_1; U_2) \) and \( K_2 \in B(Y_2; U_1) \). By the dynamic feedback connection \( \Sigma_3 \) of \( \Sigma_1 \) and \( \Sigma_2 \) we mean the static output feedback connection of \( \Sigma_1 \times \Sigma_2 \) with feedback operator \( \begin{bmatrix} 0 & 1_U \\ 1_Y & 0 \end{bmatrix} \), i.e., the i/s/o node or system \( \Sigma_3 = (S_3; [X_1], [Y], [U]) \) where \( X = [X_1] \) and

\[
(2.3.32) \quad \text{gph}(S_3) = \begin{bmatrix}
1_{X_1} & 0 & 0 & 0 \\
0 & 1_Y & 0 & 0 \\
0 & 0 & 1_{X_1} & 0 \\
0 & \begin{bmatrix} 1_U \\ 0 \end{bmatrix} & 0 & 1_{X_1} \\
\end{bmatrix} \text{gph}(S_\times),
\]

where \( S_\times \) is the generating operator of \( \Sigma_1 \times \Sigma_2 \).

Since the dynamic feedback \( \Sigma_3 \) of \( \Sigma_1 \) and \( \Sigma_2 \) is (by definition) a particular static output feedback for \( \Sigma_1 \times \Sigma_2 \), results analogous to Lemmas 2.3.24 and 2.3.25 hold for dynamic feedbacks with \( \Sigma_1 \) replaced by \( \Sigma_3 \), \( \Sigma \) replaced by \( \Sigma_1 \times \Sigma_2 \), and \( K \) replaced by \( \begin{bmatrix} 0 & -1_U \\ -1_Y & 0 \end{bmatrix} \).
2.4. Properties of Trajectories of I/S/O Systems (Jan 02, 2016)

Below we present i/s/o versions of the many of the definitions, results, and examples given in Chapter 1. We omit proofs of those results that are obvious modifications of the corresponding result in Chapter 1. It is possible to give formal proofs of these results by appealing to the correspondence between the trajectories of an i/s/o system and the trajectories of the corresponding s/s system described in Theorem 2.2.33.

2.4.1. Basic properties of the sets of classical and generalized trajectories.

2.4.1. Lemma (cf. Lemma 1.3.2). Let \( \Sigma = (D; X, U, Y) \) be an i/s/o system, and let \( I \subset \mathbb{R} \) be an interval.

(i) Every classical trajectory of \( \Sigma \) on \( I \) is also a generalized trajectory of \( \Sigma \) on \( I \).

(ii) If \( \Sigma \) is closed, then the set of classical trajectories of \( \Sigma \) on \( I \) is a closed subspace of \( C^1(I; X) \) and the set of generalized trajectories of \( \Sigma \) on \( I \) is a closed subspace of \( L^1_{loc}(I; X) \).

(iii) The restriction of a classical or generalized trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) to any subinterval \( I' \) of \( I \) is a classical respectively generalized trajectory of \( \Sigma \) on \( I' \).

(iv) If \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma \) on some interval \( I \), then, for all \( t \in \mathbb{R} \), the shifted triple of functions \( \begin{bmatrix} \tau^t x \\ \tau^t u \\ \tau^t y \end{bmatrix} := \begin{bmatrix} s \rightarrow x(t + s) \\ s \rightarrow u(t + s) \\ s \rightarrow y(t + s) \end{bmatrix} \) is a classical respectively generalized trajectory of \( \Sigma \) on the shifted interval \( \tau^t I := \{ s \in \mathbb{R} \mid t + s \in I \} \).

(v) The sets of past, future, and two-sided classical or generalized trajectories have the following invariance properties:

(a) The sets of all future classical trajectories and all future generalized trajectories of \( \Sigma \) are left-shift invariant,

(b) The sets of all past classical trajectories and all past generalized trajectories of \( \Sigma \) are right-shift invariant,

(c) The sets of all two-sided classical trajectories and all two-sided generalized trajectories of \( \Sigma \) are both left and right shift invariant.

(vi) Let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are intervals satisfying \( I_1 \cap I_2 = \{ t_0 \} \) and \( t_0 \) is both the right end-point of \( I_1 \) and the left end-point of \( I_2 \), and let \( \begin{bmatrix} x \\ u \end{bmatrix} \) be a classical trajectory of \( \Sigma \) on \( I_i \), \( i = 1, 2 \). Define \( \begin{bmatrix} x \\ u \end{bmatrix} \) on \( I \) by

\[
(2.4.1) \begin{cases}
\begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} = \begin{cases}
\begin{bmatrix} x_1(t) \\ u_1(t) \\ y_1(t) \end{bmatrix} & t \in I, \ t < t_0, \\
\begin{bmatrix} x_2(t) \\ u_2(t) \\ y_2(t) \end{bmatrix} & t \in I, \ t \geq t_0,
\end{cases}
\end{cases}
\]
Then \[
\begin{bmatrix}
x_{1}(t_{0}) \\
y_{1}(t_{0})
\end{bmatrix}
\] is a classical trajectory of \(\Sigma\) on \(I\) if and only if 
\[
\begin{bmatrix}
x_{2}(t_{0}) \\
y_{2}(t_{0})
\end{bmatrix}
\] is a classical trajectory of \(\Sigma\) on \(I\).

### 2.4.2. Solvability and the uniqueness property.

#### 2.4.2. Lemma (cf. Lemma 1.3.6).
Let \(\Sigma = (S; X, Y)\) be a solvable \(i/s/o\) system. Then

(i) If \(I\) is an interval with left end-point \(a\) in \(I\), then for every \(z_{a} = [x_{a} \ y_{a}] \in \text{dom}(S)\) and \(z_{a} = [x_{a} \ y_{a}] \in \text{gph}(S)\) there exists a classical trajectory \(\begin{bmatrix} x \ y \end{bmatrix}\) of \(\Sigma\) on \(I\) satisfying \(\begin{bmatrix} x(a) \\ y(a) \end{bmatrix} = [x_{a} \ y_{a}]\) and \(\begin{bmatrix} \dot{x}(a) \\ \dot{y}(a) \end{bmatrix} = [\dot{x}_{a} \ \dot{y}_{a}]\).

(ii) Every classical trajectory of \(\Sigma\) on some interval \(I\) with finite right end-point \(b\) in \(I\) may be continued to a classical trajectory of \(\Sigma\) on \(I \cup [b, \infty)\).

#### 2.4.3. Lemma (cf. Lemma 1.3.7).
An \(i/s/o\) system \(\Sigma = (S; X, Y)\) is solvable if and only if

\[
\text{gph}(S) = \left\{ \begin{bmatrix} x(0) \\ y(0) \\ x(0) \\ u(0) \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} \text{ is a classical future trajectory of } \Sigma \right\}.
\]

#### 2.4.4. Corollary (cf. Corollary 1.3.8).
If a closed \(i/s/o\) system \(\Sigma = (S; X, Y)\) is solvable, then its system operator \(S\) is uniquely determined by its set of classical trajectories.

#### 2.4.5. Lemma.
A closed solvable \(i/s/o\) system \(\Sigma = (S; X, Y)\) is bounded if and only if it is true for some nontrivial interval \([0, T]\) that for every \(x^{0} \in X\) and for every \(u \in C([0, T]; U)\) there exists a unique classical trajectory \(\begin{bmatrix} x \\ u \end{bmatrix}\) of \(\Sigma\) on \([0, T]\) with initial state \(x(0) = x^{0}\) and input function \(u\).

**Proof.** In one direction this follows from Lemmas 2.1.18 and 2.4.2. Conversely, suppose that for every \(x^{0} \in X\) and for every \(u \in C([0, T]; U)\) there exists a unique classical trajectory \(\begin{bmatrix} x \\ u \end{bmatrix}\) of \(\Sigma\) on \([0, T]\). Then it follows from Lemmas 2.1.18 and 2.4.3 that \(S\) is a single-valued operator with \(\text{dom}(S) = \text{gph}(S)\). Since \(S\) is assumed to be closed, this implies that \(S\) is bounded.

#### 2.4.6. Lemma (cf. Lemma 1.3.9).
Let \(\Sigma = (S; X, Y)\) be an \(i/s/o\) system, and let the \(i/s/o\) system \(\Sigma_{P, Q, R}\) be \((P, Q, R)\)-similar to \(\Sigma\). Then the following claims are true:

(i) \(\Sigma\) is solvable if and only if \(\Sigma_{P, Q, R}\) is solvable.

(ii) \(\Sigma\) has the uniqueness property if and only if \(\Sigma_{P, Q, R}\) has the uniqueness property.

(iii) \(\Sigma\) has the continuation property if and only if \(\Sigma_{1}\) has the continuation property.

#### 2.4.7. Lemma (cf. 1.3.10).
Let \(\Sigma\) and \(\Sigma_{1}\) be two solvable \(s/s\) systems. If \(P\) and \(Q\) are bicontinuous linear operators, and if it is true that \(\begin{bmatrix} x \\ u \end{bmatrix}\) is a classical future trajectory of \(\Sigma\) if and only if \(\begin{bmatrix} P x \\ Q u \end{bmatrix}\) is a classical future trajectory of \(\Sigma_{1}\), then \(\Sigma_{1}\) is \((P, Q)\)-similar to \(\Sigma\).
2.4.8. **Remark** (cf. Remark \[1.3.12\]). All the results that we have presented above about forward solvability or the forward uniqueness property remain valid for backward or two-sided classical solvability, and for the backward and two-sided uniqueness property with some obvious modifications. All the relevant results in the backward case can be obtained from the forward case by a time reflection (see Definition \[2.3.1\]). The results for the two-sided case can be obtained by combining the forward and backward cases, as will be described in more detail below.

2.4.9. **Lemma** (cf. Lemma \[1.3.13\]). An i/s/o system \(\Sigma\) is two-sided solvable if and only if it is both forward and backward solvable.

2.4.10. **Lemma** (cf. Lemma \[1.3.14\]). Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be an i/s/o system, and let \(\Sigma^R = (S^R; \mathcal{X}^R, \mathcal{U}, \mathcal{Y})\) be the time reflection of \(\Sigma\).

(i) \(\Sigma\) is forward solvable if and only if \(\Sigma^R\) is backward solvable.

(ii) \(\Sigma\) has the forward uniqueness property if and only if \(\Sigma^R\) has the backward uniqueness property.

(iii) \(\Sigma\) has the forward continuation property if and only if \(\Sigma^R\) has the backward continuation property.

2.4.11. **Lemma** (cf. Lemma \[1.3.15\]). Let \(\Sigma\) be a solvable system, and let \(\Sigma_1\) be backward a solvable system. If it is true that \([w]\) is a classical future trajectory of \(\Sigma\) if and only if \([\overline{w}]\) is a classical past trajectory of \(\Sigma_1\), then \(\Sigma_1\) is the time reflection of \(\Sigma\).

2.4.12. **Example.** The s/s system \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) in Example \[1.4.1\] can be reinterpreted as a regular i/s/o system \(\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) with the same state space \(\mathcal{X} = L^2(\mathbb{R}^+)\) and \(\mathcal{U} = \mathcal{Y} = \{0\}\). Here

\[
S \varphi := \varphi', \quad \varphi \in \text{dom } (S) := W^{1,2}(\mathbb{R}^+)\]

The classical and generalized trajectories of \(\Sigma_{i/s/o}\) coincide with the classical and generalized trajectories of \(\Sigma\). Thus, \(\Sigma_{i/s/o}\) is two-sided solvable and it has the forward uniqueness property, but not the backward uniqueness property.

2.4.13. **Example.** The s/s system \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) in Example \[1.4.2\] can be reinterpreted as a regular i/s/o system \(\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) with the same state space \(\mathcal{X} = L^2(\mathbb{R}^+)\) and \(\mathcal{U} = \mathcal{Y} = \{0\}\). Here

\[
S \varphi := -\varphi', \quad \varphi \in \text{dom } (S) := W^{1,2}(\mathbb{R}^+)\]

The classical and generalized trajectories of \(\Sigma_{i/s/o}\) coincide with the classical and generalized trajectories of \(\Sigma\). Thus, \(\Sigma_{i/s/o}\) has the two-sided uniqueness property and it is forward solvable but not backward solvable.

2.4.14. **Example.** The s/s system \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) in Example \[1.4.3\] can be reinterpreted as a regular i/s/o system \(\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) with the same state space \(\mathcal{X} = L^2(\mathbb{R}^+)\) and \(\mathcal{U} = \mathcal{Y} = \{0\}\). This is the time reflection of Example \[2.4.13\]. Thus, \(\Sigma_{i/s/o}\) has the two-sided uniqueness property and it is backward solvable, but not forward solvable.

2.4.15. **Example.** The s/s system \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) in Example \[1.4.4\] can be reinterpreted as a regular i/s/o system \(\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) with the same state space \(\mathcal{X} = L^2(\mathbb{R}^+)\) and \(\mathcal{U} = \mathcal{Y} = \{0\}\). This is the time reflection of Example \[2.4.13\]. Thus, \(\Sigma_{i/s/o}\) is two-sided solvable and it has the backward uniqueness property, but not the forward uniqueness property.
2.4.16. Example. The s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ in Example 1.4.5 can be reinterpreted as a regular i/s/o system $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with the same state space $\mathcal{X} = L^2(\mathbb{R}^+), \mathcal{U} = \{0\}$, and $\mathcal{Y} = \mathbb{C}$. Here

$$S\varphi := \begin{bmatrix} \varphi' \\ \varphi(0) \end{bmatrix}, \quad \varphi \in \text{dom}(S) := W^{1,2}(\mathbb{R}^+).$$

The classical and generalized trajectories of $\Sigma_{i/s/o}$ coincide with the classical and generalized trajectories of $\Sigma$ when we take $y = w$. Thus, $\Sigma_{i/s/o}$ is two-sided uniquely solvable.

2.4.17. Example. The s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ in Example 1.4.6 can be reinterpreted as a regular i/s/o system $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with the same state space $\mathcal{X} = L^2(\mathbb{R}^+), \mathcal{U} = \mathbb{C}$, and $\mathcal{Y} = \{0\}$. Here

$$S \left[ \begin{array}{c} \varphi \\ u \end{array} \right] := -\varphi', \quad \varphi \in \text{dom}(S) := \left\{ \begin{bmatrix} \varphi(0) \\ \varphi \in W^{1,2}(\mathbb{R}^+) \end{bmatrix} \right\}.$$

This is the time reflection of Example 2.4.16. Thus, $\Sigma_{i/s/o}$ is two-sided uniquely solvable.

2.4.3. Connections between classical, generalized, and mild trajectories.

2.4.18. Lemma (cf. Lemma 1.3.17). Let $\begin{bmatrix} x \\ u \end{bmatrix}$ be a generalized trajectory of the closed i/s/o system $\Sigma$ on some interval $I$. Then, for all closed finite intervals $[t_1, t_2] \subset I$ we have

$$\int_{t_1}^{t_2} x(s) \, ds \in \text{dom}(S)$$

and

$$\int_{t_1}^{t_2} u(s) \, ds \in S \left[ \int_{t_1}^{t_2} x(s) \, ds \right].$$

2.4.19. Definition (cf. Definition 1.3.18). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o system. We call $\begin{bmatrix} x \\ u \end{bmatrix} \in \left[ C(I;\mathcal{X}) \atop L^1_{loc}(I;\mathcal{Y}) \right]$ a mild trajectory of $\Sigma$ on the interval $I$ if (2.4.3) and (2.4.4) hold for all closed finite subintervals $[t_1, t_2]$ of $I$.

2.4.20. Lemma (cf. Lemma 1.3.19). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a closed i/s/o system. Then every generalized trajectory of $\Sigma$ on some interval $I$ is also a mild trajectory of $\Sigma$ on $I$. In particular, every classical trajectory of $\Sigma$ on $I$ is also a mild trajectory of $\Sigma$ on $I$.

By Lemma 2.4.27 below, the converse of Lemma 2.4.20 is also true when at least one of the end-points of $I$ does not belong to $I$ (i.e., $I$ is semi-open or open).

2.4.21. Lemma (cf. Lemma 1.3.20). Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o system, let $\Sigma_{\text{ext}} = (S_{\text{ext}}; \mathcal{X}, \left[ \begin{array}{c} x \\ 1 \end{array} \right], \mathcal{Y})$ be the bounded input extension of $\Sigma$ with control operator $1_X$ (see Definition 2.3.27), and let $\begin{bmatrix} x \\ u \end{bmatrix} \in \left[ C(I;\mathcal{X}) \atop L^1_{loc}(I;\mathcal{Y}) \right]$ for some interval $I \subset \mathbb{R}$. Then the following conditions are equivalent:

(i) $\begin{bmatrix} x \\ u \end{bmatrix}$ is a mild trajectory of $\Sigma$ on $I$;
(ii) For some \( t_0 \in I \) the pair \( [x_{\text{ext}}^u \ y_{\text{ext}}^y] \) is a classical trajectory of \( \Sigma_{\text{ext}} \) on \( I \), where

\[
x_{\text{ext}}(t) = \int_{t_0}^t x(s) \, ds, \quad u_{\text{ext}}(t) = \left[ \int_{t_0}^t u(s) \, ds \right], \quad \text{and} \quad y_{\text{ext}}(t) = \int_{t_0}^t y(s) \, ds,
\]

\( t \in I \).

(iii) Claim (ii) is true for all \( t_0 \in I \).

2.4.22. **Lemma** (cf. Lemma 1.3.21). Let \( [x \ y] \) be a generalized trajectory of the closed i/s/o system \( \Sigma \) on some interval \( I \), and let \( t_0 \in I \). If \( x(t_0) = 0 \), then \( [x_1 \ y_1] \) defined by

\[
\begin{bmatrix}
x_1(t) \\
u_1(t) \\
y_1(t)
\end{bmatrix} = \int_{t_0}^t \begin{bmatrix} x(s) \\ u(s) \\ y(s) \end{bmatrix} \, ds, \quad t \in I,
\]

is a classical trajectory of \( \Sigma \) on \( I \) with \( x_1(t_0) = 0, \ u_1(t_0) = 0, \) and \( y_1(t_0) = 0 \) and with \([x_1^u \ y_1^y] \in W_c^{1,2}(I; [y_1^y])\) (and hence, in particular, \([x \ y]\) is the derivative in the distribution sense of the classical trajectory \([x_1 \ y_1]\)).

2.4.23. **Lemma** (cf. Lemma 1.3.22). A closed i/s/o system \( \Sigma \) has the uniqueness property if and only if for every \( T > 0 \), every \( x^0 \in X \), and every \([y^1]\) in \( L^1([0,T]; [y^y])\) there is at most one generalized trajectory \([x \ y]\) on \([0,T]\) with the given input \( u \), output \( y \), and initial state \( x(0) = x^0 \).

2.4.24. **Lemma** (cf. Lemma 1.3.23). Let \( \Sigma = (S; X, U, Y) \) be a closed i/s/o system, and let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are intervals satisfying \( I_1 \cap I_2 = \{t_0\} \) and \( t_0 \) is both the right end-point of \( I_1 \) and the left end-point of \( I_2 \). Let \([x_1 \ y_1]\) be a mild trajectory of \( \Sigma \) on \( I_i \), \( i = 1, 2 \). Define \([x \ y]\) on \( I \) by (2.4.1). The \([x \ y]\) is a mild trajectory of \( \Sigma \) on \( I \) if and only if \( x_1(t_0) = x_2(t_0) \).

2.4.25. **Lemma** (cf. 1.3.24). The following claims are true for every closed i/s/o system \( \Sigma = (S; X, U, Y) \):

(i) Let \( I \) be an interval with right end-point \( +\infty \), and let \([x \ y]\) be a generalized trajectory of \( \Sigma \) on \( I \). For each \( n \in \mathbb{N} \), define \([x_n \ y_n]\) by

\[
\begin{bmatrix}
x_n(t) \\
u_n(t) \\
y_n(t)
\end{bmatrix} := n \int_{t_0}^{t+1/n} \begin{bmatrix} x(s) \\ u(s) \\ y(s) \end{bmatrix} \, ds, \quad t \in I.
\]

Then \([x_n \ y_n]\) is a classical trajectory of \( \Sigma \) on \( I \), and \([x_n \ y_n] \to [x \ y]\) in \( L^{1}(X) \) as \( n \to \infty \).

(ii) Let \( I \) be an interval with finite right end-point \( b \), and let \([x \ y]\) be a generalized trajectory of \( \Sigma \) on \( I \cup [b, b+\epsilon) \) for some \( \epsilon > 0 \). For each \( n > 1/\epsilon \), define \([x_n \ y_n]\) by (2.4.5). Then \([x_n \ y_n]\) is a classical trajectory of \( \Sigma \) on \( I \), and \([x_n \ y_n] \to [x \ y]\) in \( L^{1}(X) \) as \( n \to \infty \).
(iii) Let \( I \) be an interval with left end-point \(-\infty\), and let \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) be a generalized trajectory of \( \Sigma \) on \( I \). For each \( n \in \mathbb{N} \), define \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) by

\[
\begin{bmatrix} x_n(t) \\ u_n(t) \\ y_n(t) \end{bmatrix} := n \int_{t-1/n}^{t} \begin{bmatrix} x(s) \\ u(s) \\ y(s) \end{bmatrix} \, ds, \quad t \in I.
\]

Then \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) is a classical trajectory of \( \Sigma \) on \( I \), and \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) in \( \left[ C(I;X), L_{loc}^1(I;\mathcal{U}), L_{loc}^1(I;\mathcal{Y}) \right] \) as \( n \to \infty \).

(iv) Let \( I \) be an interval with finite left end-point \( a \), and let \( \begin{bmatrix} x \\ y \end{bmatrix} \) be a generalized trajectory of \( \Sigma \) on \( [a-\epsilon, a] \cup I \) for some \( \epsilon > 0 \). For each \( n > 1/\epsilon \), define \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) by (2.4.6). Then \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) is a classical trajectory of \( \Sigma \) on \( I \), and \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) in \( \left[ C(I;X), L_{loc}^1(I;\mathcal{U}), L_{loc}^1(I;\mathcal{Y}) \right] \) as \( n \to \infty \).

2.4.26. **Corollary** (cf. Corollary 1.3.25). Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a closed i/s/o system, and let \( I \) be an infinite interval. Then a triple of function \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \in \left[ C(I;X), L_{loc}^1(I;\mathcal{U}), L_{loc}^1(I;\mathcal{Y}) \right] \) is a generalized trajectory of \( \Sigma \) on \( I \) if and only if there exists a sequence \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) of classical trajectories of \( \Sigma \) which converges to \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) in \( \left[ C(I;X), L_{loc}^1(I;\mathcal{U}), L_{loc}^1(I;\mathcal{Y}) \right] \) as \( n \to \infty \).

2.4.27. **Lemma** (cf. Lemma 1.3.26). Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a closed i/s/o system, and let \( I \) be an interval for which at least one of the end point of \( I \) does not belong to \( I \) (i.e., \( I \) is unbounded or semi-open or open). Then every mild trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) is also a generalized trajectory of \( \Sigma \) on \( I \). (That the converse is also true follows from Lemma 2.4.26.)

2.4.28. **Lemma** (cf. Lemma 1.3.27). Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a closed i/s/o system, and let \( I = [t_0, t_1] \) be a finite closed interval. If \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is mild trajectory of \( \Sigma \) on \( I \) and \( c \in (0, t_1-t_0) \), then \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) can be extended to a mild trajectory on the interval \( [t_0, t_1+c] \) or the interval \( [t_0-c, t_1] \).

2.4.29. **Lemma** (cf. Lemma 1.3.28). Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a closed s/o system, and let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are intervals satisfying \( I_1 \cap I_2 = \{t_0\} \) and \( t_0 \) is both the right end-point of \( I_1 \) and the left end-point of \( I_2 \). Let \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) be a generalized trajectory of \( \Sigma \) on \( I_i \), \( i = 1, 2 \), and suppose that at least one of the following conditions hold:

(i) At least one of the intervals \( I_1 \) and \( I_2 \) is infinite or semi-open (i.e., the left end-point of \( I_1 \) does not belong to \( I_1 \), or the right end-point of \( I_2 \) does not belong to \( I_2 \)).
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(ii) \( \begin{bmatrix} x_2 \\ u_2 \\ y_2 \end{bmatrix} \) can be continued to a generalized trajectory on the interval \([t_1 - \epsilon, t_0]\) or \( \begin{bmatrix} x_2 \\ u_2 \\ y_2 \end{bmatrix} \) can be continued to a generalized trajectory on the interval \([t_0, t_2 + \epsilon]\) for some \( \epsilon > 0. \)

Define \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) on \( I \) by (2.4.1). The \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a generalized trajectory of \( \Sigma \) on \( I \) if and only if \( x_1(t_0) = x_2(t_0). \)

2.4.30. Lemma (cf. Lemma 1.3.29). The following claims are true for every closed i/s/o system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \):

(i) Let \( I \) be an interval with finite left end-point \( a \in I \). Then every generalized trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) satisfying \( x(a) = 0 \) can be extended to a generalized trajectory of \( \Sigma \) on \( (-\infty, a) \cup I \) by taking \( \begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} \) = 0 for \( t < a \).

(ii) Let \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) be the extended trajectory of \( \Sigma \) on \( (-\infty, a) \cup I \) considered in (i) which vanishes on \( (-\infty, a) \). For each \( n \in \mathbb{N} \), define \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) by (2.4.6) with \( I \) replaced by \( (-\infty, a) \cup I \). Then \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) is a classical trajectory of \( \Sigma \) on \( (-\infty, a) \cup I, x_n(t) = 0 \) and \( \begin{bmatrix} x_n(t) \\ u_n(t) \\ y_n(t) \end{bmatrix} \) = 0 for all \( t \leq a \) and all \( n \in \mathbb{N} \), and \( \begin{bmatrix} x_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} \) in \( \mathcal{C}((-\infty, a) \cup I; \mathcal{X}) \) with \( I \) replaced by \( (-\infty, a) \cup I \) as \( n \to \infty \).

(iii) Let \( I \) be an interval with finite right end-point \( b \in I \). Then every generalized trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) satisfying \( x(b) = 0 \) can be extended to a generalized trajectory of \( \Sigma \) on \( I \cup [b, \infty) \) by taking \( \begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} \) = 0 for \( t > b \).

(iv) Let \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) be the extended trajectory of \( \Sigma \) on \( I \cup [b, \infty) \) considered in (i) which vanishes on \( [b, \infty) \). For each \( n \in \mathbb{N} \), define \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) by (2.4.5) with \( I \) replaced by \( I \cup [b, \infty) \). Then \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) is a classical trajectory of \( \Sigma \) on \( I \cup [b, \infty), x_n(t) = 0 \) for all \( t \geq b \) and all \( n \in \mathbb{N} \), and \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) in \( \mathcal{C}([b, \infty); \mathcal{X}) \) with \( I \) replaced by \( [b, \infty) \) as \( n \to \infty \).

2.4.31. Lemma (cf. Lemma 1.3.30). Let \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) be a generalized trajectory of the i/s/o system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) on the interval \([a, b]\).

(i) If for some \( t \in [a, b] \) both

\[
(2.4.7) \quad z_t := \lim_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} (x(s) - x(t)) ds \quad \text{and} \quad \begin{bmatrix} u_t \\ y_t \end{bmatrix} := \lim_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} ds
\]

exist, then \( \begin{bmatrix} x(t) \\ u_t \\ y_t \end{bmatrix} \in \text{dom}(S) \) and \( z_t = S \left[ \begin{bmatrix} x(t) \\ u_t \\ y_t \end{bmatrix} \right] \).

(ii) If for some \( t \in [a, b] \) both

\[
(2.4.8) \quad z_t := \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} (x(s) - x(t - h)) ds \quad \text{and} \quad \begin{bmatrix} u_t \\ y_t \end{bmatrix} := \lim_{h \to 0^+} \frac{1}{h} \int_{t-h}^{t} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} ds
\]
exist, then \([x(t)]_{t_u} \in \text{dom}(S)\) and \([z_t]_{t_u} = S [x(t)]_{t_u}\).

2.4.32. **Theorem** (cf. Theorem 1.3.31). Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a closed i/s/o system. Then a generalized trajectory \([u_y]\) of \(\Sigma\) on some interval \(I\) is classical if and only if \([u_y] \in \left[ C^1(I; \mathcal{X}) \atop C(I; \mathcal{U}) \atop C(I; \mathcal{Y}) \right] \) (i.e., if and only if it has the necessary smoothness in order to be a classical trajectory).

2.4.33. **Corollary** (cf. Corollary 1.3.32). The system operator \(S\) of a closed solvable i/s/o system \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) is uniquely determined by the set of generalized future trajectories of \(\Sigma\).
2.5. Dynamical Properties of Input/State/Output Systems (Jan 02, 2016)

In this section we discuss a number of properties of i/s/o systems which defined with the help of the set of classical and generalized trajectories of the system, such as controllability, observability, intertwinements, restrictions, projections, and compressions. These properties are allowed to depend on the direction of time, i.e., they need not be preserved under a time reflection.

2.5.1. Controllability and observability of i/s/o systems.

2.5.1. Definition (cf. Definition 1.5.1). Let \( \Sigma = (S; X, U, Y) \) be an i/s/o system.

(i) A state vector \( x^0 \in X \) is called (forward) classically reachable if there exists a classical past trajectory \( [x u y] \) of \( \Sigma \) with compact support such that \( x(0) = x^0 \).

(ii) A state vector \( x^0 \in X \) is called (forward) exactly reachable if there exists a generalized past trajectory \( [x u y] \) of \( \Sigma \) with compact support such that \( x(0) = x^0 \).

(iii) A classical or generalized future trajectory \( [x u y] \) of \( \Sigma \) is called (forward) unobservable if both \( u = 0 \) and \( y = 0 \).

(iv) A state vector \( x^0 \in X \) is called (forward) classically unobservable if there exists a classical future unobservable trajectory \( [x 0 y] \) of \( \Sigma \) with \( x(0) = x^0 \).

(v) A state vector \( x^0 \in X \) is called (forward) unobservable if there exists a generalized future unobservable trajectory \( [x 0 y] \) of \( \Sigma \) with \( x(0) = x^0 \).

2.5.2. Lemma. Lemma 1.5.2 remains valid in the i/s/o setting.

2.5.3. Definition (cf. Definition 1.5.3). Let \( \Sigma = (S; X, U, Y) \) be an i/s/o system.

(i) The subspace of all classically reachable states of \( \Sigma \) is called the classically reachable subspace of \( \Sigma \) and it is denoted by \( \mathcal{R}_\Sigma^{\text{class}} \).

(ii) The subspace of all exactly reachable states of \( \Sigma \) is called the exactly reachable subspace of \( \Sigma \) and it is denoted by \( \mathcal{R}_\Sigma^{\text{exact}} \).

(iii) The closure of \( \mathcal{R}_\Sigma^{\text{class}} \) is called the (approximately) reachable subspace of \( \Sigma \) and it is denoted by \( \mathcal{R}_\Sigma \).

(iv) The subspace of all classically unobservable states of \( \Sigma \) is called the classically unobservable subspace of \( \Sigma \) and it is denoted by \( \mathcal{U}_\Sigma^{\text{class}} \).

(v) The subspace of all unobservable states of \( \Sigma \) is called the unobservable subspace of \( \Sigma \), and it is denoted by \( \mathcal{U}_\Sigma \).

2.5.4. Lemma (cf. Lemma 1.5.4). If \( \Sigma = (S; X, U, X) \) is closed, then the subspaces defined above have the following properties:

(i) \( \mathcal{R}_\Sigma^{\text{class}} \subset \mathcal{R}_\Sigma^{\text{exact}} \subset \mathcal{R}_\Sigma \). Thus \( \mathcal{R}_\Sigma \) is also the closure of \( \mathcal{R}_\Sigma^{\text{exact}} \).

(ii) \( \mathcal{U}_\Sigma^{\text{class}} \subset \mathcal{U}_\Sigma \) and \( \mathcal{U}_\Sigma \subset \mathcal{U}_\Sigma^{\text{class}} \).

2.5.5. Lemma (cf. Lemma 1.5.5). For each i/s/o system \( \Sigma = (S; X, U, Y) \) the following claims are true:
2.15.7. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a solvable i/s/o system. Then \( x^0 \in \mathcal{R}_\Sigma^{class} \) if and only if there exists a classical trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on some interval \([0, T]\) with \( x(0) = 0, u(0) = 0, \) and \( y(0) = 0 \) such that \( x^0 = x(T) \).

2.15.8. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a solvable i/s/o system. Then \( x^0 \in \mathcal{R}_\Sigma^{exact} \) if and only if there exists a classical trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on some interval \([0, T]\) with \( x(0) = 0, u(0) = 0, \) and \( y(0) = 0 \) such that \( x^0 = x(T) \).

2.15.9. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a closed i/s/o system with exactly reachable subspace \( \mathcal{R}_\Sigma^{exact} \) and unobservable subspace \( \mathcal{U}_\Sigma \).

2.15.10. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a closed i/s/o system.

(i) For each closed subspace \( \mathcal{Z} \) of \( \mathcal{X} \) there exists a unique minimal closed strongly invariant subspace \( \mathcal{Z}_{min} \) for \( \Sigma \) which contains \( \mathcal{Z} \), i.e., \( \mathcal{Z}_{min} \) is closed and strongly invariant for \( \Sigma \) and \( \mathcal{Z}_{min} \) is contained in every other closed strongly invariant closed subspace for \( \Sigma \) which contains \( \mathcal{Z} \). Moreover, \( \mathcal{R}_\Sigma \subseteq \mathcal{Z}_{min} \), where \( \mathcal{R}_\Sigma \) is the (approximately) reachable subspace of \( \Sigma \).
(ii) For each subspace $Z$ of $X$ there exists a unique maximal unobservably invariant subspace $Z_{\text{max}}$ for $\Sigma$ which is contained in $Z$, i.e., $Z_{\text{max}}$ is unobservably invariant for $\Sigma$ and $Z_{\text{max}}$ contains every other unobservably invariant subspace for $\Sigma$ which is contained in $Z$. Moreover, $Z_{\text{max}} \subseteq \mathcal{U}_\Sigma$, where $\mathcal{U}_\Sigma$ is the unobservable subspace $\Sigma$.

2.5.11. Definition (cf. Definition 1.5.11). Let $\Sigma = (S; X, U, Y)$ be an i/s/o system.

(i) $\Sigma$ is (forward approximately) controllable if $\mathcal{R}_\Sigma = X$.

(ii) $\Sigma$ is called (forward approximately) observable if $\mathcal{U}_\Sigma = \{0\}$.

2.5.12. Corollary (cf. Corollary 1.5.12). Let $\Sigma = (S; X, U, Y)$ be a closed i/s/o system.

(i) If $\Sigma$ is controllable, then $X$ does not contain any proper closed strongly invariant subspace (proper means that this subspace is strictly contained in $X$).

(ii) If $\Sigma$ is observable, then $X$ does not contain any nonzero unobservably invariant subspace.

2.5.13. Lemma. Lemma 1.5.13 remains valid (with obvious modifications) in the i/s/o setting.

2.5.14. Definition (cf. Definition 1.5.14). Let $\Sigma = (S; X, U, Y)$ be an i/s/o system.

(i) By the backward reachable subspace and the backward unobservable subspace of $\Sigma$ we mean the reachable subspace respectively the unobservable subspace of the time reflection $\Sigma^R$ of $\Sigma$.

(ii) A subspace $Z$ of $X$ is backward strongly invariant or backward unobservably invariant for $\Sigma$ if $Z$ is strongly invariant respectively unobservably invariant for the time reflection $\Sigma^R$ of $\Sigma$.

(iii) $\Sigma$ is backward controllable or backward observable if the time reflection $\Sigma^R$ of $\Sigma$ is controllable respectively observable.

2.5.15. Example (cf. Example 1.5.15). The i/s/o system $\Sigma_{i/s/o} = (S; X, U, Y)$ in Example 2.4.12 with $X = L^2(\mathbb{R}^+)$ and $U = Y = \{0\}$ has the same classical and generalized trajectories as the s/s system $\Sigma = (V; X, W)$ in Examples 1.4.7 and 1.5.15. Thus, this example is neither forward nor backward observable and it is not forward controllable, but it is backward controllable, in spite of the fact that its signal space is $\{0\}$. Recall that this example does not have the backward uniqueness property.

2.5.16. Example (cf. Example 1.5.16). The i/s/o system $\Sigma_{i/s/o} = (S; X, U, Y)$ in Example 2.4.13 with $X = L^2(\mathbb{R}^+)$ and $U = Y = \{0\}$ has the same classical and generalized trajectories as the s/s system $\Sigma = (V; X, W)$ in Examples 1.4.2 and 1.5.16. Thus, this example is neither forward nor backward controllable and it is not forward observable, but it is backward observable, in spite of the fact that its signal space is $\{0\}$. Recall that this example is not backward solvable.

2.5.17. Example (cf. Example 1.5.17). The i/s/o system $\Sigma_{i/s/o} = (S; X, U, Y)$ in Example 2.4.14 with $X = L^2(\mathbb{R}^+)$ and $U = Y = \{0\}$ has the same classical and generalized trajectories as the s/s system $\Sigma = (V; X, W)$ in Examples 1.4.3 and 1.5.17. Thus, this example is neither forward nor backward controllable and it is
not backward observable, but it is forward observable, in spite of the fact that its signal space is \{0\}. Recall that this example is not forward solvable.

2.5.18. **Example (cf. Example 1.5.18).** The i/s/o system \(\Sigma_{i/s/o} = (S; X, U, Y)\) in Example 2.4.17 with \(X = L^2(\mathbb{R}^+)\) and \(U = \{0\}\) has the same classical and generalized trajectories as the s/s system \(\Sigma = (V; X, W)\) in Examples 1.4.4 and 1.5.18. Thus, this example is neither forward nor backward observable and it is not backward controllable, but it is forward controllable, in spite of the fact that its signal space is \{0\}. Recall that this example does not have the forward uniqueness property.

2.5.19. **Example (cf. Example 1.5.19).** The i/s/o system \(\Sigma_{i/s/o} = (S; X, U, Y)\) in Example 2.4.16 with \(X = L^2(\mathbb{R}^+)\), \(U = \{0\}\) and \(Y = \mathbb{C}\) has the same classical and generalized trajectories as the s/s system \(\Sigma = (V; X, W)\) in Examples 1.4.5 and 1.5.19. Thus, this system is forward observable and backward controllable, but it is not forward controllable and not backward observable. Recall that this example does not have the forward uniqueness property.

2.5.20. **Example (cf. Example 1.5.20).** The i/s/o system \(\Sigma_{i/s/o} = (S; X, U, Y)\) in Example 2.4.17 with \(X = L^2(\mathbb{R}^+)\), \(U = \{0\}\) and \(Y = \mathbb{C}\) has the same classical and generalized trajectories as the s/s system \(\Sigma = (V; X, W)\) in Examples 1.4.6 and 1.5.19. Thus, this example is forward controllable and backward observable, but it is not forward observable and not backward controllable. Recall that this example is two-sided uniquely solvable. This system is therefore controllable but not observable.

2.5.2. **Intertwinements and compressions of input/state/output systems.**

2.5.21. **Definition (cf. Definition 1.5.21).** Two i/s/o systems \(\Sigma_i = (S_i; X_i, U, Y), i = 1, 2\), (with the same input and output spaces) are *externally equivalent* if they satisfy the following condition for all intervals \(I\) of the type \([0, T]\) (where \(T > 0\)) as well as for \(I = \mathbb{R}^+\): \(\Sigma_1\) has a generalized trajectory \(\begin{bmatrix} x_1 \\ u \\ y \end{bmatrix}\) on \(I\) with zero initial state \(x_1(0) = 0\), input function \(u \in L_{loc}^1(I; U)\), and output function \(y \in L_{loc}^1(I; Y)\) if and only if \(\Sigma_2\) has a generalized trajectory \(\begin{bmatrix} x_2 \\ u \\ y \end{bmatrix}\) on \(I\) with zero initial state \(x_2(0) = 0\) and the same input and output functions \(u\) and \(y\).

2.5.22. **Definition (cf. Definition 1.5.22).** Let \(\Sigma_i = (S_i; X_i, U, Y), i = 1, 2\), be two i/s/o systems (with the input and output spaces), and let \(P \in \mathcal{ML}(X_1; X_2)\). We say that \(\Sigma_1\) and \(\Sigma_2\) are *intertwined by \(P\)* if the following two conditions hold for all intervals \(I\) of the type \([0, T]\) (where \(T > 0\)) and for \(I = \mathbb{R}^+\):

(i) If \(\begin{bmatrix} x_1 \\ u \\ y \end{bmatrix}\) is a generalized trajectory of \(\Sigma_1\) on \(I\) with \(x_1(0) \in \text{dom}(P)\), then for every \(x_2^0 \in P x_1(0)\) there exists a generalized trajectory \(\begin{bmatrix} x_2 \\ u \\ y \end{bmatrix}\) of \(\Sigma_2\) on \(I\) satisfying \(x_2(0) = x_2^0\) and \(x_2(t) \in Px_1(t)\) for all \(t \in \mathbb{R}^+\).

(ii) Condition (i) above also holds if we interchange \(\Sigma_1\) and \(\Sigma_2\) and replace \(P\) by \(P^{-1}\). In other words, if \(\begin{bmatrix} x_2 \\ u \\ y \end{bmatrix}\) is a generalized trajectory of \(\Sigma_2\) on \(I\) with \(x_2(0) \in \text{rng}(P)\), then for every \(x_1^0 \in P^{-1}x_2(0)\) there exists a generalized trajectory \(\begin{bmatrix} x_1 \\ u \\ y \end{bmatrix}\) of \(\Sigma_1\) on \(I\) satisfying \(x_1(0) = x_1^0\) and \(x_2(t) \in Px_1(t)\) for all \(t \in \mathbb{R}^+\).
2.5.23. **Definition** (cf. Definition 2.5.23). Let \( \Sigma_i = (S_i; X_i, U, Y) \), \( i = 1, 2 \), be two i/s/o systems (with the same input and output spaces). We say that \( \Sigma_1 \) and \( \Sigma_2 \) are **pseudo-similar** if they are intertwined by a closed injective (single-valued) linear operator \( P: X \to X_1 \) with dense domain and dense range, called the **pseudo-similarity operator**.

2.5.24. **Lemma** (cf. Lemma 1.5.24). Let \( \Sigma_i = (S_i; X_i, U, Y) \), \( i = 1, 2 \), be two i/s/o systems (with the same input and output spaces).

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \in \mathcal{ML}(X_1; X_2) \) if and only if \( \Sigma_2 \) and \( \Sigma_1 \) are intertwined by \( P^{-1} \).

(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar with pseudo-similarity operator \( P \) if and only if \( \Sigma_2 \) and \( \Sigma_1 \) are pseudo-similar with pseudo-similarity operator \( P^{-1} \).

**Proof.** This follows directly from Definition 2.5.23.

2.5.25. **Lemma** (cf. Lemma 1.5.25). Let \( \Sigma_i = (S_i; X_i, U, Y) \), \( i = 1, 2, 3 \), be three i/s/o systems. If \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P_1 \in \mathcal{ML}(X_1; X_2) \) and \( \Sigma_2 \) and \( \Sigma_3 \) are intertwined by \( P_2 \in \mathcal{ML}(X_2; X_3) \), then \( \Sigma_1 \) and \( \Sigma_3 \) are intertwined by \( P_3 := P_2 P_1 \).

**Proof.** The proof is analogous to the proof of Lemma 1.5.25.

2.5.26. **Lemma** (cf. Lemma 2.5.26). Let \( \Sigma_i = (S_i; X_i, U, Y) \) be two solvable i/s/o systems, and let \( P \in \mathcal{B}(X_1; X_2) \) have an inverse in \( \mathcal{B}(X_2; X_1) \). Then \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \) if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are \((P, 1_{tr}, 1_{tr'})\)-similar (see Definition 2.3.11).

**Proof.** The proof is analogous to the proof of Lemma 1.5.26.

2.5.27. **Lemma** (cf. Lemma 2.5.27). Let \( \Sigma_i = (S_i; X_i, U, Y) \) be two closed i/s/o systems with exactly reachable subspaces \( R_{\Sigma_i}^{\text{exact}} \) and unobservable subspaces \( U_{\Sigma_i} \), \( i = 1, 2 \). If \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \in \mathcal{ML}(X_1; X_2) \), then the following claims hold:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

(ii) \( \text{dom}(P) \) is strongly invariant for \( \Sigma_1 \). In particular, \( R_{\Sigma_1}^{\text{exact}} \subset \text{dom}(P) \).

(iii) \( \text{rng}(P) \) is strongly invariant for \( \Sigma_2 \). In particular, \( R_{\Sigma_2}^{\text{exact}} \subset \text{rng}(P) \).

(iv) \( \text{ker}(P) \) is unobservably invariant for \( \Sigma_1 \). In particular, \( \text{ker}(P) \subset U_{\Sigma_1} \).

(v) \( \text{mul}(P) \) is unobservably invariant for \( \Sigma_2 \). In particular, \( \text{mul}(P) \subset U_{\Sigma_2} \).

**Proof.** The proof is analogous to the proof of Lemma 1.5.27.

2.5.28. **Definition** (cf. Definition 1.5.28). Let \( \Sigma_i = (S_i; X_i, U, Y) \), \( i = 1, 2 \), be two i/s/o systems (with the same input and output spaces), where \( X_1 \) is a closed subspace of \( X_2 \), and let \( Z_1 \) be a direct complement to \( X_1 \) in \( X_2 \). We call \( \Sigma_1 \) a **compression** of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \), and we call \( \Sigma_2 \) a **dilation** of \( \Sigma_1 \) along \( Z_1 \), if the following two condition holds for all intervals \( I \) of the type \( I = [0, T] \) (where \( T > 0 \)) and for \( I = \mathbb{R}^+ \):

(i) If \( \begin{bmatrix} x_2 \\ u \\ y \end{bmatrix} \) is a generalized trajectory of \( \Sigma_2 \) on \( I \) with \( x_2(0) \in X_1 \), then
\[
\begin{bmatrix} x_{2,T} \\ u \\ y \end{bmatrix} = P_{X_1}^{I} \begin{bmatrix} x_2 \\ u \\ y \end{bmatrix}
\]
is a generalized trajectory of \( \Sigma_1 \) on \( I \).
holds for all intervals $I$ of $S_i$ on $I$ there exists some generalized trajectory $[\begin{bmatrix} x_1 \\ u \\ y \end{bmatrix}]$ of $S_2$ on $I$ with $x_2(0) = x_1(0) \in X_1$ such that $x_1 = P_{X_1}^S x_2$.

2.5.29. **Lemma** (cf. Lemma 1.5.29). If the i/s/o system $S_1 = (S_1; X_1, U, Y)$ is the compression of the i/s/o system $S_2 = (S_2; X_2, U, Y)$ onto $X_1$ along $Z_i$, then $S_2$ and $S_1$ are externally equivalent.

**Proof.** This follows immediately from Definitions 2.5.21 and 2.5.28.

2.5.30. **Lemma** (cf. Lemma 1.5.30). Let $S_i = (S_i; X_i, U, Y)$ be three i/s/o systems (with the same input and output spaces). If $S_2$ is the compression of $S_3$ onto $X_2$, and if $S_1$ is the compression of $S_2$ onto $X_1$, then $S_1$ is the compression of $S_3$ onto $X_1$ along $Z_1 = Z_2$.

**Proof.** The proof is analogous to the proof of Lemma 1.5.30.

2.5.31. **Lemma** (cf. Lemma 1.5.31). Let $S_2 = (S_2; X_2, U, Y)$ be a closed i/s/o system, and let $X_2 = X_1 + Z_1$.

(i) $S_2$ has at most one closed solvable compression onto $X_1$ along $Z_1$.

(ii) If $S_2$ has the uniqueness property, then every closed compression of $S_2$ onto $X_1$ along $Z_1$ has the uniqueness property.

**Proof.** The proof is analogous to the proof of Lemma 1.5.31.

2.5.32. **Lemma** (cf. Lemma 1.5.32). Let the closed i/s/o system $S_1$ be a compression of the closed i/s/o system $S_2$ onto $X_1$ along $Z_1$. For $i = 1, 2$ we denote the unobservable subspace of $S_i$ by $U_{X_i} = X_i$, the exactly reachable subspace of $S_i$ by $R_{X_i}\Sigma_1$, and the reachable subspace of $S_i$ by $\mathcal{R}_{X_i}$. Then

(i) $U_{X_1} = U_{X_2} \cap X_1$. In particular, $S_1$ is observable whenever $S_2$ is observable.

(ii) $R_{X_1}^{\Sigma_1} = P_{X_1}^S R_{X_2}^{\Sigma_2}$ and $\mathcal{R}_{X_1} = P_{X_1}^S \mathcal{R}_{X_2}$. In particular, $S_1$ is controllable whenever $S_2$ is controllable.

**Proof.** The proof is analogous to the proof of Lemma 1.5.32.

2.5.33. **Definition** (cf. Definition 1.5.33). Let $S_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two i/s/o systems (with the same input and output spaces), where $X_1$ is a closed subspace of $X_2$. We call $S_i$ a restriction of $S_2$ to $X_1$ if the following two condition holds for all intervals $I$ of the type $[0, T]$, where $T > 0$, and for $I = \mathbb{R}^+$:

(i) Every generalized trajectory of $S_1$ on $I$ is also a generalized trajectory of $S_2$ on $I$.

(ii) If $[\begin{bmatrix} x_2 \\ u \\ y \end{bmatrix}]$ is a generalized trajectory of $S_2$ on $I$ with $x_2(0) \in X_1$, then $x_2(t) \in X_1$ for all $t \in I$, and $[\begin{bmatrix} x_2 \\ u \\ y \end{bmatrix}]$ is also a generalized trajectory of $S_1$ on $I$.

2.5.34. **Lemma** (cf. Lemma 1.5.34). Let $S_i = (S_i; X_i, U, Y)$ be three i/s/o systems. If $S_2$ is the restriction of $S_3$ to $X_2$ and $S_1$ is the restriction of $S_2$ to $X_1$, then $S_1$ is the restriction of $S_3$ to $X_1$.

**Proof.** The proof is analogous to the proof of Lemmas 1.5.30 and 1.5.34.
2.5.35. Lemma (cf. Lemma 1.5.35). Let the i/s/o system \( \Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) be the restriction to \( \mathcal{X}_1 \) of the i/s/o system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), and suppose that \( \Sigma \) is closed and solvable and that \( \Sigma_1 \) is closed. Then \( \Sigma \) is solvable and

\[
(2.5.1) \quad \text{gph}(S_1) = \text{gph}(S) \cap \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{U} \end{array} \right] = \text{gph}(S) \cap \left[ \begin{array}{c} \mathcal{X} \\ \mathcal{U} \end{array} \right].
\]

Thus, in particular, \( \Sigma_1 \) is the part of \( \Sigma \) in \( \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{U} \end{array} \right] \).

\textbf{Proof.} The proof is analogous to the proof of Lemma 2.5.35.

\boxend{2.5.35}

2.5.36. Lemma (cf. Lemma 1.5.36). Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \), \( i = 1, 2 \), be two closed i/s/o systems (with the same input and output spaces), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \). Let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) is a restriction of \( \Sigma_2 \) to \( \mathcal{X}_1 \).

(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by the embedding operator \( \mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \).

(iii) \( \mathcal{X}_1 \) is a strongly invariant subspace for \( \Sigma_2 \), and \( \Sigma_1 \) is the compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

\textbf{Proof.} The proof is analogous to the proof of Lemma 1.5.36.

\boxend{2.5.36}

2.5.37. Definition (cf. Definition 1.5.37). Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \), \( i = 1, 2 \), be two i/s/o systems (with the same input and output spaces), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), and let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X} \). We call \( \Sigma_1 \) a (dynamic) projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) if the following two condition holds for all intervals \( I \) of the type \( I = [0, T] \) (where \( T > 0 \)) and for \( \mathcal{Z} = \mathbb{R}^+ \):

(i) If \( \left[ \begin{array}{c} x_2 \\ u \\ y \end{array} \right] \) is a generalized trajectory of \( \Sigma_2 \) on \( I \), then \( \left[ \begin{array}{c} P^x_{\mathcal{X}_1} x_2 \\ u \\ y \end{array} \right] \) is a generalized trajectory of \( \Sigma_1 \) on \( I \).

(ii) If \( \left[ \begin{array}{c} z_2 \\ u \\ y \end{array} \right] \) is a generalized trajectory of \( \Sigma_1 \) on \( I \), then for each \( x_2^0 \in \mathcal{X}_2 \) satisfying \( P^x_{\mathcal{X}_1} x_2^0 = x_1(0) \) there exists a generalized trajectory \( \left[ \begin{array}{c} x_2^0 \\ u \\ y \end{array} \right] \) of \( \Sigma_2 \) on \( I \) satisfying \( x_2(0) = x_2^0 \) and \( P^x_{\mathcal{X}_1} x_2 = x_1 \).

2.5.38. Lemma (cf. Lemma 1.5.38). Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \) be three i/s/o systems. If \( \Sigma_2 \) is the projection of \( \Sigma_3 \) onto \( \mathcal{X}_2 \) along \( \mathcal{Z}_2 \) and \( \Sigma_1 \) is the projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \), then \( \Sigma_1 \) is the projection of \( \Sigma_3 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 + \mathcal{Z}_2 \).

\textbf{Proof.} The proof is analogous to the proof of Lemmas 1.5.30 and 1.5.38.

\boxend{2.5.38}

2.5.39. Lemma (cf. Lemma 1.5.40). Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \), \( i = 1, 2 \), be two closed i/s/o systems (with the same input and output spaces), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), and let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) is a projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(ii) \( \Sigma_2 \) and \( \Sigma_1 \) are intertwined by the projection operator \( P^x_{\mathcal{X}_1} \).

(iii) \( \mathcal{Z}_1 \) is an unobservably invariant subspace for \( \Sigma_2 \), and \( \Sigma_1 \) is the compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

\textbf{Proof.} The proof is analogous to the proof of Lemma 1.5.40.

\boxend{2.5.39}
2.5.40. Lemma (cf. Lemma 1.5.39). Let the i/s/o system $\Sigma_1 = (S_1; X_1, U, Y)$ be the projection onto $X_1$ along $Z_1$ of the i/s/o system $\Sigma = (S; X, U, Y)$, and suppose that $\Sigma$ is solvable and $\Sigma_1$ is closed. Then

$$\text{graph}(S_1) \subset \begin{bmatrix} P_{X_1} & 0 & 0 & 0 \\ 0 & 1_Y & 0 & 0 \\ 0 & 0 & P_{Z_1} & 0 \\ 0 & 0 & 0 & 1_U \end{bmatrix} \text{graph}(S).$$

Proof. The proof is analogous to the proof of Lemma 1.5.39. □

2.5.41. Definition (cf. Definition 1.5.41). (i) An i/s/o system $\Sigma = (S; X, U, Y)$ is minimal if it does not have any non-trivial compression. (ii) By a minimal compression of an i/s/o system $\Sigma$ we mean a compression $\Sigma_1$ of $\Sigma$ which is minimal (i.e., $\Sigma_1$ does not have any further non-trivial compressions).

Existence of minimal compressions will be proved later for two special classes of i/s/o systems, namely bounded i/s/o systems (see Chapter 3), and well-posed i/s/o systems (see Chapter 8).

2.5.42. Definition (cf. Definition 1.5.42). Let $\Sigma = (S; X, U, Y)$ and $\Sigma_i = (S_i; X_i, U_i, Y_i)$, $i = 1, 2$, be i/s/o systems. The following “backward” notions are defined by applying the corresponding “forward” definitions to the time reflected systems $\Sigma_R$ and $\Sigma_i R$ respectively $\Sigma_i$, $i = 1, 2$:

(i) backward external equivalence of $\Sigma_1$ and $\Sigma_2$ (cf. Definition 2.5.21);
(ii) backward intertwinement of $\Sigma_1$ and $\Sigma_2$ (cf. Definition 2.5.22);
(iii) backward bounded intertwinement and backward pseudo-similarity of $\Sigma_1$ and $\Sigma_2$ (cf. Definition 2.5.23);
(iv) backward compression of $\Sigma_2$ onto $X_1$ along $Z_1$ (cf. Definition 2.5.28);
(v) backward restriction of $\Sigma_2$ onto $X_1$ (cf. Definition 2.5.33);
(vi) backward projection of $\Sigma_2$ onto $X_1$ along $Z_1$ (cf. Definition 2.5.33);
(vii) backward minimality of $\Sigma$ (cf. Definition 2.5.41);

2.5.3. Consequences of the continuation property.

2.5.43. Definition (cf. Definition 1.5.43). By the future behavior of an i/s/o system $\Sigma = (S; X, U, Y)$ we mean the set of all pairs $[x; y] \in L^1_{\text{loc}}(\mathbb{R}^+; X) \times L^1_{\text{loc}}(\mathbb{R}^+; Y)$ for which there exists some $x \in C_0(\mathbb{R}^+; X)$ (with $x(0) = 0$) such that $[x; y]$ is a generalized future trajectory of $\Sigma$.

2.5.44. Lemma (cf. Lemma 1.5.44). Let $\Sigma = (S; X, U, Y)$ and $\Sigma_i = (S_i; X_i, U_i, Y_i)$, $i = 1, 2$, be two closed i/s/o systems with the continuation property.

(i) $x^0 \in P_{\Sigma_i}^{\text{exact}}$ if and only if there exists an generalized future trajectory $[x; y]$ of $\Sigma$ with $x(0) = 0$ such that $x^0 = x(t)$ for some $t \in \mathbb{R}^+$.
(ii) A subspace $Z$ of $X$ is strongly invariant for $\Sigma$ if and only if every generalized future trajectory $[x; y]$ of $\Sigma$ with $x(0) \in Z$ satisfies $x(t) \in Z$ for all $t \in \mathbb{R}^+$.
(iii) $\Sigma_1$ and $\Sigma_2$ are externally equivalent if and only if they have the same future behavior.
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PROOF. This follows more or less directly from the relevant definitions. □

2.5.45. Lemma (cf. Lemma 1.5.45). Let \( \Sigma_i = (S_i; X_i, U, \mathcal{Y}) \), \( i = 1, 2 \), be two closed i/s/o systems, and let \( P \in \text{ML}(X_1; X_2) \).

(i) If \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \), then the following claims are true:
   (a) If \( \Sigma_1 \) has the continuation property, then every generalized trajectory \( \begin{bmatrix} x_1 \\ y_1 \\ u_1 \\ y_2 \end{bmatrix} \) of \( \Sigma_2 \) on some interval \([0, T]\) with \( x_2(0) \in \text{rng}(P) \) can be continued to a generalized trajectory of \( \Sigma_1 \) on \( \mathbb{R}^+ \), and conversely, if \( \Sigma_2 \) has the continuation property, then every generalized trajectory \( \begin{bmatrix} x_1 \\ y_1 \\ u_1 \\ y_2 \end{bmatrix} \) of \( \Sigma_1 \) on some interval \([0, T]\) with \( x_1(0) \in \text{dom}(P) \) can be continued to a generalized trajectory of \( \Sigma_1 \) on \( \mathbb{R}^+ \).

(b) If \( \Sigma_1 \) has the continuation property and \( \text{rng}(P) = X_2 \), then \( \Sigma_2 \) has the continuation property, and conversely, if \( \Sigma_2 \) has the continuation property and \( \text{dom}(P) = X_1 \), then \( \Sigma_1 \) has the continuation property.

(ii) If both \( \Sigma_1 \) and \( \Sigma_2 \) have the continuation property, then \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some \( P \in \text{ML}(X_1; X_2) \) if and only if conditions (i) and (ii) in Definition 2.5.28 hold for \( I = \mathbb{R}^+ \).

PROOF. The proof is analogous to the proof of Lemma 2.5.45. □

2.5.46. Lemma (cf. Lemma 1.5.46). Let \( \Sigma = (S; X, U, \mathcal{Y}) \) and \( \Sigma_i = (S_i; X_i, U, \mathcal{Y}) \), \( i = 1, 2 \), be closed i/s/o systems, where \( X_1 \) is a closed subspace of \( X_2 \) with a direct complement \( Z_1 \).

(i) If \( \Sigma_2 \) has the continuation property and \( \Sigma_1 \) is a restriction of \( \Sigma_2 \) to \( X_1 \), or a projection or compression of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \), then \( \Sigma_1 \) has the continuation property.

(ii) If both \( \Sigma_1 \) and \( \Sigma_2 \) have the continuation property, then
   (a) \( \Sigma_1 \) is the compression of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if conditions (i) and (ii) in Definition 2.5.28 hold for \( I = \mathbb{R}^+ \).
   (b) \( \Sigma_1 \) is the restriction of \( \Sigma_2 \) to \( X_1 \) if and only if conditions (i) and (ii) in Definition 2.5.28 hold for \( I = \mathbb{R}^+ \).
   (c) \( \Sigma_1 \) is the projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if conditions (i) and (ii) in Definition 2.5.28 hold for \( I = \mathbb{R}^+ \).

PROOF. The proof is analogous to the proof of Lemma 1.5.46. □

2.5.47. Lemma (cf. Lemma 1.5.47). Let \( \Sigma = (S; X, U, \mathcal{Y}) \) be a closed i/s/o system which has either the forward or the backward continuation property. Let \( I = I_1 \cup I_2 \), where \( I_1 \) and \( I_2 \) are closed intervals satisfying \( I_1 \cap I_2 = \{t_0\} \) and \( t_0 \) is both the right end-point of \( I_1 \) and the left end-point of \( I_2 \). Let \( \begin{bmatrix} x_i \\ u_i \\ y_i \end{bmatrix} \) be a generalized trajectory of \( \Sigma \) on \( I_i \), \( i = 1, 2 \). Define \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) on \( I \) by (2.4.1). Then \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a generalized trajectory of \( \Sigma \) on \( I \) if and only if \( x_i(t_0) = x_2(t_0) \).

PROOF. The proof if analogous to the proof of Lemma 1.5.47. □

2.5.48. Lemma (cf. Lemma 1.5.48). Let \( \Sigma = (S; X, U, \mathcal{Y}) \) be a closed i/s/o system with the continuation property, and let \( X_1 \) be a closed strongly invariant
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(5.3) \( \text{gph}(S_1) = \text{gph}(S) \cap \begin{bmatrix} X_1 \\ Y \\ X_1 \\ U \end{bmatrix} = \left\{ \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \begin{bmatrix} X_1 \\ Y \\ X_1 \\ U \end{bmatrix} : \begin{bmatrix} z \\ y \end{bmatrix} \in S \begin{bmatrix} x \\ u \end{bmatrix} \right\} \).

Then \( \Sigma_1 \) is closed, \( \Sigma_1 \) is a restriction of \( \Sigma \) to \( X_1 \), and \( \Sigma_1 \) has the continuation property. If, in addition, \( \Sigma \) is solvable, then \( \Sigma_1 \) is solvable, and \( \Sigma_1 \) is the unique solvable restriction of \( \Sigma \) to \( X_1 \).

PROOF. The proof is analogous to the proof of Lemma 1.5.48.

2.5.4. Connections between s/s and i/s/o notions. Below we summarize the most important connections between s/s systems and their i/s/o representations. For simplicity we restrict ourselves to the case where the s/s system and its i/s/o representations are closed.

2.5.49. PROPOSITION. Let \( \Sigma_{i/s/o} = (S; X, U, Y) \) be an i/s/o representation of the closed s/s system \( \Sigma = (V; X, W) \). Then the following claims hold:

(i) \( \begin{bmatrix} x_0 \\ w \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma \) on some closed interval \( I \) if and only if \( \begin{bmatrix} u \\ v \end{bmatrix} \) is a classical or generalized trajectory of \( \Sigma_{i/s/o} \) on \( I \), where \( u = P_U^I v \) and \( y = P_Y^I w \).

(ii) \( \Sigma \) is solvable if and only if \( \Sigma_{i/s/o} \) is solvable.

(iii) \( \Sigma \) has the uniqueness property if and only if \( \Sigma_{i/s/o} \) has the uniqueness property.

(iv) \( \Sigma \) has the continuation property if and only if \( \Sigma_{i/s/o} \) has the continuation property.

(v) A state vector \( x^0 \in X \) is classically or exactly reachable for \( \Sigma \) if and only if it is classically or exactly reachable for \( \Sigma_{i/s/o} \).

(vi) \( \begin{bmatrix} z_0 \\ 0 \\ 0 \end{bmatrix} \) is a classical or generalized unobservable trajectory of \( \Sigma \) if and only if \( \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \) is a classical or generalized unobservable trajectory of \( \Sigma_{i/s/o} \).

(vii) A state vector \( x^0 \in X \) is unobservable or classically unobservable for \( \Sigma \) if and only if it is unobservable or classically unobservable for \( \Sigma_{i/s/o} \).

(viii) The classically reachable subspace, the exactly reachable subspace, and the reachable subspace of \( \Sigma \) coincide with the classically reachable subspace, the exactly reachable subspace, and the reachable subspace of \( \Sigma_{i/s/o} \).

(ix) \( \Sigma \) is controllable or observable if and only if \( \Sigma_{i/s/o} \) is controllable or observable.

(x) A subspace \( Z \) of \( X \) is strongly invariant for \( \Sigma \) if and only if it is strongly invariant for \( \Sigma_{i/s/o} \).

(xi) A subspace \( Z \) of \( X \) is unobservably invariant for \( \Sigma \) if and only if it is unobservably invariant for \( \Sigma_{i/s/o} \).

(xii) \( \Sigma \) is minimal if and only if \( \Sigma_{i/s/o} \) is minimal.

(xiii) The “backward” versions of claims (ii)–(xii) are also true (these are the claims that one gets by applying (ii)–(xii) to the time-reflected systems).

(xiv) The future behavior of \( \Sigma_{i/s/o} \) can be interpreted as a coordinate representation of the future behavior of \( \Sigma \) (with respect to the coordinate representation \( L^1_{\text{loc}}(\mathbb{R}^+; \begin{bmatrix} U \\ Y \end{bmatrix} \)) of \( L^1_{\text{loc}}(\mathbb{R}^+; W) \)).
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The relationships between various transformations of \( \Sigma \) and analogous transformations of \( \Sigma_{i/s/o} \) are discussed in Section 2.3.

**Proof.** This follows from Theorems 2.2.2 and 2.2.13 and the relevant definitions.

2.5.50. **Proposition.** Let \( \Sigma_{i/s/o} = (S; X, U, Y) \) and \( \Sigma^j_{i/s/o} = (S_j; X, U, Y_j) \), \( j = 1, 2 \), be i/s/o representations (with the same input and output spaces) of the closed s/s system \( \Sigma = (V; X, W) \) respectively \( \Sigma_j = (V_j; X_j, W) \) (with the same signal space). Then the following claims hold:

(i) \( \Sigma_2 \) is the time reflection of \( \Sigma_1 \) if and only if \( \Sigma^2_{i/s/o} \) is the time reflection of \( \Sigma^1_{i/s/o} \).

(ii) \( \Sigma_2 \) is the exponential \( \alpha \)-weighting of \( \Sigma_1 \) if and only if \( \Sigma^2_{i/s/o} \) is the exponential \( \alpha \)-weighting of \( \Sigma^1_{i/s/o} \).

(iii) \( \Sigma_1 \) and \( \Sigma_2 \) are \( P \)-similar if and only if \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are \( P \)-similar.

(iv) \( \Sigma \) is the cross product of \( \Sigma_1 \) and \( \Sigma_2 \) if and only if \( \Sigma_{i/s/o} \) is the cross product of \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \).

(v) \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent if and only if \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are externally equivalent.

(vi) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \in \mathcal{ML}(X_1; X_2) \) if and only if \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are intertwined by \( P \).

(vii) \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar if and only if \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are pseudo-similar (with the same pseudo-similarity operator \( P \)).

(viii) \( \Sigma_1 \) is the compression of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if \( \Sigma^1_{i/s/o} \) is the compression of \( \Sigma^2_{i/s/o} \) onto \( X_1 \) along \( Z_1 \).

(ix) \( \Sigma_1 \) is the restriction of \( \Sigma_2 \) to \( X_1 \) if and only if \( \Sigma^1_{i/s/o} \) is the restriction of \( \Sigma^2_{i/s/o} \) to \( X_1 \).

(x) \( \Sigma_1 \) is the projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if \( \Sigma^1_{i/s/o} \) is the projection of \( \Sigma^2_{i/s/o} \) onto \( X_1 \) along \( Z_1 \).

(xi) The “backward” versions of claims (v)–(ix) are also true (these are the claims that one gets by applying (v)–(ix) to the time-reflected systems).

Proof. This follows from Theorems 2.2.2 and 2.2.13 and the relevant definitions.
2.6. Notes and Comments (Jan 02, 2016)

All results in this chapter remain valid if we throughout replace \( L^1 \) by \( L^p \) where \( 1 \leq p < \infty \).

All results in this chapter remain valid if we allow the state spaces of the s/s systems to be \( B \)-spaces instead of \( H \)-spaces. Most of the results also remain valid if we allow the signal spaces of the s/s systems to be \( B \)-spaces instead of \( H \)-spaces. Then exceptions are Theorem 2.2.18, 2.2.27, and 2.2.29 and Corollary 2.2.28. Also there results remain valied in a \( B \)-space setting under the additional assumption that the subspace \( W_0 \) has a direct complement in \( W \) (or in the case of Theorem 2.2.18 that \( \overline{W_0} \) has a direct complement in \( W \)).
In this chapter we develop the theory presented in Chapters 1 and 2 further in the special case where the s/s system or i/s/o system is bounded. In particular, finite/dimensional s/s systems and i/s/o systems are of this type. As was shown in Chapter 2, a s/s system is bounded if and only if it has a bounded i/s/o representation. The boundedness of the system operator of such a representation makes it relatively straightforward to prove a number of additional results for bounded s/s and i/s/o systems. Many of these results will later be extended to the classes of semi-bounded i/s/o and s/s systems in Chapter 4 to the class of well-posed i/s/o systems in Chapter 8 and to the class of well-posed s/s systems in Chapter 9. These results are also useful ingredients in the frequency domain theory of s/s and i/s/o systems presented in Chapters 5–7.
3.1. Intertwinements and Compressions of Bounded Linear Operators

(3.1.1) Before studying how the notions of intertwinements, compressions, and dilations of i/o and s/s system look like in the bounded case we first look at the where the systems have no inputs and outputs. In the i/o setting this means that both the input space $U$ and the output space $Y$ is $\{0\}$, and in the s/s setting it means that the signal space $W$ are $\{0\}$. The trajectory of such a system on some interval $I$ consists only of a state component $x$ which satisfies the differential equation

$$\dot{x}(t) = Ax(t), \quad t \in I,$$

where $A$ is a bounded linear operator in the state space. We already gave the most basic results for this class of systems in Section 2.1.2, and here we develop that theory further.

3.1.1. The resolvent of a linear operator.

3.1.1. Definition. Let $A$ be a linear operator in an $H$-space $\mathcal{X}$.

(i) The resolvent set of $A$ consists of the set of points $\lambda \in \mathbb{C}$ for which the operator $\lambda - A$ has an inverse in $\mathcal{B}(\mathcal{X})$. This set is denoted by $\rho(A)$.

(ii) The $\mathcal{B}(\mathcal{X})$-valued function $\lambda \mapsto (\lambda - A)^{-1}$, defined on $\rho(A)$, is called the resolvent of $A$.

(iii) The complement of $\rho(A)$ is called the spectrum of $A$, and it is denoted by $\sigma(A)$.

3.1.2. Lemma. Let $A \in \mathcal{B}(\mathcal{X})$, where $\mathcal{X}$ is an $H$-space.

(i) Let $\|\cdot\|_{\mathcal{X}}$ be an admissible norm in $\mathcal{X}$, and denote the corresponding operator norm in $\mathcal{B}(\mathcal{X})$ by $\|A\|_{\mathcal{B}(\mathcal{X})}$. Then $\lim_{n \to \infty} \left(\|A^n\|_{\mathcal{B}(\mathcal{X})}\right)^{1/n}$ exists, and

$$r_{\infty}(A) := \inf_{n \in \mathbb{N}} \left(\|A^n\|_{\mathcal{B}(\mathcal{X})}\right)^{1/n} = \lim_{n \to \infty} \left(\|A^n\|_{\mathcal{B}(\mathcal{X})}\right)^{1/n},$$

In particular, $0 \leq r_{\infty}(A) \leq \|A\|_{\mathcal{B}(\mathcal{X})}$, and $r_{\infty}(A)$ does not depend on the choice of the admissible norm $\|\cdot\|_{\mathcal{X}}$ in $\mathcal{X}$.

(ii) For every $z \in \mathbb{C}$ with $|z| < 1/r_{\infty}(A)$ the power series $\sum_{n=0}^{\infty} (zA)^n$ converges absolutely, and

$$F(z) := (1 - zA)^{-1} = \sum_{n=0}^{\infty} (zA)^n, \quad |z| < 1/r_{\infty}(A).$$

In particular, the function $F(z) := (1 - zA)^{-1}$ is analytic at zero and $F^{(n)}(0) = n! A^n$.

(iii) For every $\lambda \in \mathbb{C}$ with $|\lambda| > r_{\infty}(A)$ the power series $\sum_{n=0}^{\infty} \left(\frac{1}{\lambda}A\right)^n$ converges absolutely, and

$$F(z) := (\lambda - A)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda}A\right)^n, \quad |\lambda| > r_{\infty}(A).$$

In particular,

$$\lim_{\lambda \to \infty} (\lambda - A)^{-1} = 0, \quad \lim_{\lambda \to \infty} \lambda(\lambda - A)^{-1} = 1_{\mathcal{X}}.$$
PROOF. All of the results listed above are well-known and easy to find in the existing literature.

Denote \( \Omega := \{ \lambda \in \mathbb{C} \mid |\lambda| > \|A\|_{\mathcal{B}(\mathcal{X})} \} \). For each \( \lambda \in \Omega \) the power series on the right-hand side of (3.1.4) converges absolutely, and it defines an analytic \( \mathcal{B}(\mathcal{X}) \)-valued function \( \hat{A} \) in \( \Omega \). By multiplying this series by \( \lambda - A \) to the left and to the right one finds that \( (\lambda - A)\hat{A}(\lambda) = \hat{A}(\lambda)(\lambda - A) \). This implies that \( \lambda - A \) has the inverse \( (\lambda - A)^{-1} = \hat{A}(\lambda) \). \( \square \)

3.1.3. Lemma. Let \( A \in \text{LIN}\mathcal{X} \), where \( \mathcal{X} \) is an \( H \)-space, and suppose that \( \rho(A) \neq 0 \).

(i) The spectrum \( \sigma(A) \) of \( A \) is a nonempty closed and bounded set. Thus, the resolvent set \( \rho(A) \) of \( A \) is open, and it has exactly one unbounded component.

(ii) The resolvent of \( A \) satisfies the resolvent identity

\[
(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - \lambda)(\mu - A)^{-1}(\lambda - A)^{-1}, \quad \lambda, \mu \in \rho(A).
\]

(iii) The resolvent of \( A \) is an analytic \( \mathcal{B}(\mathcal{X}) \)-valued function in \( \rho(A) \), and

\[
\frac{d^n}{d\lambda^n} (\lambda - A)^{-1} = (-1)^n n! (\lambda - A)^{-(n+1)}, \quad \lambda \in \rho(A), \quad n \in \mathbb{Z}^+.
\]

PROOF. This is a special case of Theorem 3.2.3 (where the result same is proved for multi-valued linear operators). \( \square \)

Note that by Lemma 3.1.2 \( \rho(A) \) has exactly one unbounded component.

3.1.4. Notation. Let \( A \in \mathcal{B}(\mathcal{X}) \), where \( \mathcal{X} \) is an \( H \)-space.

(i) The unbounded component of \( \rho(A) \) is denoted by \( \rho_\infty(A) \).

(ii) By the spectral radius of \( A \) we mean the number \( r_\infty(A) \) in (3.1.2).

Clearly \( \rho_\infty(A) = \rho(A) \) whenever \( \rho(A) \) is connected. This is true, e.g., if \( A \) is compact (and hence it is true whenever \( \mathcal{X} \) is finite-dimensional). However, \( \rho(A) \) need not always be connected. For example, if \( A \) is the left-shift operator in \( \ell^2(\mathbb{Z}) \) defined by \( (Au)(n) = Au(n + 1), \) \( u \in \ell^2(\mathbb{Z}) \), the \( \sigma(A) \) is the unit circle \( \mathbb{T} = \{ \lambda \in \mathbb{C} \mid |\lambda| = 0 \} \), and \( \rho_\infty(A) = \{ \lambda \in \mathbb{C} \mid |\lambda| > 0 \} \).

3.1.2. Invariant subspaces of bounded linear operators and semigroups. At this point the reader may want to recall the definitions of a semigroup and group of bounded linear operators given in Definition 2.1.10

3.1.5. Definition. Let \( \mathcal{X} \) be an \( H \)-space, and let \( \mathcal{Z} \) be a subspace of \( \mathcal{X} \).

(i) \( \mathcal{Z} \) is invariant subspace for a bounded linear operator \( A \in \mathcal{B}(\mathcal{X}) \) if \( A\mathcal{Z} \subset \mathcal{Z} \). In this case we also say that \( \mathcal{Z} \) is an \( A \)-invariant subspace of \( \mathcal{X} \).

(ii) \( \mathcal{Z} \) is invariant subspace for a semigroup \( \mathcal{A} \) in \( \mathcal{X} \) if \( \mathcal{A}^t\mathcal{Z} \subset \mathcal{Z} \) for all \( t \in \mathbb{R}^+ \).

In this case we also say that \( \mathcal{Z} \) is an \( \mathcal{A} \)-invariant subspace of \( \mathcal{X} \).

(iii) \( \mathcal{Z} \) is invariant subspace for a group \( \mathcal{A} \) in \( \mathcal{X} \) if \( \mathcal{A}^t\mathcal{Z} \subset \mathcal{Z} \) for all \( t \in \mathbb{R} \). In this case we also say that \( \mathcal{Z} \) is an \( \mathcal{A} \)-invariant subspace of \( \mathcal{X} \).

Thus, \( \mathcal{Z} \) is an invariant subspace for a semigroup \( \mathcal{A} \) if and only if \( \mathcal{Z} \) is an \( \mathcal{A}^t \)-invariant subspace of \( \mathcal{X} \) for all \( t \in \mathbb{R}^+ \), and \( \mathcal{Z} \) is an invariant subspace for a group \( \mathcal{A} \) if and only if \( \mathcal{Z} \) is an \( \mathcal{A}^t \)-invariant subspace of \( \mathcal{X} \) for all \( t \in \mathbb{R} \). Part (i) of above definition is extended to the case of a closed operator \( A \) in Definition 4.1.24.
3.1.6. Lemma. Let $A \in B(\mathcal{X})$ where $\mathcal{X}$ is an $H$-space.

(i) If the subspace $Z$ of $\mathcal{X}$ is $A$-invariant, then the closure of $Z$ is also $A$-invariant.

(ii) If both $Z_1$ and $Z_2$ are $A$-invariant, then $Z_1 + Z_2$, $Z_1 \cup Z_2$, and $Z_1 \cap Z_2$ are $A$-invariant.

Proof. That the closure of $Z$ is $A$-invariant whenever $Z$ is $A$-invariant follows from the continuity of $A$. The $A$-invariance of $Z_1 + Z_2$ and $Z_1 \cap Z_2$ in (ii) follows directly from Definition 3.1.5. Since $Z_1 \cup Z_2$ is the closure of $Z_1 + Z_2$, it follows from (i) that also $Z_1 \cup Z_2$ is $A$-invariant. $\square$

3.1.7. Lemma. Let $\mathcal{X}$ be an $H$-space, let $Z$ be a closed subspace of $\mathcal{X}$, and let $\mathfrak{A}$ be a uniformly continuous group in $\mathcal{X}$ with generator $A \in B(\mathcal{X})$. Then the following conditions are equivalent:

(i) $AZ \subset Z$;
(ii) $\mathfrak{A}'Z = Z$ for all $t \in \mathbb{R}$;
(iii) $\mathfrak{A}'Z \subset Z$ for all $t \in \mathbb{R}^+$;
(iv) $\mathfrak{A}'Z \subset \mathfrak{A}Z$ for all $t \in \mathbb{R}^-$;
(v) $(\lambda - A)^{-1}Z = Z$ for all $\lambda \in \rho_\infty(A)$ (where $\rho_\infty(A)$ is the unbounded component of $\rho(A)$);
(vi) $(\lambda - A)^{-1}Z \subset Z$ for some $\lambda \in \rho_\infty(A)$.

An extension of this result to the case where $\mathfrak{A}$ is a $C_0$ semigroup with generator $A \in \mathcal{L}(\mathcal{X})$ is given in Theorem 4.1.26 below.

Proof of Lemma 3.1.7. (i) $\Rightarrow$ (ii): Assume that (i) holds. Since $AZ \subset Z$, we have $A^tZ \subset Z$ for all $t \in \mathbb{Z}^+$, and it follows from the representation (2.1.13) of $\mathfrak{A}^t$ that $\mathfrak{A}^tZ \subset Z$ for all $t \in \mathbb{R}$. Replacing $t$ by $-t$ we get $\mathfrak{A}^{-t}Z \subset Z$ for all $t \in \mathbb{R}$, and consequently $\mathcal{Z} = \mathfrak{A}'Z = \mathfrak{A}^{-t}Z \subset \mathfrak{A}Z$, $t \in \mathbb{R}$. Thus $\mathfrak{A}'Z = Z$ for all $t \in \mathbb{R}$.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv): This is trivial.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds. Let $z \in Z$. Then $\mathfrak{A}t - 1 \mathfrak{A}z \in Z$ for all $t > 0$. Letting $t \downarrow 0$ and using (2.1.15) we find that

$$Az = \lim_{t \downarrow 0} \frac{1}{t} (\mathfrak{A}t - 1 \mathfrak{A}z)z \subset Z.$$

(iv) $\Rightarrow$ (i): The proof of this implication is the same as the proof of the implication (iii) $\Rightarrow$ (i) above with $A$ replaced by $-A$ and $\mathfrak{A}'$ replaced by $\mathfrak{A}^{-t} = e^{At}$, $t \in \mathbb{R}$.

(i) $\Rightarrow$ (vi): Let $\lambda \in C$ with $|\lambda| > r_\infty(A)$, where $r_\infty(A)$ is the spectral radius of $A$. Then it follows from (3.1.4) that $(\lambda - A)^{-1}Z \subset Z$ since the $A$-invariance of $Z$ implies that $A^nZ \subset Z$ for all $n \in \mathbb{Z}^+$.

(vi) $\Rightarrow$ (v): Let $\lambda \in \rho_\infty(A)$, and suppose that $(\lambda - A)^{-1}Z \subset Z$. Then $(\lambda - A)^{-n}Z \subset Z$ for all $n \in \mathbb{Z}^+$, or equivalently, $\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}Z \subset Z$ for all $n \in \mathbb{Z}^+$. By A.3.6, $(\lambda - A)^{-1}Z \subset Z$ for all $\lambda \in \rho_\infty(A)$. To prove the converse inclusion it suffices to observe that every $z \in Z$ is of the form $z = (\lambda - A)^{-1}x$ where $x = (\lambda - A)z \in \mathcal{Z}$ because of the $A$-invariance of $Z$.

(v) $\Rightarrow$ (i): Suppose that (v) holds. The operator $A$ can be recovered from $(\lambda - A)$ from the formula

$$A = \lim_{\lambda \to \infty} A \left(1_{\mathcal{X}} - \frac{1}{\lambda} A\right)^{-1} = \lim_{\lambda \to \infty} \lambda A(\lambda - A)^{-1} = \lim_{\lambda \to \infty} (\lambda^2(\lambda - A)^{-1} - \lambda).$$
Here \((\lambda^2(\lambda-A)^{-1}-\lambda)Z \subset Z\) for all sufficiently large \(|\lambda|\), and consequently \(AZ \subset Z\).

\[ \square \]

3.1.3. The restriction of a bounded linear operator to an invariant subspace. If \(A \in \mathcal{B}(\mathcal{X})\) for some \(H\)-space \(\mathcal{X}\) and \(Z\) is a closed \(A\)-invariant subspace, then by Lemma 3.1.7, \(Z\) is also an invariant subspace for \((\lambda-A)^{-1}\) for all \(\lambda \in \rho_\infty(A)\), and it is also an invariant subspace for the uniformly continuous group \(\mathfrak{A}\) generated by \(A\). This implies that the ranges of the restrictions \(A_{|Z}\), \((\lambda-A)^{-1}_{|Z}\), \(\lambda \in \rho_\infty(A)\), and \(\mathfrak{A}|_{Z}, t \in \mathbb{R}\), are contained in \(Z\), and therefore these operators can be interpreted as operators in \(\mathcal{B}(Z)\) (as opposed to operators in \(\mathcal{B}(Z;\mathcal{X})\)). Below we investigate the relationships between these restricted operators.

3.1.8. Lemma. Let \(\mathcal{X}\) be an \(H\)-space, and let \(A \in \mathcal{B}(\mathcal{X})\), and \(Z\) be a closed \(A\)-invariant subspace of \(\mathcal{X}\). Then, for each \(\lambda \in \rho(A)\) the following claims are equivalent:

(i) \((\lambda-A)^{-1}Z \subset Z\),
(ii) \(\lambda \in \rho(A_{|Z})\),

If these equivalent conditions hold, then

(iii) \((\lambda-A)^{-1}Z = Z\),
(iv) \((\lambda-A)_{|Z}^{-1} = (\lambda-A)^{-1}_{|Z}\).

An extension of this result to the case where \(A\) is a closed linear operator is given in Lemma 3.1.25.

Proof of Lemma 3.1.8. Let us begin by observing that \(A_{|Z} \in \mathcal{B}(Z)\) since \(Z\) is \(A\)-invariant. Throughout the rest of this proof we assume that \(\lambda \in \rho(A)\).

We next observe that for all \(z \in Z\) we have \((\lambda-A_{|Z})z \in Z\) and

\[ (\lambda-A)^{-1}_{|Z}(\lambda-A_{|Z})z = (\lambda-A)^{-1}_{|Z}z = z. \]

Thus, \((\lambda-A)^{-1}_{|Z}\) is always a left-inverse of \((\lambda-A_{|Z})\) (whenever \(Z\) is \(A\)-invariant and \(\lambda \in \rho(A)\)).

(i) \(\Rightarrow\) (ii) and (i) \(\Rightarrow\) (iv): We claim that if (i) holds, then \((\lambda-A)^{-1}_{|Z}\) is a right-inverse of \((\lambda-A_{|Z})\) as well. This is true because for all \(z \in Z\) we have \((\lambda-A)^{-1}_{|Z}z \in Z\) and

\[ (\lambda-A)_{|Z}((\lambda-A)^{-1}_{|Z})z = (\lambda-A)(\lambda-A)^{-1}_{|Z}z = z. \]

Thus \((\lambda-A)^{-1}_{|Z}\) is an inverse of \((\lambda-A_{|Z})\), and hence \(\lambda \in \rho(A_{|Z})\). This shows that (i) implies both (ii) and (iv).

(ii) \(\Rightarrow\) (i) and (ii) \(\Rightarrow\) (iii): If (ii) holds, then \((\lambda-A_{|Z})\) maps \(Z\) one-to-one onto itself, and hence also \((\lambda-A)\) maps \(Z\) one-to-one onto itself. This implies (i) and (iv). \(\square\)

3.1.9. Theorem. Let \(\mathcal{X}\) be an \(H\)-space, let \(Z\) be a closed subspace of \(\mathcal{X}\), and let \(\mathfrak{A}\) be a uniformly continuous group in \(\mathcal{X}\) with generator \(A \in \mathcal{B}(\mathcal{X})\). Then the following conditions are equivalent:

(i) \(Z\) is an invariant subspace for \(A\);
(ii) \(\mathfrak{A}tZ = Z\) for all \(t \in \mathbb{R}\);
(iii) The family \(\mathfrak{A}|_{Z}, t \in \mathbb{R}\), is a uniformly continuous group in \(Z\) with generator \(A_{|Z}\);
(iv) \((\lambda-A)^{-1}Z = Z\) for all \(\lambda \in \rho_\infty(A)\);
(v) \((\lambda-A)^{-1}Z \subset Z\) for some \(\lambda \in \rho_\infty(A)\).
If these equivalent conditions hold then the following claims are also true:

(vi) \((\lambda - A|_Z)^{-1} = (\lambda - A)^{-1}|_Z\) for all \(\lambda \in \rho(A|_Z) \cap \rho(A)\).

(vii) \(\rho_\infty(A) \subset \rho(A|_Z)\), and hence \((\lambda - A|_Z)^{-1} = (\lambda - A)^{-1}|_Z\) for all \(\lambda \in \rho_\infty(A)\).

An extension of this result to the case where \(\mathfrak{A}\) is a \(C_0\) semigroup with generator \(A \in \mathcal{L}(\mathcal{X})\) will be given in Theorem 4.1.26 below.

**Proof of Theorem 3.1.9** (i) \(\iff\) (ii) \(\iff\) (iv) \(\iff\) (v): This follows from Lemma 3.1.10.

(ii) \(\iff\) (iii): A direct inspection shows that (ii) and (iii) are equivalent.

Proof of (vi): That (vi) holds follows from Lemma 3.1.8.

Proof of (vii): That (vii) holds follows from (iv) and Lemma 3.1.8. □

Motivated by Theorem 3.1.9 we make the following definition.

3.1.10. **Definition.** Let \(\mathcal{X}\) be a \(H\)-space, let \(Z\) be a closed subspace of \(\mathcal{X}\), let \(A \in \mathcal{B}(\mathcal{X})\), and let \(\mathfrak{A}\) be a uniformly continuous group in \(\mathcal{X}\).

(i) An operator \(A_1 \in \mathcal{B}(Z)\) is called a *restriction of \(A\) in \(\mathcal{B}(Z)\)* if \(A_1 = A|_Z\).

(ii) A uniformly continuous group \(\mathfrak{A}_1\) in \(Z\) is called a *restriction of \(\mathfrak{A}\) to \(Z\)* if \(\mathfrak{A}^t_1 = \mathfrak{A}^t|_Z\) for all \(t \in \mathbb{R}\).

Thus, by Theorem 3.1.9 a necessary and sufficient condition for \(A \in \mathcal{B}(\mathcal{X})\) to have a restriction in \(\mathcal{B}(Z)\) is that \(Z\) is a closed \(A\)-invariant subspace of \(\mathcal{X}\), and a necessary and sufficient condition for a uniformly continuous group in \(\mathcal{X}\) to have a restriction to \(Z\) (which is a group in \(Z\)) is that \(Z\) is a closed \(\mathfrak{A}\)-invariant subspace of \(\mathcal{X}\). These restrictions are determined uniquely by \(Z\) and \(A\) respectively \(\mathfrak{A}\). We warn the reader that a “restriction of \(A\) in \(\mathcal{B}(Z)\)” is not the same thing as the “restriction of \(A\) to \(Z\)”. The restriction \(A|_Z\) of \(A \in \mathcal{B}(\mathcal{X})\) always exists as an operator in \(\mathcal{B}(Z;\mathcal{X})\). Only in the case where \(Z\) is \(A\)-invariant it is true that \(A|_Z\) is a restriction of \(A\) in \(\mathcal{B}(Z)\). Likewise, it is always possible to define the operator family \(t \mapsto \mathfrak{A}^t|_Z\), \(t \in \mathbb{R}\), but this family is not a group unless \(Z\) is \(\mathfrak{A}\)-invariant.

3.1.11. **Definition.** Let \(\mathcal{X}\) be an \(H\)-space, let \(A \in \mathcal{B}(\mathcal{X})\), and let \(\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1\) be a direct sum decomposition of \(\mathcal{X}\). An operator \(A_{proj} \in \mathcal{B}(\mathcal{X}_1)\) is called a *projection of \(A\) in \(\mathcal{B}(\mathcal{X}_1)\) along \(\mathcal{Z}_1\)* if

\[
A_{proj} P_{\mathcal{X}_1} = P_{\mathcal{X}_1} A.
\]

(3.1.7)

Note that if \(A\) has a projection \(A_{proj} \in \mathcal{B}(\mathcal{X}_1)\) along \(\mathcal{Z}_1\), then it follows from (3.1.7) that \(A_{proj}\) is uniquely determined by \(A\), \(\mathcal{X}_1\), and \(\mathcal{Z}_1\) by

\[
A_{proj} = P_{\mathcal{X}_1} A|_{\mathcal{X}_1}.
\]

(3.1.8)

However, as the following lemma shows, it is not true that an operator \(A \in \mathcal{B}(\mathcal{X})\) always has a projection in \(\mathcal{B}(\mathcal{X}_1)\) along \(\mathcal{Z}_1\) if \(\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1\) is an arbitrary direct sum decomposition of \(\mathcal{X}\).\[1\]

\[1\]Here the words “in \(\mathcal{B}(\mathcal{X}_1)\)” are significant. In Definitions 4.1.28 and 6.4.23 we extend Definition 3.1.11 by introducing the notion of a “projection onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\)”. The “projection onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\)” always exists for every direct sum decomposition \(\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1\) of \(\mathcal{X}\), but it may be multi-valued or unbounded or not closed.
3.1.12. Lemma. Let $\mathcal{X}$ be an $H$-space, and let $A \in \mathcal{B}(\mathcal{X})$, and let $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$ be a direct sum decomposition of $\mathcal{X}$. Then the following conditions are equivalent:

(i) $\mathcal{Z}_1$ is an invariant subspace for $A$;
(ii) $P_{\mathcal{X}_1}^{\mathcal{Z}_1}AP_{\mathcal{Z}_1}=0$;
(iii) $P_{\mathcal{X}_1}^{\mathcal{Z}_1}A = P_{\mathcal{X}_1}^{\mathcal{Z}_1}AP_{\mathcal{Z}_1}$;
(iv) $A$ has a projection in $\mathcal{B}(\mathcal{X}_1)$ along $\mathcal{Z}_1$.

Proof. (i) $\iff$ (ii): This is obvious.

(ii) $\iff$ (iii): This follows from the fact that $1_{\mathcal{X}} = P_{\mathcal{X}_1}^{\mathcal{Z}_1} + P_{\mathcal{Z}_1}^{\mathcal{Z}_1}$.

(ii) $\iff$ (iv): Define $A_{\text{proj}} = P_{\mathcal{X}_1}^{\mathcal{Z}_1}A|_{\mathcal{X}_1}$. If (ii) holds, then

$$A_{\text{proj}}P_{\mathcal{X}_1}^{\mathcal{Z}_1} = P_{\mathcal{X}_1}^{\mathcal{Z}_1}AP_{\mathcal{X}_1}^{\mathcal{Z}_1} = P_{\mathcal{X}_1}^{\mathcal{Z}_1}A(P_{\mathcal{X}_1}^{\mathcal{Z}_1} + P_{\mathcal{Z}_1}^{\mathcal{Z}_1}) = P_{\mathcal{X}_1}^{\mathcal{Z}_1}A.$$

Thus $A_{\text{proj}}$ is a projection of $A$ in $\mathcal{B}(\mathcal{X}_1)$ along $\mathcal{Z}_1$. Conversely, if $A_{\text{proj}}$ is a projection of $A$ in $\mathcal{B}(\mathcal{X}_1)$ along $\mathcal{Z}_1$, then

$$P_{\mathcal{X}_1}^{\mathcal{Z}_1}AP_{\mathcal{X}_1}^{\mathcal{Z}_1} = A_1P_{\mathcal{X}_1}^{\mathcal{Z}_1}P_{\mathcal{Z}_1}^{\mathcal{Z}_1} = 0,$$

and hence (ii) holds. \hfill \Box

3.1.13. Lemma. Let $\mathcal{X}$ be an $H$-space, and let $A \in \mathcal{B}(\mathcal{X})$, and let $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$ be a direct sum decomposition of $\mathcal{X}$. In addition, suppose that $\mathcal{Z}_1$ is invariant for $A$. Then, for each $\lambda \in \rho(A)$ the following claims are equivalent:

(i) $(\lambda - A)^{-1}\mathcal{Z}_1 \subset \mathcal{Z}_1$;
(ii) $\lambda \in \rho(A_{\text{proj}})$, where $A_{\text{proj}}$ is the projection in $\mathcal{B}(\mathcal{X}_1)$ along $\mathcal{Z}_1$ of $A$.

If these equivalent conditions hold, then

(iii) $(\lambda - A_{\text{proj}})^{-1} = (\lambda - A)^{-1}_{\text{proj}},$

where $(\lambda - A)^{-1}_{\text{proj}}$ is the projection of $(\lambda - A)^{-1}$ in $\mathcal{B}(\mathcal{X}_1)$ along $\mathcal{Z}_1$.

An extension of this result to the case where $A$ is a closed linear operator is given in Lemma 4.1.30.

Proof of Lemma 3.1.13 Let us first observe that by Lemma 3.1.12 $A$ has a projection $A_{\text{proj}} \in \mathcal{B}(\mathcal{X}_1)$ along $\mathcal{Z}_1$. Throughout the rest of this proof we assume that $\lambda \in \rho(A)$.

We next observe that for all $x \in \mathcal{X}_1$ we have

$$(\lambda - A_{\text{proj}})(\lambda - A)^{-1}_{\text{proj}}x = (\lambda - A_{\text{proj}})P_{\mathcal{X}_1}^{\mathcal{Z}_1}(\lambda - A)^{-1}x$$

$$= P_{\mathcal{X}_1}^{\mathcal{Z}_1}(\lambda - A)(\lambda - A)^{-1}x = x.$$

Thus $(\lambda - A)^{-1}_{\text{proj}}$ is always a right-inverse of $(\lambda - A_{\text{proj}})$ (whenever $\mathcal{Z}$ is $A$-invariant and $\lambda \in \rho(A)$).

(i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii): We claim that if (i) holds, then $(\lambda - A)^{-1}_{\text{proj}}$ is a left-inverse of $(\lambda - A_{\text{proj}})$ as well. This is true because for $x \in \mathcal{X}_1$ we have

$$(\lambda - A)^{-1}_{\text{proj}}(\lambda - A_{\text{proj}})x = P_{\mathcal{X}_1}^{\mathcal{Z}_1}(\lambda - A)^{-1}P_{\mathcal{X}_1}^{\mathcal{Z}_1}(\lambda - A)x$$

$$= P_{\mathcal{X}_1}^{\mathcal{Z}_1}(\lambda - A)^{-1}(P_{\mathcal{X}_1}^{\mathcal{Z}_1} + P_{\mathcal{Z}_1}^{\mathcal{Z}_1})(\lambda - A)x$$

$$= P_{\mathcal{X}_1}^{\mathcal{Z}_1}(\lambda - A)^{-1}(\lambda - A)x = x.$$

Thus $(\lambda - A)^{-1}_{\text{proj}}$ is an inverse of $(\lambda - A_{\text{proj}})$, and hence $\lambda \in \rho(A_{\text{proj}})$. This shows that (i) implies both (ii) and (iii).
(ii) \Rightarrow (i): For all \( z \in Z_1 \) we have
\[
(\lambda - A_{\text{proj}})P_{\lambda_1}(\lambda - A)^{-1}z = P_{\lambda_1}(\lambda - A)(\lambda - A)^{-1}z = P_{\lambda_1}z = 0.
\]
If (ii) holds, then \( \lambda - A_{\text{proj}} \) is injective, and consequently \( P_{\lambda_1}(\lambda - A)^{-1}z = 0 \), i.e.,
\[
(\lambda - A)^{-1}z \subset Z.
\]
Thus (ii) \( \Rightarrow \) (i). \( \square \)

By combining Lemmas 3.1.8 and 3.1.13 we get the following result:

3.1.14. Lemma. Let \( \mathcal{X} \) be an \( H \)-space, and let \( A \in B(\mathcal{X}) \), and let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \)
be a direct sum decomposition of \( \mathcal{X} \). In addition, suppose that \( Z_1 \) is invariant for \( A \). Then, for each \( \lambda \in \rho(A) \) the following claims are equivalent:

(i) \( (\lambda - A)^{-1}Z_1 \subset Z_1 \),
(ii) \( \lambda \in \rho(A|_{Z_1}) \),
(iii) \( \lambda \in \rho(A_{\text{proj}}) \), where \( A_{\text{proj}} \) is the projection in \( B(\mathcal{X}_1) \) along \( Z_1 \) of \( A \).

If these equivalent conditions hold, then

(iv) \( (\lambda - A)^{-1}Z_1 = Z_1 \),
(v) \( (\lambda - A|_{Z_1})^{-1} = (\lambda - A)^{-1}|_{Z_1} \),
(vi) \( (\lambda - A_{\text{proj}})^{-1} = (\lambda - A)^{-1}_{\text{proj}} \),

where \( (\lambda - A)^{-1}_{\text{proj}} \) is the projection of \( (\lambda - A)^{-1} \) in \( B(\mathcal{X}_1) \) along \( Z_1 \).

Proof. This follows from Lemmas 3.1.8 and 3.1.13 \( \square \)

3.1.15. Theorem. Let \( \mathcal{X} \) be an \( H \)-space with a direct sum decomposition \( \mathcal{X} = \mathcal{X}_1 + Z_1 \), and let \( \mathfrak{A} \) be a uniformly continuous group in \( \mathcal{X} \) with generator \( A \in B(\mathcal{X}) \). Then the following conditions are equivalent:

(i) \( Z_1 \) is an invariant subspace for \( A \);
(ii) \( A \) has a projection in \( B(\mathcal{X}_1) \) along \( Z_1 \);
(iii) \( \mathfrak{A}_t \) has a projection in \( B(\mathcal{X}_1) \) along \( Z_1 \) for all \( t \in \mathbb{R} \);
(iv) \( (\lambda - A)^{-1} \) has a projection in \( B(\mathcal{X}_1) \) along \( Z_1 \) for all \( \lambda \in \rho_\infty(A) \);
(v) \( (\lambda - A)^{-1} \) has a projection in \( B(\mathcal{X}_1) \) along \( Z_1 \) for some \( \lambda \in \rho_\infty(A) \).

Suppose that these equivalent conditions hold, and denote the projections of \( A, \mathfrak{A}_t, \) and \( (\lambda - A)^{-1} \) by \( A_{\text{proj}}, \mathfrak{A}_t_{\text{proj}}, \) and \( (\lambda - A)^{-1}_{\text{proj}} \), respectively. Then

(vi) \( (\lambda - A_{\text{proj}})^{-1} = (\lambda - A)^{-1}_{\text{proj}} \) for all \( \lambda \in \rho(A_{\text{proj}}) \cap \rho(A) \);
(vii) \( \rho_\infty(A) \subset \rho(A_{\text{proj}}) \);
(viii) \( A_{\text{proj}}^n P_{\lambda_1} = P_{\lambda_1} A^n \) for all \( n \in \mathbb{Z}^+ \);
(ix) The family \( \mathfrak{A}_{\text{proj}} : t \mapsto \mathfrak{A}_t_{\text{proj}} \) is a uniformly continuous group in \( Z \) with generator \( A_{\text{proj}} \).

An extension of this result to the case where \( \mathfrak{A} \) is a \( C_0 \) semigroup with generator \( A \in \mathcal{L}(\mathcal{X}) \) be given in Theorem 4.1.32

Proof of Theorem 3.1.15 (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv) \( \iff \) (v): This follows from Lemmas 3.1.7 and 3.1.12

Proof of (vi): That (vi) holds follows from Lemmas 3.1.12 and 3.1.13

Proof of (vii): That (vii) holds follows from (iv) and Lemmas 3.1.12 and 3.1.13

Proof of (viii): The identity (3.1.13) can be proved by induction over \( n \) starting from (3.1.7).

Proof of (ix): This follows from (viii) and (2.1.13). \( \square \)

Motivated by Theorem 3.1.15 we make the following definition.
3.1.16. Definition. Let $X$ be a $H$-space, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, and let $A$ be a uniformly continuous group in $X$. A uniformly continuous group $\mathfrak{A}$ in $X$ is called a projection of $A$ in $\mathcal{B}(X_1)$ along $Z_1$ if

$$\mathfrak{A}^t P_{X_1}^t P_{X_1}^2 = P_{X_1}^2 \mathfrak{A}^t, \quad t \in \mathbb{R}.$$ 

Thus by Theorem 3.1.15 $\mathfrak{A}$ has a projection onto $X_1$ along $Z_1$ (which is a group in $X_1$) if and only if $Z_1$ is $\mathfrak{A}$-invariant.

3.1.15. Intertwinements of bounded linear operators. Let $A \in \mathcal{B}(X)$, where $X$ is an $H$-space. Up to now we have in this section discussed restrictions of $A$ to some closed invariant subspace $X_1$ and projections of $A$ in $\mathcal{B}(X_1)$ along some invariant subspace $Z_1$ which is a direct complement to $X_1$ in $X$. Both of these notions can be regarded as special cases of intertwinements of bounded linear operators. In this more general notion it is no longer required that $X_1$ is a subspace of $X$.

3.1.17. Definition. Let $A_1 \in \mathcal{B}(X_1)$ and $A_2 \in \mathcal{B}(X_2)$, where $X_1$ and $X_2$ are $H$-spaces, and let $P \in \mathcal{ML}(X_1; X_2)$.

(i) We say that $A_1$ and $A_2$ are intertwined by $P$ if

$$\begin{bmatrix} A_2x_2 \\ A_1x_1 \end{bmatrix} \in gph(P) \text{ whenever } \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \in gph(P),$$

or equivalently, if $A_2x_2 \in PA_1x_1$ for every $x_2 \in Px_1$.

(ii) $A_1$ and $A_2$ are pseudo-similar if $A_1$ and $A_2$ are intertwined by an closed single-valued injective linear operator $P: X \to X_1$ with dense domain and dense range, called the pseudo-similarity operator.

(iii) $A_1$ and $A_2$ are similar if $A_1$ and $A_2$ are intertwined by a bounded linear operator $P \in \mathcal{B}(X_1; X_2)$ with a bounded inverse $P^{-1} \in \mathcal{B}(X_2; X_1)$, called the similarity operator.

In the study of the intertwinement of two bounded operators $A_1$ and $A_2$ it is useful to introduce the notion of the cross product of $A_1$ and $A_2$.

3.1.18. Definition. Let $A_i$ be a linear operators in the $H$-space $X_i$, $i = 1, 2$. By the cross product of $A_1$ and $A_2$ we mean the operator $A_1 \times A_2$ defined by

$$\begin{bmatrix} A_1 \times A_2 \\ x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} A_1 \\ 0 \\ A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1x_1 \\ A_2x_2 \end{bmatrix},$$

where $x_1 \in \text{dom}(A_1 \times A_2) = \begin{bmatrix} \text{dom}(A_1) \\ \text{dom}(A_2) \end{bmatrix}$.

3.1.19. Lemma. Let $A_1 \in \mathcal{B}(X_1)$ and $A_2 \in \mathcal{B}(X_2)$ where $X_1$ and $X_2$ are $H$-spaces, and let $P \in \mathcal{ML}(X_1; X_2)$. Then $A_1$ and $A_2$ are intertwined by $P$ if and only $gph(P)$ is an invariant subspace for $A_2 \times A_1$ of $A_2$ and $A_1$.

Proof. This follows immediately from Definitions 3.1.5 and 3.1.17.

3.1.20. Lemma. Let $A = A_1 \times A_2$ be the cross product of the two operators $A_i \in \mathcal{B}(X_i)$, $i = 1, 2$, and denote $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. Then the following claims are true:

(i) $A_1 \times A_2 \in \mathcal{B}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right)$. 

3.1.21. Lemma. Let $A_1 \in \mathcal{B}(X_1)$ and $A_2 \in \mathcal{B}(X_2)$ where $X_1$ and $X_2$ are $H$-spaces, and let $P \in \mathcal{ML}(X_1; X_2)$.

(i) $A_1$ and $A_2$ are intertwined by $P$ if and only if $A_1$ and $A_2$ are intertwined by $P^{-1}$.

(ii) If $A_1$ and $A_2$ are intertwined by $P$, then both $\text{dom}(P)$ and $\text{ker}(P)$ are invariant subspaces for $A_1$, and both $\text{rng}(P)$ and $\text{mul}(P)$ are invariant subspaces for $A_2$.

(iii) If $A_1$ and $A_2$ are intertwined by $P$, then $A_1$ and $A_2$ are also intertwined by the closure of $P$.

(iv) If $P$ is single-valued, then $A_1$ and $A_2$ are intertwined by $P$ if and only if $A_2P^{-1} = PA_1$ for all $x_1 \in \text{dom}(P)$. In particular, if $P \in \mathcal{B}(X_1; X_2)$, then $A_1$ and $A_2$ are intertwined by $P$ if and only if $A_2P = PA_1$.

Proof. (i) That (i) holds follows immediately from Definition 3.1.17 and the definition of the inverse of a multi-valued operator.

(ii) Let $x_1 \in \text{dom}(P)$, and take any $x_2 \in Px_1$. Then $[x_2 x_1] \in \text{gph}(P)$, and consequently $[x_2 x_1] \in \text{gph}(P)$. This implies that $A_1x_1 \in \text{dom}(P)$. This shows that $\text{dom}(P)$ is invariant for $A_1$. If instead $x_1 \in \text{ker}(P)$, then $[x_1 x_1] = [xA_1] \in \text{gph}(P)$. Thus, $x_1 \in \text{ker}(A_1)$. This shows that also $\text{ker}(P)$ is invariant for $A_1$. After interchanging $A_1$ and $A_2$ and replacing $P$ by $P^{-1}$ we find in the same way that both $\text{rng}(P)$ and $\text{mul}(P)$ are invariant subspaces for $A_2$.

(iii) That (iii) holds follows from Lemmas 3.1.6 and 3.1.19.

(iv) Claim (iv) follows directly from Definition 3.1.17.

3.1.22. Lemma. Let $A_i \in \mathcal{B}(X_i)$ where $X_i$ are $H$-spaces, $i = 1, 2, 3$. If $A_1$ and $A_2$ are intertwined by $P_1 \in \mathcal{ML}(X_1; X_2)$ and $A_2$ and $A_3$ are intertwined by $P_2 \in \mathcal{ML}(X_2; X_3)$, then $A_1$ and $A_3$ are intertwined by $P_3 := P_2P_1$, and hence also by the closure of $P_3$.

Proof. The first claim follows from Definition 3.1.17 and the definition of the composition of two multi-valued operators. The second claim follows from Lemma 3.1.21.

3.1.23. Lemma. Let $X$ be a $H$-space with a direct sum decomposition $X = X_1 + X_2$, and let $A \in \mathcal{B}(X)$ and $A_1 \in \mathcal{B}(X_1)$. Then the following claims are true:

(i) $A_1$ is a restriction of $A$ in $\mathcal{B}(X_1)$ if and only if $A_1$ and $A_2$ are intertwined by the embedding operator $X_1 \hookrightarrow X$.
(ii) $A_1$ is a projection of $A$ in $\mathcal{B}(X_1)$ if and only if $A$ and $A_1$ are intertwined by the projection operator $P^Z_{X_1}$, interpreted as an operator in $\mathcal{B}(X;X_1)$.

**Proof.** Claim (i) follows from Definitions 3.1.10 and 3.1.17 and claim (ii) from Theorem 3.1.15 and Definition 3.1.17.

3.1.24. **Theorem.** Let $A_1 \in \mathcal{B}(X_1)$ and $A_2 \in \mathcal{B}(X_2)$ where $X_1$ and $X_2$ are $H$-spaces, and let $P \in \mathcal{ML}(X_1;X_2)$ be closed. Denote the uniformly continuous groups generated by $A_1$ and $A_2$ by $\mathfrak{A}_1$ respectively $\mathfrak{A}_2$. Then the following conditions are equivalent:

(i) $A_1$ and $A_2$ are intertwined by $P$;

(ii) $\mathfrak{A}_1^t$ and $\mathfrak{A}_2^t$ are intertwined by $P$ for all $t \in \mathbb{R}$;

(iii) $(\lambda - A_1)^{-1}$ and $(\lambda - A_2)^{-1}$ are intertwined by $P$ for all $\lambda$ in the unbounded component of $\rho(A_1) \cap \rho(A_2)$;

(iv) $(\lambda - A_1)^{-1}$ and $(\lambda - A_2)^{-1}$ are intertwined by $P$ for some $\lambda$ in the unbounded component of $\rho(A_1) \cap \rho(A_2)$.

**Proof.** This follows from Lemma 3.1.7, Definition 3.1.18, and Lemma 3.1.19.

Motivated by Theorem 3.1.24, we make the following definition.

3.1.25. **Definition.** Let $X_1$ and $X_2$ be $H$-spaces, let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be uniformly continuous groups in $X_1$ respectively $X_2$, and let $P \in \mathcal{ML}(X_1;X_2)$. We say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are intertwined by $P$ if $\mathfrak{A}_1^t$ and $\mathfrak{A}_2^t$ are intertwined by $P$ for all $t \in \mathbb{R}$.

3.1.6. **Compressions of bounded linear operators.** Earlier in this section we have defined what we mean by a restriction in $\mathcal{B}(X_1)$ of an operator $A \in \mathcal{B}(X)$ when $X_1$ is a closed subspace of $X$, and we also defined what we mean by a projection of $A$ in $\mathcal{B}(X_1)$ along some direct complement $Z_1$ to $X_1$ in $X$. We proved that a restriction in $\mathcal{B}(X_1)$ of $A$ exists if and only if $X_1$ is an $A$ invariant subspace, in which case this restriction is equal to the restriction $A|_Z$ of $A$ to $X_1$ and satisfies

$$
(A|_Z)^n = A^n|_Z, \quad n \in \mathbb{Z}^+.
$$

We also proved that $A$ has a projection $A_{\text{proj}}$ in $\mathcal{B}(X_1)$ along $Z_1$ if and only if $Z_1$ is an invariant subspace of $X$, in which case $A_{\text{proj}}$ satisfies

$$
A_{\text{proj}}^n P^Z_{X_1} = P^Z_{X_1} A^n, \quad n \in \mathbb{Z}^+.
$$

Both of these formulas can be interpreted as special cases of the formula

$$
A^n_{\text{cmp}} = P^Z_{X_1} A^n|_{X_1}, \quad n \in \mathbb{Z}^+,
$$

or equivalently,

$$
A^n_{\text{cmp}} P^Z_{X_1} = P^Z_{X_1} A^n P^Z_{X_1}, \quad n \in \mathbb{Z}^+,
$$

3.1.26. **Definition.** Let $X$ be a $H$-space, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. An operator $A_{\text{cmp}} \in \mathcal{B}(X_1)$ is called a compression of the operator $A \in \mathcal{B}(X)$ onto $X_1$ along $Z_1$ if (3.1.14) holds. (Note, in particular, that (3.1.14) implies that $A_{\text{cmp}} = P^Z_{X_1} A|_{X_1}$.) In this case we also call $A$ a dilation of $A_{\text{cmp}}$ from $X_1$ into $X$ along $Z_1$. 
As we noticed above, a compression $A_{cmp}$ of $A$ onto $X_1$ along $Z_1$ is determined uniquely by $A$, $X_1$, and $Z_1$ by the formula

$$A_{cmp} = P_{X_1}^Z A|_{X_1}.$$  \hfill (3.1.16)

However, as the following lemma shows, it is not true that an operator $A \in B(X)$ always has a compression onto $X_1$ along $Z_1$ if $X = X_1 + Z_1$ is an arbitrary direct sum decomposition of $X$.

3.1.27. Lemma. Let $X$ be a $H$-space, and let $X = X_{cmp} + Z_1$ be a direct sum decomposition of $X$. Then $A_{cmp} \in B(X_1)$ is a compression of $A$ onto $X_1$ along $Z_1$ if and only if $A_{cmp} = P_{X_1}^Z A|_{X_1}$ and

$$P_{X_1}^Z A^{m+n} P_{X_1}^Z = P_{X_1}^Z A^m P_{X_1}^Z A^n P_{X_1}^Z, \quad n, m \in \mathbb{Z}^+.$$  \hfill (3.1.17)

Here condition (3.1.17) can alternatively be replaced by the equivalent condition

$$P_{X_1}^Z A^m P_{X_1}^Z A^n P_{X_1}^Z = 0, \quad n, m \in \mathbb{Z}^+.$$  \hfill (3.1.18)

Proof. (i) If $A_{cmp}$ is a compression of $A$ onto $X_1$ along $Z_1$, then it follows from (3.1.14) both that (3.1.17) holds (take $n = 1$) and that

$$P_{X_1}^Z A^{m+n} P_{Z_1}^Z = A_{cmp}^n P_{X_1}^Z A^m P_{X_1}^Z P_{Z_1}^Z = P_{X_1}^Z A^m P_{X_1}^Z A^n P_{Z_1}^Z.$$  

Thus if $A_{cmp}$ is a compression of $A$ onto $X_1$ along $Z_1$, then both (3.1.14) and (3.1.17) hold. Conversely, suppose that (3.1.14) and (3.1.17) hold. Then for all $n \in \mathbb{Z}^+$ we have

$$A_{cmp}^n = (P_{X_1}^Z A)^n|_{X_1} = P_{X_1}^Z A P_{X_1}^Z A \cdots P_{X_1}^Z A|_{X_1} = P_{X_1}^Z A^n|_{X_1}.$$  

This implies that $A_{cmp}$ is a compression of $A$ onto $X_1$ along $X_1$.

Trivially, (3.1.17) and (3.1.18) are equivalent. \hfill $\Box$

3.1.28. Lemma. Let $X = X_1 + Z_1$ and $X_1 = X_2 + Z_2$ (and hence $X_2 = X_1 + Z_2 + Z_2$) be $H$-spaces. If $A_1 \in B(X_1)$ is a compression of $A \in B(X)$ onto $X_1$ along $Z_1$ and $A_2 \in B(X_2)$ is a compression of $A_1$ onto $X_2$ along $Z_2$, then $A_2$ is a compression of $A$ onto $X_2$ along $Z_2 + Z_2$.

Proof. This follows from Definition 3.1.26 since $P_{X_2}^{Z_1 + Z_2} = P_{X_2}^{Z_2} P_{X_1}^{Z_1}$. \hfill $\Box$

3.1.29. Lemma. Let $X$ be an $H$-space, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, and let $A \in B(X)$ and $A_1 \in B(X_1)$.

(i) The following conditions are equivalent:

(a) $A_1$ is a restriction of $A$ in $B(X_1)$;

(b) $X_1$ is an invariant subspace for $A$ and $A_1$ is a compression of $A$ onto $X_1$ along $Z_1$.

(ii) The following conditions are equivalent:

(a) $A_1$ is a projection of $A$ in $B(X_1)$ along $Z_1$;

(b) $Z_1$ is an invariant subspace for $A$ and $A_1$ is a compression of $A$ onto $X_1$ along $Z_1$.

Proof. Part (i) of this lemma follows from Theorem 3.1.9 and Definition 3.1.10 and part (ii) from Definition 3.1.11 and Theorem 3.1.15. \hfill $\Box$

3.1.30. Lemma. Let $X$ be an $H$-space with a direct sum decomposition $X = X_1 + Z_1$, and let $A$ be a uniformly continuous group in $X$ with generator $A \in B(X)$. Then the following conditions are equivalent:
(i) $A$ has a compression onto $X_1$ along $Z_1$;
(ii) the family $P_{X_1}^s A^t |X_1$, $t \in \mathbb{R}$, is a uniformly continuous group in $X$;
(iii) for each $t \in \mathbb{R}$, the operator $A^t$ has a compression onto $X_1$ along $Z_1$;
(iv) the group $\mathfrak{A}$ satisfies the condition
$$
P_{X_1}^s A^t \big| X_1 = P_{X_1}^s A^s P_{Z_1}^s A^t \big| X_1, \quad s, \ t \in \mathbb{R};$$
(v) the group $\mathfrak{A}$ satisfies the condition
$$
P_{X_1}^s A^t P_{Z_1}^s A^t \big| X_1 = 0, \quad s, \ t \in \mathbb{R}.
$$
If these equivalent conditions hold, then the generator of the uniformly continuous group $t \mapsto P_{X_1}^s A^t \big| X_1$ is the compression of $A$ onto $X_1$ along $Z_1$.

An extension of this result to the case where $\mathfrak{A}$ is a $C_0$ semigroup with generator $A \in \mathcal{L}(X)$ is given in Lemma 4.1.42.

**Proof of Lemma 3.1.30** (i) $\Leftrightarrow$ (ii): Suppose that (i) holds, and define $A_1 := P_{X_1}^s A \big| X_1$ and $A_1^t := P_{X_1}^s A^t \big| X_1$, $t \in \mathbb{R}$. It follows from (2.1.13) and (3.1.14) that for all $x \in X_1$ and $t \in \mathbb{R},$
$$A_1^t x = P_{X_1}^s A^t x = P_{X_1}^s \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k x = \sum_{k=0}^{\infty} \frac{1}{k!} P_{X_1}^s A^k x = \sum_{k=0}^{\infty} \frac{1}{k!} (A_1 t)^k x.
$$
This implies that $\mathfrak{A}_1$ is a uniformly continuous group with generator $A_1$. Conversely, suppose that $\mathfrak{A}_1$ is a uniformly continuous group generated by some operator $A_1 \in \mathcal{B}(X_1)$. Then by (2.1.14), for all $x \in X_1$ and all $n \in \mathbb{Z}^+$,
$$A_1^n x = \left( \frac{d^n}{dt^n} A_1^t x \right) \big|_{t=0} = \left( \frac{d^n}{dt^n} P_{X_1}^s A^t x \right) \big|_{t=0} = P_{X_1}^s \left( \frac{d^n}{dt^n} A^t x \right) \big|_{t=0} = P_{X_1}^s A^n x.
$$
This implies that $A_1$ is a compression of $A$ onto $X_1$ along $Z_1$.

(ii) $\Rightarrow$ (iii): Suppose that (ii) holds, and denote the group in (ii) by $\mathfrak{A}_1$. The semigroup properties of $\mathfrak{A}_1$ and $\mathfrak{A}$ give for $x \in X$, all $t \in \mathbb{R}$, and all and all $n \in \mathbb{Z}^+$,
$$A_1^n x = A_1^t A_1^t x = A_1^{nt} x = P_{X_1}^s A^{nt} x = P_{X_1}^s (A^t)^n x.$$
This means that $\mathfrak{A}_1$ is a compression of $A^t$ onto $X_1$ along $Z_1$ for all $t \in \mathbb{R}$.

(iii) $\Rightarrow$ (ii): Suppose that for all $t \in \mathbb{R}$ it is true that $\mathfrak{A}_1^t$ is a compression of $A^t$ onto $X_1$ along $Z_1$. If follows from the definition of $\mathfrak{A}_1$ that $\mathfrak{A}_1^0 = 1_{X_1}$ and that $\mathfrak{A}_1^t$ is a continuous function of $t$ in $\mathcal{B}(X_1)$. Let $s, t \in \mathbb{R}$, and suppose in addition that the quotient $s/t$ is rational (or that $t = 0$). Then there exists $n, m \in \mathbb{Z}^+$ and some $r \in \mathbb{R}$ such that $s = nr$ and $t = mr$. Since by assumption $\mathfrak{A}_1^t$ is a compression of $A^t$ onto $X_1$ along $Z_1$, it follows from (3.1.17) that for all $x \in X_1$,
$$A_1^n \mathfrak{A}_1^t x = \mathfrak{A}_1^{nr} A^{mr} x = (A_1^t)^n (A_1^t)^r x = P_{X_1}^s (A^t)^n P_{X_1}^s (A^t)^r x = P_{X_1}^s (A^t)^n P_{X_1}^s A^t x = P_{X_1}^s A^{nt} x = \mathfrak{A}_1^{nt} x = A_1^{nt} x.$$
Thus, $\mathfrak{A}_1^t \mathfrak{A}_1^t = \mathfrak{A}_1^{s+t}$ whenever $s/t$ is rational. Since $\mathfrak{A}_1^t$ is a continuous function of $t$, it follows that $\mathfrak{A}_1^t \mathfrak{A}_1^t = \mathfrak{A}_1^{s+t}$ for all $s, t \in \mathbb{R}$.

(ii) $\Leftrightarrow$ (iv): Denote $\mathfrak{A}_1^t := P_{X_1}^s A^t \big| X_1$, $t \in \mathbb{R}$. If (ii) holds, then for all $s, t \in \mathbb{R}$ we have
$$P_{X_1}^s A^t P_{X_1}^s A^t \big| X_1 = P_{X_1}^s A^{s+t} P_{X_1}^s A^t \big| X_1 = \mathfrak{A}_1^s \mathfrak{A}_1^t = P_{X_1}^s \mathfrak{A}_1^s P_{X_1}^s \mathfrak{A}_1^t \big| X_1.
$$
Thus (ii) implies (iii). Conversely, if (iii) holds, then it is easy to see that $\mathfrak{A}_1$ inherits all the properties of a uniformly continuous group $\mathfrak{A}$.
Motivated by Lemma 3.1.30 we make the following definition.

3.1.31. Definition. Let $\mathcal{X}$ be a $H$-space, let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\mathfrak{A}$ be a uniformly continuous group in $\mathcal{X}$. A uniformly continuous group $A_1$ in $X_1$ is called the compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$ if $A_1^t = P_{X_1}^Z A^t|_{X_1}$ for all $t \in \mathbb{R}$.

Thus by Lemma 3.1.30 a necessary and sufficient condition for a uniformly continuous group $\mathfrak{A}$ in $\mathcal{X}$ to have a compression onto a closed subspace $X_1$ along the direct complement $Z_1$ is that the family $t \mapsto P_{X_1}^Z A^t|_{X_1}$ is a uniformly continuous group in $X_1$.

3.1.32. Lemma. Let $\mathcal{X}$ be an $H$-space, let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\mathfrak{A}$ and $\mathfrak{A}_1$ be uniformly continuous groups in $\mathcal{X}$ respectively $X_1$ with generators $A \in \mathcal{B}(\mathcal{X})$ and $A_1 \in \mathcal{B}(X_1)$.

(i) The following conditions are equivalent:
- (a) $A_1$ is a compression of $A$ onto $X_1$ along $Z_1$;
- (b) $\mathfrak{A}_1$ is a compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$.

(ii) The following conditions are equivalent:
- (a) $A_1$ is a restriction of $A$ in $\mathcal{B}(X_1)$;
- (b) $X_1$ is $A$-invariant and $A_1$ is a compression of $A$ onto $X_1$ along $Z_1$;
- (c) $\mathfrak{A}_1$ is a restriction of $\mathfrak{A}$ in $\mathcal{B}(X_1)$.

(iii) The following conditions are equivalent:
- (a) $A_1$ is a projection of $A$ in $\mathcal{B}(X_1)$ along $Z_1$;
- (b) $Z_1$ is $A$-invariant and $A_1$ is a compression of $A$ onto $X_1$ along $Z_1$;
- (c) $\mathfrak{A}_1$ is a projection of $\mathfrak{A}$ in $\mathcal{B}(X_1)$ along $Z_1$;
- (d) $\mathfrak{A}_1$ is $\mathfrak{A}$-invariant subspace and $\mathfrak{A}_1$ is a compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$.

Proof. This follows from Theorems 3.1.9, 3.1.15 and 3.1.30 and Definitions 3.1.10, 3.1.16 and 3.1.31. 

3.1.33. Lemma. Let $\mathcal{X}$ be a $H$-space with a direct sum decomposition $\mathcal{X} = X_1 + Z_1$, and $A \in \mathcal{B}(\mathcal{X})$. Then the following conditions are equivalent:

(i) $A$ has a compression onto $X_1$ along $Z_1$;

(ii) for all $\lambda \in \rho_{\infty}(A)$ the operator $(\lambda - A)^{-1}$ has a compression in $\mathcal{B}(X_1)$ along $Z$;

(iii) for some $\lambda \in \rho_{\infty}(A)$ the operator $(\lambda - A)^{-1}$ has a compression onto $X_1$ along $Z$;

(iv) The resolvent of $A$ satisfies the condition

\begin{equation}
(3.1.21) \quad P_{X_1}^Z (\lambda - A)^{-1}(\mu - A)^{-1}|_{X_1} = P_{X_1}^Z (\lambda - A)^{-1}P_{X_1}^Z (\mu - A)^{-1}|_{X_1}
\end{equation}

for all $\lambda, \mu \in \rho_{\infty}(A)$;

(v) The resolvent of $A$ satisfies the condition

\begin{equation}
(3.1.22) \quad P_{X_1}^Z (\lambda - A)^{-1}P_{Z_1}^X (\mu - A)^{-1}|_{X_1} = 0
\end{equation}

for all $\lambda, \mu \in \rho_{\infty}(A)$;
Thus, also this function is identically zero in $\rho$

Suppose that the above equivalent conditions hold, and denote the compressions of (3.1.5) in $\rho$

For all $\lambda$, $\mu$ in $\rho(A)$.

An extension of this result to the case where $A$ and $(\lambda - A)^{-1}$ by $A_{\text{cmp}}$ respectively $(\lambda - A)_{\text{cmp}}^{-1}$, $\lambda$ in $\rho(A)$.

Proof of Lemma 3.1.33 (i) $\Rightarrow$ (iii): Denote $A_{\text{cmp}} := P^{Z_1}_{X_1} A|_{X_1}$ and let $\Omega$ be the set

For all $\lambda$ in $\Omega$ we can apply (3.1.4) to both $A$ and $A_{\text{cmp}}$, and it follows from (3.1.14) that

By differentiating this identity $n - 1$ times with respect to $\lambda$ and using (3.1.6) we find that

for all $\lambda$ in $\Omega$, i.e., the operator $(\lambda - A_{\text{cmp}})^{-1}$ is a compression of $(\lambda - A)^{-1}$ in $B(X_1)$ along $Z_1$ for all $\lambda$ in $\Omega$. In particular, (iii) holds.

(iii) $\Rightarrow$ (v): Denote the particular $\lambda$ for which (iii) holds by $\lambda_0$. By Lemma 3.1.3 the resolvent of $A$ is analytic in $\rho(A)$, and consequently, for every $m$ in $\mathbb{N}$ the function

The value of the derivative of order $n$ in $\mathbb{Z}^+$ with respect to $\mu$ of this function at the point $\mu = \lambda_0$ is equal to

which according to Lemma 3.1.27 is equal to zero since (iii) holds. Therefore this function vanishes identically in $\rho(A)$, i.e.,

Likewise, the function $\lambda \mapsto P^{Z_1}_{X_1}(\mu - A)^{-1}P^{X_1}_{Z_1}(\lambda_0 - A)^{-m}|_{X_1}$ is analytic in $\rho(A)$, and all its derivatives of order $m$ in $\mathbb{Z}^+$ with respect to $\lambda$ vanish at the point $\lambda = \lambda_0$. Thus, also this function is identically zero in $\rho(A)$. Thus (v) holds.

(iv) $\Leftrightarrow$ (v): This is obvious.

(iv) $\Leftrightarrow$ (vi): This is true since the resolvent of $A$ satisfies the resolvent identity (3.1.5) in $\rho(A)$.
(iv) ⇒ (i): Suppose that (iv) holds. Then for all \( \lambda \in \rho_\infty(A) \) we have
\[
P^Z_{\lambda_1} \left( A \left( 1 - \frac{1}{\lambda} A \right)^{-1} \right)^2 \big|_{\lambda_1} = P^Z_{\lambda_1} \left( \lambda \left( 1 - \frac{1}{\lambda} A \right)^{-1} - 1, A \right) \big|_{\lambda_1} - P^Z_{\lambda_1} \left( \lambda \left( 1 - \frac{1}{\lambda} A \right)^{-1} - 1, A \right) \big|_{\lambda_1} = \left( P^Z_{\lambda_1} \left( A \left( 1 - \frac{1}{\lambda} A \right)^{-1} \right)^2 \big|_{\lambda_1} - 1 \right)^2.
\]
This implies that for all \( n \in \mathbb{N} \) we have
\[
P^Z_{\lambda_1} \left( A \left( 1 - \frac{1}{\lambda} A \right)^{-1} \right)^n \big|_{\lambda_1} = \left( P^Z_{\lambda_1} \left( A \left( 1 - \frac{1}{\lambda} A \right)^{-1} \right)^2 \big|_{\lambda_1} - 1 \right)^n.
\]
Letting \( \lambda \to \infty \) and using the fact that \( A \left( 1 - \frac{1}{\lambda} A \right)^{-1} \to A \) in \( B(\mathcal{X}) \) we get
\[
P^Z_{\lambda_1, A} \big|_{\lambda_1} = \left( P^Z_{\lambda_1, A} \big|_{\lambda_1} \right)^n. \]
Thus \( A_{\text{cmp}} := P^Z_{\lambda_1, A} \big|_{\lambda_1} \) is a compression of \( A \) in \( B(\mathcal{X}) \) along \( \mathcal{Z}_1 \).

(vi) ⇒ (vii): This is trivial.

(vii) ⇒ (i): Suppose that, for example, that (vii) holds with \( \rho_\infty(A) \) replaced by \( \Omega \). Then the analyticity of function in (3.1.26) implies that (3.1.26) holds for all \( \lambda \in \rho_\infty(A) \), and thus (vii) holds. As we saw above, this implies (i). The proofs for the cases where (v) or (vi) holds with \( \rho_\infty(A) \) replaced by \( \Omega \) is analogous.

Proof of (viii): In the proof of the implication (i) ⇒ (iii) we observed that (3.1.26) holds for all \( \lambda \in \Omega \). Since both sides of this identity are analytic in \( \rho(A) \cap \rho(A_{\text{cmp}}) \) we find that (3.1.26) holds for all \( \lambda \) in the unbounded component of \( \rho(A) \cap \rho(A_{\text{cmp}}) \). By Theorem 3.1.34, \( (\lambda - A_{\text{cmp}})^{-1} \|_{B(\mathcal{Z}_1)} \to \infty \) as \( \lambda \) tends to the boundary of \( \rho(A_{\text{cmp}}) \). Since
\[
\| (\lambda - A_{\text{cmp}})^{-1} \|_{B(\mathcal{Z}_1)} \leq \| P^Z_{\lambda_1} \|_{B(\mathcal{Z}_1)} \| (\lambda - A)^{-1} \|_{B(\mathcal{Z})} \leq \| P^Z_{\lambda_1} \|_{B(\mathcal{Z}_1)} \| (\lambda - A)^{-1} \|_{B(\mathcal{X})}
\]
and \( (\lambda - A)^{-1} \|_{B(\mathcal{X})} \) is locally bounded in \( \rho_\infty(A) \), the set \( \rho_\infty(A) \) cannot contain any boundary points of \( \rho(A_{\text{cmp}}) \). This implies that \( \rho_\infty(A) \subset \rho(A_{\text{cmp}}) \). By the analyticity of both sides of (3.1.26), this identity must hold for all \( \lambda \in \rho_\infty(A) \) since it holds for all \( \lambda \in \Omega \). Thus (viii) holds.

By combining Lemmas 3.1.30 and 3.1.33 we get the following theorem.

3.1.34. Theorem. Let \( \mathcal{X} \) be an \( H \)-space with a direct sum decomposition \( \mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1 \), and let \( \mathfrak{A} \) be a uniformly continuous group in \( \mathcal{X} \) with generator \( A \in B(\mathcal{X}) \). Then the following conditions are equivalent:

(i) \( A \) has a compression onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \);
(ii) the family \( t \to P^Z_{\lambda_1} \mathfrak{A}^t \big|_{\lambda_1} \) is a uniformly continuous group in \( \mathcal{X}_1 \);
(iii) \( \mathfrak{A}^t \) has a compression onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) for all \( t \in \mathbb{R} \);
(iv) \( (\lambda - A)^{-1} \) has a compression onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) for all \( \lambda \in \rho_\infty(A) \);
(v) \( (\lambda - A)^{-1} \) has a compression onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) for some \( \lambda \in \rho_\infty(A) \);
(vi) The function \( \lambda \mapsto P^Z_{\lambda_1} (\lambda - A)^{-1} \big|_{\lambda_1} \) satisfies the resolvent identity (3.1.5) in \( \rho_\infty(A) \).
Suppose that these equivalent conditions hold, and denote the compressions of $A$ and $(\lambda - A)^{-1}$ by $A_{\text{cmp}}$ respectively $(\lambda - A)^{-1}_{\text{cmp}}$. Then

(vii) $\rho_{\infty}(A) \subset \rho(A_{\text{cmp}})$ and $(\lambda - A)^{-1}_{\text{cmp}} = (\lambda - A)^{-1}$ for all $\lambda \in \rho_{\infty}(A)$.

**Proof.** This follows from Lemmas 3.1.30 and 3.1.33. □

### 3.1.7. The general structure of a compression of a bounded linear operator.

It will be shown below that for bounded linear operators as well as for $C_0$ groups and semigroups in some $H$-space $X$ every compression onto some subspace be obtained as a superposition of a restriction and a projection, or alternatively, as a superposition of a projection and a restriction. This will first be shown for the case of bounded linear operators. We start with a preliminary lemma.

#### 3.1.35. Lemma.

Let $X$ be a $H$-space with the direct sum decomposition $X = X_1 + Z_1$, and let $X_2$ be a closed subspace of $X$ which contains $X_1$. Then

(3.1.28)

$$X_2 = X_1 + (X_2 \cap Z_1),$$

and the complementary projections $P_{X_1 \cap Z_1}$ and $P_{X_1 \cap Z_1}^X$ in $X_2$ are given by

(3.1.29)

$$P_{X_1 \cap Z_1} = P_{X_1 \cap Z_1}^X,$$

$$P_{X_1 \cap Z_1}^X = P_{X_1 \cap Z_1}^X.$$

**Proof.** Since $P_{X_1 \cap Z_1}^X + P_{X_1 \cap Z_1}^X = 1_X$, we have $P_{X_1 \cap Z_1}^X |_{X_2} + P_{X_1 \cap Z_1}^X |_{X_2} = 1_{X_2}$.

The operator $P_{X_1 \cap Z_1}^X |_{X_2}$ is a projection in $X_2$ since $P_{X_1 \cap Z_1}^X$ is a projection in $X$ and $X_1 \subset X_2$, and therefore also $P_{X_1 \cap Z_1}^X |_{X_2} = 1_{X_2} - P_{X_1 \cap Z_1}^X |_{X_2}$ is a projection in $X_2$. These two projections are complementary since their sum is the identity. The kernel of $P_{X_1 \cap Z_1}^X |_{X_2}$ is $X_1$, and the kernel of $P_{X_1 \cap Z_1}^X |_{X_2}$ is $X_2 \cap Z_1$, and hence the range of $P_{X_1 \cap Z_1}^X |_{X_2}$ is $X_2 \cap Z_1$ and the range of $P_{X_1 \cap Z_1}^X |_{X_2}$ is $X_1$. Thus (3.1.28) and (3.1.29) hold. □

#### 3.1.36. Lemma.

Let $X$ be a $H$-space with the direct sum decomposition $X = X_1 + Z_1$, and let $A \in B(X)$.

(i) Define

(3.1.30)

$$\mathcal{X}_{\text{min}} := \bigvee_{n \in \mathbb{Z}^+} A^n \mathcal{X}_1.$$

Then $\mathcal{X}_{\text{min}}$ is the minimal closed $A$-invariant subspace which contains $X_1$ (i.e., $\mathcal{X}_{\text{min}}$ is closed and $A$-invariant, and $\mathcal{X}_{\text{min}}$ is contained in every other closed $A$-invariant subspace which contains $X_1$).

(ii) The space $\mathcal{X}_{\text{min}}$ has the direct sum decomposition $\mathcal{X}_{\text{min}} = X_1 + Z_{\text{min}}$,

where

(3.1.31)

$$Z_{\text{min}} = \mathcal{X}_{\text{min}} \cap Z_1 = P_{Z_1}^X \mathcal{X}_{\text{min}}.$$

(iii) Define

(3.1.32)

$$Z_{\text{max}} := \bigcap_{n \in \mathbb{Z}^+} \{ x \in X \mid A^n x \in Z_1 \}.$$

Then $Z_{\text{max}}$ is the maximal $A$-invariant subspace which is contained in $Z_1$ (i.e., $Z_{\text{max}}$ is $A$-invariant, and $Z_{\text{max}}$ contains every other $A$-invariant subspace which is contained in $Z_1$).

Note that the space $X_{\text{min}}$ in (3.1.30) depends only on $X_1$ (and not on $Z_1$), and that the space $Z_{\text{max}}$ in (3.1.32) depends only on $Z_1$ (and not on $X_1$).
Proof of Lemma 3.1.36: Proofs of (i) and (iii): It is easy to see that both $\mathcal{X}_{\text{min}}$ and $\mathcal{Z}_{\text{max}}$ are closed $A$-invariant subspaces of $\mathcal{X}$, and that $\mathcal{X}_1 \subset \mathcal{X}_{\text{min}}$ and $\mathcal{Z}_{\text{max}} \subset \mathcal{Z}_1$.

If $\mathcal{X}_2$ is an arbitrary closed $A$-invariant subspace which contains $\mathcal{X}_1$, then $A^n \mathcal{X}_1 \subset \mathcal{X}_2$ for all $n \in \mathbb{Z}^+$, and thus also $\mathcal{X}_{\text{min}} = \bigvee_{n \in \mathbb{Z}^+} A^n \mathcal{X}_1 \subset \mathcal{X}_2$. Thus $\mathcal{X}_{\text{min}}$ is minimal among all closed $A$-invariant subspaces which contain $\mathcal{X}_1$. Likewise, if $\mathcal{Z}_2$ is an arbitrary $A$-invariant subspace which is contained in $\mathcal{Z}_1$, then necessarily $A^n \mathcal{Z}_2 \subset \mathcal{Z}_2 \subset \mathcal{Z}_1$ for all $n \in \mathbb{Z}^+$, and thus $\mathcal{Z}_2 \subset \mathcal{Z}_{\text{max}}$. Thus $\mathcal{Z}_{\text{max}}$ is maximal among all $A$-invariant subspaces which are contained in $\mathcal{X}_1$.

Proof of (ii): Define $\mathcal{Z}_{\text{min}}$ by $\mathcal{Z}_{\text{min}} = \mathcal{X}_{\text{min}} \cap \mathcal{Z}_1$. Then by Lemma 3.1.35, $\mathcal{X}_{\text{min}} = \mathcal{X}_1 + \mathcal{Z}_{\text{min}}$ and $\mathcal{Z}_{\text{min}} = \mathcal{X}_{\text{min}} \cap \mathcal{Z}_1 = P_{\mathcal{X}_1} \mathcal{X}_{\text{min}}$.

3.1.37. Lemma. Let $\mathcal{X}$ be a $H$-space with the direct sum decomposition $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$, and let $A \in \mathcal{B}(\mathcal{X})$. Let $\Omega'$ be an arbitrary subset of $\rho_{\infty}(A)$ which has a cluster point in $\rho_{\infty}(A)$, and let $\lambda_0 \in \rho_{\infty}(A)$. Then the following claims are true:

(i) The subspace $\mathcal{X}_{\text{min}}$ defined in (3.1.30) can be computed in the following alternative ways:

$$\mathcal{X}_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng} \left( \mathcal{A}^t \big|_{\mathcal{X}_1} \right) = \bigvee_{t \in \mathbb{R}^-} \text{rng} \left( \mathcal{A}^t \big|_{\mathcal{X}_1} \right) = \bigvee_{n \in \mathbb{Z}^+} \text{rng} \left( A^n \big|_{\mathcal{X}_1} \right)$$

(3.1.33)

(ii) The subspace $\mathcal{Z}_{\text{max}}$ defined in (3.1.32) can be computed in the following alternative ways:

$$\mathcal{Z}_{\text{max}} = \bigcap_{t \in \mathbb{R}^+} \{ x \in \mathcal{X} \mid \mathcal{A}^t x \in \mathcal{Z}_1 \} = \bigcap_{t \in \mathbb{R}^-} \{ x \in \mathcal{X} \mid \mathcal{A}^t x \in \mathcal{Z}_1 \} = \bigcap_{n \in \mathbb{Z}^+} \{ x \in \mathcal{X} \mid A^n x \in \mathcal{Z}_1 \}$$

(3.1.34)

Proof. Proof of (i): The function $t \mapsto \mathcal{A}^t$ can be extended to an entire $\mathcal{B}(\mathcal{X})$-valued function by allowing the parameter $t$ in (2.1.13) to take complex values. The derivative of this function of order $n$ at the point zero is equal to $A^n$. Therefore by Lemma A.3.6

$$\bigvee_{t \in \mathbb{R}^+} \text{rng} \left( \mathcal{A}^t \big|_{\mathcal{X}_1} \right) = \bigvee_{t \in \mathbb{R}^-} \text{rng} \left( \mathcal{A}^t \big|_{\mathcal{X}_1} \right) = \bigvee_{n \in \mathbb{Z}^+} \text{rng} \left( A^n \big|_{\mathcal{X}_1} \right).$$

By Lemma 3.1.36 the function $\lambda \mapsto (\lambda - A)^{-1}$ is analytic in $\rho_{\infty}(A)$, and the derivative of this function of order $n$ at the point $\lambda_0$ is equal to $(-1)^n n! (\lambda_0 - A)^{-(n+1)}$. Therefore by Lemma A.3.6
Therefore by Lemma \textbf{A.3.6}
\[
\bigvee_{\lambda \in \rho_\infty(A)} \text{rng} \left( \left( \lambda - A \right)^{-1} |_{X_1} \right) = \bigvee_{\lambda \in \Omega'} \text{rng} \left( \left( \lambda - A \right)^{-1} \right)_{X_1} = \bigvee_{n \in \mathbb{Z}^+} \text{rng} \left( (\lambda_0 - A)^{-(n+1)} |_{X_1} \right).
\]

By replacing $1/\lambda$ by $z$ in \textbf{3.1.4} we find that the function $z \mapsto (1 - zA)^{-1} |_{X_1} = \sum n \in \mathbb{Z}^+(zA)^n |_{X_1}$ is analytic in the neighborhood $\Lambda_0(A) := \{0\} \cup \{ z \in \mathbb{C} \mid 1/z \in \rho_\infty(A) \}$ of zero. The derivative of order $n$ of this function at zero is equal to $n!A^n |_{X_1}$. Therefore, by Lemma \textbf{A.3.6}
\[
\bigvee_{n \in \mathbb{Z}^+} \text{rng} (A^n |_{X_1}) = \bigvee_{1/z \in \rho_\infty(A)} (1 - zA)^{-1} |_{X_1} = \bigvee_{\lambda \in \rho_\infty(A)} (\lambda - A)^{-1} |_{X_1} = \bigvee_{n \in \mathbb{Z}^+} (\lambda - A)^{-(n+1)} |_{X_1}.
\]

This proves that all the different expressions in \textbf{3.1.33} are equal to each other.

\textit{Proof of (ii):} The proof of (ii) is analogous to the proof of (i). One way to carry out this proof is to first rewrite \textbf{3.1.34} in the equivalent form
\[
Z_{\max} = \bigvee_{t \in \mathbb{R}^+} \ker \left( P_{X_1}^t \right) = \bigvee_{t \in \mathbb{R}^-} \ker \left( P_{X_1}^t \right) = \bigvee_{n \in \mathbb{Z}^+} \ker \left( P_{X_1}^{t} A^n \right)
\]
\[
= \bigvee_{\lambda \in \rho_\infty(A)} \ker \left( P_{X_1}^a (\lambda - A)^{-1} \right) = \bigvee_{\lambda \in \Omega'} \ker \left( P_{X_1}^{a} (\lambda - A)^{-1} \right)
\]
\[
= \bigvee_{n \in \mathbb{Z}^+} \ker \left( P_{X_1}^{a} (\lambda_0 - A)^{-(n+1)} \right).
\]

and to proceed in the same was in the proof of (i) above, replacing \textbf{3.1.34} by \textbf{A.3.6}. (The reason why we have used \textbf{3.1.34} in the formulation of Lemma \textbf{3.1.37} is that it is obvious from \textbf{3.1.34} that $Z_{\max}$ does not depend on $X_1$, but only on $Z_1$.)

\textbf{3.1.38. Theorem.} Let $X$ be a $H$-space with a direct sum decomposition $X = X_1 + Z_1$, and let $A \in B(X)$. Then the following conditions are equivalent:

(i) $A$ has a compression onto $X_1$ along $Z_1$.

(ii) There exists a closed subspace $Z$ of $X_1$ such that both $Z$ and $X_1 + Z$ are $A$-invariant subspaces.

(iii) $Z_{\min}$ in an $A$-invariant subspace where $Z_{\min}$ is given by \textbf{3.1.31}.

(iv) $X_1 + Z_{\max}$ is an $A$-invariant subspace, where $Z_{\max}$ is given by \textbf{3.1.32}.

(v) $Z_{\min} \subset Z_{\max}$.

Two possible choices of the subspace $Z$ in (ii) are $Z = Z_{\min}$ and $Z = Z_{\max}$, and every possible subspace $Z$ in (ii) satisfies $Z_{\min} \subset Z \subset Z_{\max}$.

\textit{Proof.} (i) $\Rightarrow$ (iii): Suppose that (i) holds, and let $A := P_{X_1}^a A |_{X_1}$ be the compression of $A$ onto $X_1$ along $Z_1$. It follows from Lemma \textbf{3.1.27} that for all $n \in \mathbb{Z}^+$ and all $x \in X_1$ we have $P_{X_1}^a A P_{X_1}^a A^n x = 0$. Taking the closed linear span over all $x \in X_1$ and all $n \in \mathbb{Z}^+$ we find that $P_{X_1}^a A P_{X_1}^a A_{\min} = 0$, where $A_{\min}$ is defined by \textbf{3.1.30}. Here $P_{X_1} A_{\min} = Z_{\min}$, so $P_{X_1} A Z_{\min} = 0$, i.e., $AZ_{\min} \subset Z_1$. 

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On the other hand, \( AZ_{\text{min}} \subset AX_{\text{min}} \subset X_{\text{min}} \) since \( Z_{\text{min}} \subset X_{\text{min}} \) and \( X_{\text{min}} \) is \( A \)-invariant. Thus, \( AZ_{\text{min}} \subset X_{\text{min}} \cap Z_1 = Z_{\text{min}} \). This proves (ii).

(iii) \( \Rightarrow \) (ii): This follows from Lemma 3.1.36 (take \( Z = X_{\text{min}} \)).

(ii) \( \Rightarrow \) (i): Suppose that (ii) holds, and define \( A_1 := P_{X_1} A_{|X_1} \). It follows from (ii) that for all \( n \in \mathbb{Z}^+ \),

\[
A^n (X_1 + Z) \subset X_1 + Z, \quad A^n Z \subset Z.
\]

Thus, \( P_{X_1} A^n X_1 \subset \mathbb{Z} \), and for all \( m \in \mathbb{Z}^+ \) we have \( A^n P_{X_1} A^n X_1 \subset \mathbb{Z} \). Therefore \( P_{X_1} A^n P_{X_1} A^n X_1 = \{0\} \). By Lemma 3.1.27 \( A_1 \) is the compression of \( A \) onto \( X_1 \) along \( Z_1 \).

(i) \( \Rightarrow \) (iv): Suppose that (i) holds. By Lemma 3.1.27 for all \( x \in X_1 \) we have \( P_{X_1} A^n P_{X_1} A x = 0 \), i.e., \( A^n P_{X_1} A x \in Z_1 \). By (3.1.32), this means that \( P_{X_1} A z \in \mathbb{Z}_{\text{max}} \), and consequently \( A x \in X_1 + \mathbb{Z}_{\text{max}} \). This shows that \( A X_1 \subset X_1 + \mathbb{Z}_{\text{max}} \). By Lemma 3.1.36 \( A \mathbb{Z}_{\text{max}} \subset \mathbb{Z}_{\text{max}} \). Thus \( A(X_1 + \mathbb{Z}_{\text{max}}) \subset X_1 + \mathbb{Z}_{\text{max}} \).

(iv) \( \Rightarrow \) (ii): If (iv) holds, then it follows from Lemma 3.1.36 that (ii) holds with \( Z = \mathbb{Z}_{\text{max}} \).

(iii) \( \Rightarrow \) (v): This follows from Lemma 3.1.36.

(v) \( \Rightarrow \) (iii): By Lemma 3.1.36 both \( X_1 + \mathbb{Z}_{\text{min}} \) and \( \mathbb{Z}_{\text{max}} \) are \( A \)-invariant. The \( A \)-invariance of \( A_1 + \mathbb{Z}_{\text{min}} \) implies that \( A^n \mathbb{Z}_{\text{min}} \subset \mathbb{Z}_{\text{min}} \) for all \( n \in \mathbb{Z}^+ \), and the \( A \)-invariance of \( \mathbb{Z}_{\text{max}} \) together with the condition \( \mathbb{Z}_{\text{min}} \subset \mathbb{Z}_{\text{max}} \) implies that \( A^n \mathbb{Z}_{\text{min}} \subset A^n \mathbb{Z}_{\text{max}} \subset \mathbb{Z}_{\text{max}} \subset \mathbb{Z}_{\text{min}} \) for all \( n \in \mathbb{Z}^+ \). Thus \( A \mathbb{Z}_{\text{min}} \subset (X_1 + \mathbb{Z}_{\text{min}}) \cap Z_1 = \mathbb{Z}_{\text{min}} \), so (iii) holds.

3.1.39. Corollary. Let \( X \) be a \( H \)-space with a direct sum decomposition \( X = X_1 + Z_1 \), let \( A \in \mathcal{B}(X) \), and define \( A_1 := P_{X_1} A_{|X_1} \). Then \( A_1 \) is the compression of \( A \) onto \( X_1 \) along \( Z_1 \) if and only if \( Z_1 \) has a direct sum decomposition \( Z_1 = Z + Z_c \) such that \( A \) has the following structure with respect to the decomposition \( X = Z + X_1 + Z_c \) of \( X \) (where we use "\( * \)" to mark irrelevant entries):

\[
A = \begin{bmatrix}
A_Z & * & * \\
0 & A_1 & * \\
0 & 0 & A_{Z_c}
\end{bmatrix}.
\]

From this structure follows that both \( Z \) and \( Z + X_1 \) are \( \mathcal{A} \)-invariant, that \( t \mapsto \mathcal{A}_Z^t \) is the restriction of \( \mathcal{A} \) to \( Z \), and that \( t \mapsto \mathcal{A}_{X_1}^t \) is the projection of \( \mathcal{A} \) onto \( Z_1 \) along \( X_1 + Z \). The subspace \( Z \) in this decomposition can be chosen to be the same as the subspace \( Z \) in condition (ii) in Theorem 3.1.38 and the subspace \( Z_c \) can be chosen to be an arbitrary direct complement to \( Z \) in \( Z_1 \). In particular, two possible choices of \( Z \) are \( Z = \mathbb{Z}_{\text{min}} \) and \( Z = \mathbb{Z}_{\text{max}} \), where \( \mathbb{Z}_{\text{min}} \) and \( \mathbb{Z}_{\text{max}} \) are defined by (3.1.31) and (3.1.32).

Proof. This follows from the equivalence of (i) and (ii) in Theorem 3.1.38 (take \( Z_c \) to be an arbitrary direct complement to \( Z \) in \( Z_1 \)).

3.1.40. Theorem. Let \( X \) be a \( H \)-space with a direct sum decomposition \( X = X_1 + Z_1 \), let \( A \in \mathcal{B}(X) \), define \( A_1 := P_{X_1} A_{|X_1} \), and suppose that \( A_1 \) is the compression of \( A \) onto \( X_1 \) along \( Z_1 \). Let \( Z \) satisfy the conditions listed in (ii) in Theorem 3.1.38 and let \( Z_c \) be an arbitrary direct complement to \( Z \) in \( Z_1 \). Then the following claims are true:

(i) \( A_1 \) is the projection in \( \mathcal{B}(X_1) \) along \( Z \) of the restriction of \( A \) in \( \mathcal{B}(X_1 + Z) \);
(ii) $A_1$ is the restriction in $\mathcal{B}(\mathcal{X}_1)$ of the projection of $A$ in $\mathcal{B}(\mathcal{X}_1 + Z_c)$ along $Z$.

**Proof.** This follows from Corollary 3.1.39 and Theorem 3.2.27. □

3.1.41. **Lemma.** Let $\mathcal{X}$ be a $H$-space with a direct sum decomposition $\mathcal{X} = \mathcal{X}_1 + Z_1$, let $A \in \mathcal{B}(\mathcal{X})$, and suppose that $A_1 := P_{\mathcal{X}_1} A|_{\mathcal{X}_1}$ is the compression of $A$ onto $\mathcal{X}_1$ along $Z_1$. Let $Z$ satisfy the conditions in condition (ii) in Theorem 3.1.38, and let $Z_c$ be an arbitrary direct complement to $Z$ in $Z_1$. Then $A$ and $A_1$ are intertwined by the operator $P_{\mathcal{X}_1}^{Z_1}|_{\mathcal{X}_1 + Z}$, interpreted as a continuous operator from $\mathcal{X}$ to $\mathcal{X}_1$ with closed domain $\mathcal{X}_1 + Z$.

**Proof.** This follows from Lemmas 3.1.22 and 3.1.23. □
3.2. Intertwinements and Compressions of Bounded I/S/O Systems
(Jan 02, 2016)

In this section the results developed in the previous section for bounded linear operators and the uniformly continuous groups generated by these operators will be extended to bounded i/s/o systems.

3.2.1. Strongly invariant and unobservably invariant subspaces. At this point the reader may want to recall the notions of strongly invariant and unobservably invariant subspaces of an i/s/o system introduced in Definition 2.5.8

3.2.1. Lemma. Let Σ = (\( A \), \( B \); \( X, U, Y \)) be a bounded i/s/o system.

(i) If \( Z \) is a strongly invariant or unobservable invariant subspace for \( \Sigma \), then the closure of \( Z \) is also strongly invariant respectively unobservably invariant for \( \Sigma \).

(ii) If both \( Z_1 \) and \( Z_2 \) are strongly invariant for \( \Sigma \), then \( Z_1 + Z_2 \) and \( Z_1 \cap Z_2 \) are strongly invariant for \( \Sigma \).

(iii) If both \( Z_1 \) and \( Z_2 \) are unobservably invariant for \( \Sigma \), then \( Z_1 \cap Z_2 \) is unobservably invariant for \( \Sigma \).

Proof. This follows from Definition 2.5.8 and the representation formula (2.1.19) for trajectories of bounded i/s/o systems.

3.2.2. Lemma. Let \( \Sigma = (\left[ \begin{array}{c} A \\ B \end{array} \right] ; X, U, Y) \) be a bounded i/s/o system with evolution group \( \mathfrak{A} \), and let \( Z \) be a closed subspace of \( X \). Then the following conditions are equivalent:

(i) \( Z \) is a (forward) strongly invariant subspace for \( \Sigma \);

(ii) \( Z \) is a backward strongly invariant subspace for \( \Sigma \);

(iii) \( \text{rng}(B) \subset Z \) and \( AZ \subset Z \);

(iv) \( \text{rng}(B) \subset Z \) and \( \mathfrak{A}t Z = Z \) for all \( t \in \mathbb{R} \);

(v) \( \text{rng}(B) \subset Z \) and \( (\lambda - A)^{-1} Z = Z \) for all \( \lambda \in \rho_{\infty}(A) \);

(vi) \( \text{rng}(B) \subset Z \) and \( (\lambda - A)^{-1} \subset Z \) for some \( \lambda \in \rho_{\infty}(A) \);

(vii) \( \text{rng}((\lambda - A)^{-1} B) \subset Z \) and \( (\lambda - A)^{-1} Z = Z \) for all \( \lambda \in \rho_{\infty}(A) \);

(viii) \( \text{rng}((\lambda - A)^{-1} B) \subset Z \) and \( (\lambda - A)^{-1} \subset Z \) for some \( \lambda \in \rho_{\infty}(A) \);

Proof. (iii) ⇔ (iv) ⇔ (v) ⇔ (vi): See Theorem 3.1.9

(iv) ⇒ (i) and (iv) ⇒ (ii): This follows from Definition 2.5.8 and Theorem 2.1.14

(i) ⇒ (iv): Suppose that (i) holds. We first show that \( \mathfrak{A} t Z \subset Z \) for all \( t \in \mathbb{R}^+ \)

Let \( x^0 \in \mathfrak{X} \), and define \( x(t) = \mathfrak{A} t x^0 \), \( u(t) = 0 \), and \( y(t) = C \mathfrak{A} t x^0 \), \( t \in \mathbb{R}^+ \). Then \( \left[ \begin{array}{c} x(t) \\ u(t) \\ y(t) \end{array} \right] \) is a classical future trajectory of \( \Sigma \). By the strong invariance of \( Z \), we therefore have \( x(t) \in Z \) for all \( t \in \mathbb{R}^+ \). Thus, \( \mathfrak{A} t Z \subset Z \) for all \( t \in \mathbb{R}^+ \). By Lemma 3.1.7 this implies that \( \mathfrak{A} t Z = Z \) for all \( t \in \mathbb{R} \).

To complete the proof of the claim that (i) ⇒ (iv) it still remains to show that \( \text{rng}(B) \subset Z \). To do this we fix some arbitrary \( u^0 \in U \) and, for all \( n \in \mathbb{N} \), we define

\[
(3.2.1) \quad u_n(t) = \begin{cases} n u^0, & 0 \leq t \leq 1/n, \\ 0, & t > 1/n. \end{cases}
\]

and let \( \left[ \begin{array}{c} x_n \\ u_n \\ y_n \end{array} \right] \) be the future generalized trajectory satisfying \( x(0) = 0 \) given by Theorem 2.1.14 Then by the strong invariance of \( \Sigma \) we have \( x_n(t) \in Z \) for all
\( t \in \mathbb{R}^+ \). By Theorem 2.1.14

\[(3.2.2) \quad x_n(t) = \mathfrak{t}^n \int_0^{1/n} \mathfrak{A}^{-s} Bu_0 \, ds, \quad t \geq 1/n. \]

Here \( n \int_0^{1/n} \mathfrak{A}^{-s} Bu_0 \, ds \to Bu_0 \) as \( n \to \infty \), and hence

\[(3.2.3) \quad \lim_{n \to \infty} x_n(t) = \mathfrak{t}^t Bu_0, \quad t > 0, \]

\[(3.2.3) \quad \lim_{n \to \infty} x_n(1/n) = Bu_0. \]

(The first limit in (3.2.3) cannot be locally uniform in \( t \) since \( x_n(0) = 0 \) for all \( n \).

Since \( x_n(1/n) \in \mathcal{Z} \) for all \( n \in \mathbb{Z}^+ \) and \( \mathcal{Z} \) is closed, we get \( Bu_0 \in \mathcal{Z} \). This shows that (i) \( \Rightarrow \) (iv).

(ii) \( \Rightarrow \) (iv): Apply the same argument as in the proof of the implication (i) \( \Rightarrow \) (iv) with \( A \) replaced by \( -A \) and \( \mathfrak{t}^t \) replaced by \( \mathfrak{A}^{-s} = e^{-At}, \ t \in \mathbb{R} \).

(v) \( \Leftrightarrow \) (vii): Trivially (v) \( \Rightarrow \) (vii). Conversely, suppose that (vii) holds. Then \( \lambda(\lambda - A)^{-1}Bu \subset \mathcal{Z} \) for all \( u \in \mathcal{U} \) and \( \lambda \in \rho_{\infty}(A) \), and consequently also

\[ Bu = \lim_{\lambda \to \infty} \lambda(\lambda - A)^{-1}Bu \in \mathcal{Z}, \quad u \in \mathcal{U}. \]

(vii) \( \Leftrightarrow \) (viii): Trivially (vii) \( \Rightarrow \) (viii). Conversely, suppose that (viii) holds for some \( \lambda_0 \in \rho_{\infty}(A) \). Then \( \text{rng} \left( \lambda_0(\lambda - A)^{-1}B\big|_{\lambda=\lambda_0} \right) = \text{rng} \left( (\lambda_0 - A)^{-1}B \big|_{\lambda=\lambda_0} \right) \subset \mathcal{Z} \) for all \( n \in \mathbb{N} \) and by Lemma A.3.6 \( \text{rng} \left( (\lambda - A)^{-1}B \big|_{\lambda=\lambda_0} \right) \subset \mathcal{Z} \) for all \( \lambda \in \rho_{\infty}(A) \). That \( (\lambda - A)^{-1} = \mathcal{Z} \) for all \( \lambda \in \rho_{\infty}(A) \) follows from Lemma 3.1.7 \( \Box \)

3.2.3. Lemma. Let \( \Sigma = (A \ B \ C \ D), \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a bounded i/s/o system with evolution group \( \mathfrak{A} \), and let \( \mathcal{Z} \) be a closed subspace of \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( \mathcal{Z} \) is a (forward) unobservably invariant subspace for \( \Sigma \);
(ii) \( \mathcal{Z} \) is a backward unobservably invariant subspace for \( \Sigma \);
(iii) \( \mathcal{Z} \subset \ker(C) \) and \( A \mathcal{Z} \subset \mathcal{Z} \);
(iv) \( \mathcal{Z} \subset \ker(C) \) and \( \mathfrak{A}^t \mathcal{Z} = \mathcal{Z} \) for all \( t \in \mathbb{R} \);
(v) \( \mathcal{Z} \subset \ker(C) \) and \( (\lambda - A)^{-1} \mathcal{Z} = \mathcal{Z} \) for all \( \lambda \in \rho_{\infty}(A) \);
(vi) \( \mathcal{Z} \subset \ker(C) \) and \( (\lambda - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \) for some \( \lambda \in \rho_{\infty}(A) \);
(vii) \( \mathcal{Z} \subset \ker(C(\lambda - A)^{-1}) \) and \( (\lambda - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \) for all \( \lambda \in \rho_{\infty}(A) \);
(viii) \( \mathcal{Z} \subset \ker(C(\lambda - A)^{-1}) \) and \( (\lambda - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \) for some \( \lambda \in \rho_{\infty}(A) \).

Proof. (iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v) \( \Leftrightarrow \) (vi): See Theorem 3.1.9

(iv) \( \Rightarrow \) (i) and (iv) \( \Rightarrow \) (ii): This follows from Definition 2.5.8 and Theorem 2.1.14

(i) \( \Rightarrow \) (iv): Suppose that (i) holds. Let \( z^0 \in \mathcal{Z} \). Then according to (i), there exists some generalized future trajectory \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) of \( \Sigma \) satisfying \( x(0) = z^0 \) and \( x(t) \in \mathcal{Z} \) for all \( t \in \mathbb{R}^+ \). By Theorem 2.1.14, \( x(t) = \mathfrak{t}^t z^0 \) and \( 0 = y(t) = C \mathfrak{t}^t z^0, \ t \in \mathbb{R}^+ \). In particular, \( 0 = y(0) = Cz^0 \). This shows that \( \mathfrak{A}^t \mathcal{Z} \subset \mathcal{Z} \) and that \( \mathcal{Z} \subset \ker(C) \). By Lemma 3.1.7 \( \mathfrak{A}^t \mathcal{Z} \subset \mathcal{Z} \) for all \( t \in \mathbb{R} \).

(ii) \( \Rightarrow \) (iv): Apply the same argument as in the proof of the implication (i) \( \Rightarrow \) (iv) with \( A \) replaced by \( -A \) and \( \mathfrak{t}^t \) replaced by \( \mathfrak{A}^{-s} = e^{-At}, \ t \in \mathbb{R} \).

(v) \( \Leftrightarrow \) (vii): Trivially (v) \( \Rightarrow \) (vii). Conversely, suppose that (vii) holds. Then

\[ Cz = \lim_{\lambda \to \infty} \lambda C(\lambda - A)^{-1}z = 0, \quad z \in \mathcal{Z}. \]
(vii) $\Leftrightarrow$ (viii): Trivially (vii) $\Rightarrow$ (viii). Conversely, suppose that (viii) holds for some $\lambda_0 \in \rho_\infty(A)$. Then $Z \subset \ker (C(\lambda_0 - A)^{-1}) = \ker \left( \frac{d}{dt} C(\lambda - A)^{-1} |_{\lambda = \lambda_0} \right)$ for all $n \in \mathbb{N}$, and by Lemma A.3.6 $Z \subset \ker (C(\lambda - A)^{-1})$ for all $\lambda \in \rho_\infty(A)$. That $(\lambda - A)^{-1}Z = Z$ for all $\lambda \in \rho_\infty(A)$ follows from Lemma 3.1.7.

At this point the reader may want to recall the notions of the reachable subspace and the unobservable subspace of an i/s/o system introduced in Definition 2.5.3.

3.2.4. Lemma. Let $\Sigma = ([A \ B] ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a bounded i/s/o system with evolution semigroup $\mathcal{A}$. Let $\Omega'$ be an arbitrary subset of $\rho_\infty(A)$ which has a cluster point in $\rho_\infty(A)$, and let $\lambda_0 \in \rho_\infty(A)$. Then the following claims are true:

(i) The forward and backward reachable subspaces of $\Sigma$ are the same, and the forward and backward unobservable subspaces of $\Sigma$ are also the same.

(ii) The (forward and backward) reachable subspace $\mathcal{R}_\Sigma$ of $\Sigma$ can be computed in the following alternative ways:

\[
\mathcal{R}_\Sigma = \bigvee_{t \in \mathbb{R}^+} \text{rng } (\mathcal{A}^t B) = \bigvee_{t \in \mathbb{R}^-} \text{rng } (\mathcal{A}^t B)
\]

\[
= \bigvee_{n \in \mathbb{Z}^+} \text{rng } (A^n B) = \bigvee_{\lambda \in \rho_\infty(A)} \text{rng } ((\lambda - A)^{-1}B)
\]

\[
= \bigvee_{\lambda \in \Omega'} \text{rng } ((\lambda - A)^{-1}B) = \bigvee_{n \in \mathbb{Z}^+} \text{rng } ((\lambda_0 - A)^{-1}B).
\]

(iii) The (forward and backward) unobservable subspace $\mathcal{U}_\Sigma$ of $\Sigma$ can be computed in the following alternative ways:

\[
\mathcal{U}_\Sigma = \bigcap_{t \in \mathbb{R}^+} \ker (\mathcal{A}^t C) = \bigcap_{t \in \mathbb{R}^-} \ker (\mathcal{A}^t C)
\]

\[
= \bigcap_{n \in \mathbb{Z}^+} \ker (CA^n) = \bigcap_{\lambda \in \rho_\infty(A)} \ker (C(\lambda - A)^{-1})
\]

\[
= \bigcap_{\lambda \in \Omega'} \ker (C(\lambda - A)^{-1}) = \bigcap_{n \in \mathbb{Z}^+} \ker (C(\lambda_0 - A)^{-1}).
\]

(iv) $\mathcal{R}_\Sigma$ is the minimal closed $A$-invariant subspace which contains $\text{rng } (B)$, and it is also the minimal closed strongly invariant subspace for $\Sigma$.

(v) $\mathcal{U}_\Sigma$ coincides with the classically unobservable subspace of $\Sigma$. It is the maximal $A$-invariant subspace which is contained in $\ker (C)$, and it is also the maximal unobservable invariant subspace for $\Sigma$.

Proof. Proof of (i): Claim (i) will be proved as a part of the proofs of claims (ii) and (iii).

Proof of (ii): The proof of the fact that all the different expressions in (3.2.4) are equal to each other is analogous to the proof of Lemma 3.1.37. We still have to show that these equivalent expressions are equal to the common forward and backward reachable subspace of $\Sigma$.

By Theorem 2.1.14 if $\begin{bmatrix} x \\ y \end{bmatrix}$ is a generalized trajectory of $\Sigma$ with $x(0) = 0$ on the interval $[0, T]$, then

\[
x(t) = \int_0^t \mathcal{A}^{t-s} Bu(s) \, ds \in \bigvee_{t \in \mathbb{R}^+} \text{rng } (\mathcal{A}^t B), \quad t \in [0, T].
\]
This fact combined with Lemma 2.5.7 implies that $\mathcal{R}_\Sigma \subset \bigvee_{t \in \mathbb{R}^+} \operatorname{rng}(\mathfrak{A}^t B)$. To prove the converse inclusion we define $u_n$ as in (3.2.1) and let $[\begin{smallmatrix} x_n \\ u_n \\ v_n \end{smallmatrix}]$ be the corresponding generalized future trajectory of $\Sigma$. Then by Lemma 2.5.7 $x_n(t) \in \mathcal{R}_\Sigma$ for all $t \in \mathbb{R}^+$, and by (3.2.3) $x_n(t) \to \mathfrak{A}^t Bu^0$ as $n \to \infty$. This implies that $\operatorname{rng} (\mathfrak{A}^t B) \subset \mathcal{R}_\Sigma$ for all $t \in \mathbb{R}^+$, and since $\mathcal{R}_\Sigma$ is a closed subspace also $\bigvee_{t \in \mathbb{R}^+} \operatorname{rng}(\mathfrak{A}^t B) \subset \mathcal{R}_\Sigma$.

By arguing in the same way as we did in the preceding paragraph with $A$ replaced by $-A$ and $\mathfrak{A}^t$ replaced by $\mathfrak{A}^{-t}$, $t \in \mathbb{R}$, we find that the backward reachable subspace of $\Sigma$ is equal to $\bigvee_{t \in \mathbb{R}^-} \operatorname{rng}(\mathfrak{A}^t B)$. Since all the different expressions in (3.2.4) are equal to each other, this proves that the forward and backward reachable subspaces of $\Sigma$ coincide.

Proof of (iii): The proof of (iii) is analogous to the proof of (ii) given above.

Proof of (iv): It follows from the identity $\mathcal{R}_\Sigma = \bigvee_{n \in \mathbb{Z}^+} \operatorname{rng}(A^n B)$ that $\mathcal{R}_\Sigma$ is a closed $A$-invariant subspace which contains $\operatorname{rng}(B)$. Clearly every $A$-invariant subspace which contains $\operatorname{rng}(B)$ must also contain $\operatorname{span}_{n \in \mathbb{Z}^+} \operatorname{rng}(A^n B)$ for all $n \in \mathbb{Z}$, and therefore $\mathcal{R}_\Sigma = \bigvee_{n \in \mathbb{Z}^+} \operatorname{rng}(A^n B)$ is contained in every other closed $A$-invariant subspace which contains $\operatorname{rng}(B)$, i.e., $\mathcal{R}_\Sigma$ is the minimal closed $A$-invariant subspace which contains $\operatorname{rng}(B)$. That $\mathcal{R}_\Sigma$ is also the minimal closed strongly invariant subspace of $\Sigma$ follows from Lemma 3.2.2 which says that a closed subspace of $X$ is strongly invariant for $\Sigma$ if and only if it is $A$-invariant and contains $\operatorname{rng}(B)$.

Proof of (v): The proof of (v) is analogous to the proof of (iv) with Lemma 3.2.2 replaced by Lemma 3.2.3. □

3.2.5. Lemma. Let $\mathcal{X}$ be a $H$-space with the direct sum decomposition $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$, let $A \in \mathcal{B}(\mathcal{X})$, and let $\Sigma = ([A \ B] \ : \mathcal{X}, \mathcal{X}_1, \mathcal{Z}_1)$ be the bounded i/s/o system with system operator

$$
(3.2.6) \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ P_{\mathcal{X}_1}^0 & 0 \end{bmatrix}.
$$

Then the subspace $\mathcal{X}_{\min}$ in (3.1.30) is the reachable subspace of $\Sigma$ and the subspace $\mathcal{Z}_{\max}$ in (3.1.31) is the unobservable subspace of $\Sigma$.

Proof. This follows from Lemmas 3.1.37 and 3.2.4. □

3.2.2. External equivalence of bounded i/s/o systems. At this point the reader may want to recall what we mean by the future behavior of an i/s/o system (see Definition 2.5.43).

3.2.6. Lemma. Let $\Sigma = ([A \ B] \ : \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a bounded i/s/o system with evolution group $\mathfrak{A}$. Then the future behavior of $\Sigma$ consists of all $[y] \in L_1^1([y])$ which satisfy

$$
(3.2.7) \quad y(t) = \int_0^t C\mathfrak{A}^{t-s} Bu(s) \, ds + Du(t), \quad t \in \mathbb{R}^+.
$$

(this identity should be interpreted in the $L^1$ sense, i.e., it only holds almost everywhere).

Proof. This follows from Theorem 2.1.14 and Definition 2.5.43. □

At this point the reader may want to recall what we mean by external equivalence of two i/s/o systems (see Definition 2.5.21).
3.2.7. Theorem. Let $\Sigma_i = ([A_i, B_i, C_i, D_i] ; X_i, U, Y_i), i = 1, 2$, be two bounded i/s/o systems with evolution groups $\mathfrak{A}_i$. Let $\Omega'$ be an arbitrary subset of $\rho_\infty(A_1 \times A_2)$ which has a cluster point in $\rho_\infty(A_1 \times A_2)$, and let $\lambda_0 \in \rho_\infty(A_1 \times A_2)$. Then the following conditions are equivalent:

(i) $\Sigma_1$ and $\Sigma_2$ are (forward) externally equivalent;
(ii) $\Sigma_1$ and $\Sigma_2$ are backward externally equivalent;
(iii) $D_1 = D_2$ and $C_1 \mathfrak{A}_1^{-1} B_1 = C_2 \mathfrak{A}_2^{-1} B_2$ for all $t \in \mathbb{R}$;
(iv) $D_1 = D_2$ and $C_1 A_i^1 B_1 = C_2 A_i^2 B_2$ for all $n \in \mathbb{Z}^+$;
(v) $D_1 = D_2$ and $C_1 (\lambda - A_1)^{-1} B_1 = C_2 (\lambda - A_2)^{-1} B_2$ for all $\lambda \in \rho_\infty(A_1 \times A_2)$.
(vi) $D_1 = D_2$ and $C_1 (\lambda - A_1)^{-1} B_1 = C_2 (\lambda - A_2)^{-1} B_2$ for all $\lambda \in \Omega'$.
(vii) $D_1 = D_2$ and $C_1 (\lambda - A_1)^{-n} B_1 = C_2 (\lambda - A_2)^{-n} B_2$ for all $n \in \mathbb{Z}^+$.

Proof. (i) $\Leftrightarrow$ (iii): Since every bounded i/s/o system has the continuation property it follows from Lemma 2.5.44 that $\Sigma_1$ and $\Sigma_2$ are externally equivalent if and only if $\Sigma_1$ and $\Sigma_2$ have the same future behavior. By Lemma 3.2.6 this is equivalent to the condition that for all $u \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{U})$ and almost all $t \in \mathbb{R}^+$ we have

\[(3.2.8) \quad \int_0^t C_1 \mathfrak{A}_1^{-s} B_1 u(s) \, ds + D_1 u(t) = \int_0^t C_2 \mathfrak{A}_2^{-s} B_2 u(s) \, ds + D_2 u(t).\]

This is clearly true if (iii) holds. Conversely, suppose that (3.2.8) holds for all $u \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{U})$ and almost all $t \in \mathbb{R}^+$. If $u$ is continuous, then both sides are continuous, and thus this identity holds for all $t \in \mathbb{R}^+$ (instead of almost all $t \in \mathbb{R}^+$). By taking $u$ to be an arbitrary constant in $\mathcal{U}$ and letting $t \downarrow 0$ we find that $D_1 = D_2$. Thus (3.2.8) simplifies into

\[C_1 \mathfrak{A}_1^t \int_0^t \mathfrak{A}_1^{-s} B_1 u(s) \, ds = C_2 \mathfrak{A}_2^t \int_0^t \mathfrak{A}_2^{-s} B_2 u(s) \, ds, \quad t \in \mathbb{R}^+\]

(in the almost everywhere sense, but since both sides are continuous functions of $t$ we have equality everywhere). In this identity we may, e.g., take $u = u_n$, defined in (3.2.1) for some $u^0 \in \mathcal{U}$ and all $n \in \mathbb{N}$. As $n \to \infty$ the left hand side tends to $C_1 \mathfrak{A}_1^t B_1 u^0$ and the right hand side tends to $C_2 \mathfrak{A}_2^t B_2 u^0$ for all $t \in \mathbb{R}^+$. Thus $C_1 \mathfrak{A}_1^t B_1 = C_2 \mathfrak{A}_2^t B_2$ for all $t \in \mathbb{R}^+$. Since both $C_1 \mathfrak{A}_1^t B_1$ and $C_2 \mathfrak{A}_2^t B_2$ are entire functions of $t$ and they coincide on $\mathbb{R}^+$, they must coincide on all of $\mathbb{R}$. This shows that (iii) holds.

(ii) $\Leftrightarrow$ (iii): This is the same as the proof of the equivalence (i) $\Leftrightarrow$ (iii) given above with $A_i$ replaced by $-A_i$, $\mathfrak{A}_i$ replaced by $\mathfrak{A}_i^{-1}$, and $B_i$ replaced by $-B_i$ for $i = 1, 2$.

(iii) $\Leftrightarrow$ (iv): This follows (2.1.13) (there is an one-to-one correspondence between an entire function and the coefficients in its Taylor series).

(iv) $\Rightarrow$ (v): If (iv) holds, then it follows from (3.1.4) that the identity in (v) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| > \max \{r_\infty(A_1), r_\infty(A_2)\}$, where $r_\infty(A_i)$ is the spectral radius of $A_i$, $i = 1, 2$. Since both sides are analytic in $\rho(A_1) \cap \rho(A_2)$ the same identity must therefore be true for all $\lambda$ in the unbounded component of $\rho(A_1) \cap \rho(A_2)$.

(v) $\Rightarrow$ (iv): Define $\mathcal{D}_i(\lambda) = C_i (\lambda - A_i)^{-1} B_1 + D_i$, $\lambda \in \rho(A_i)$, $i = 1, 2$, and let $F_i(0) = D_i$ and $F_i(z) = \mathcal{D}_i(1/z)$ for $0 \neq z \in \mathbb{C}$ with $1/z \in \rho(A_i)$, $i = 1, 2$. By Lemma 3.1.4 both $F_1$ and $F_2$ are analytic at zero. If (v) holds, then $F_1(z) = F_2(z)$ in some neighborhood of zero, and therefore they have the same Taylor expansion at zero. This together with (3.1.3) gives (iv).
(v) ⇔ (vi) ⇔ (vii): This follows from the analyticity of the two resolvents in \( \rho(A_1 \times A_2) \), \((3.1.6)\), and the fact that \( \rho_\infty(A_1 \times A_2) \) is connected. \( \square \)

3.2.8. Definition. Let \( \Sigma = ([A \ B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o system with evoution group \( \mathfrak{A} \).

(i) The mapping from \( u \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \) to \( y \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \) defined by \((3.2.7)\) is called the future i/o map (input/output map) of \( \Sigma \).

(ii) The \( \mathcal{B}(\mathcal{U}; \mathcal{Y}) \)-valued function

\[
\hat{D}(\lambda) := C(\lambda - A)^{-1}B = D, \quad \lambda \in \rho(A),
\]

is called the i/o (input/output) resolvent of \( \Sigma \).

3.2.9. Remark. Using the terminology of Definition \((3.2.8)\) conditions (iii) and (v) in Theorem \((3.2.7)\) can be reformulated as follows:

(iii') \( \Sigma_1 \) and \( \Sigma_2 \) have the same i/o map;

(v') The i/o resolvents of \( \Sigma_1 \) and \( \Sigma_2 \) coincide in the unbounded component of \( \rho(A_1) \cap \rho(A_2) \).

3.2.10. Lemma. Let \( \Sigma = ([A \ B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o system. Then the i/o resolvent \( \hat{D} \) of \( \Sigma \) is analytic in \( \rho(A) \), and it is also analytic at infinity. In particular, the i/o resolvent of \( \Sigma \) is analytic at infinity.

Proof. This follows from Lemmas \( 3.1.2 \) and \( 3.1.3 \) and Definition \((3.2.8)\). \( \square \)

3.2.3. Restrictions and projections of bounded i/s/o systems. At this point the reader may want to recall Definitions \( 2.5.33 \) and \( 2.5.37 \) of what we mean by a restriction and a projection of an i/s/o system.

3.2.11. Theorem. Let \( \Sigma = ([A \ B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o system, and let \( \mathcal{X}_1 \) be a closed subspace of \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( \mathcal{X}_1 \) is a (forward) strongly invariant subspace for \( \Sigma \);

(ii) \( \mathcal{X}_1 \) is a backward strongly invariant subspace for \( \Sigma \);

(iii) \( \Sigma \) has a (forward) restriction to \( \mathcal{X}_1 \);

(iv) \( \Sigma \) has a backward restriction to \( \mathcal{X}_1 \).

Suppose that these equivalent conditions hold, and define \( [A_1 \ B_1] \in \mathcal{B}([X_1] [Y_1]) \) by

\[
[A_1 \ B_1] = [A_{|X_1} \ B]_{|X_1} D_1.
\]

Then the bounded i/s/o system \( \Sigma_1 = ([A_1 \ B_1]; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) is both a forward and a backward restriction of \( \Sigma \) to \( \mathcal{X}_1 \), and \( \Sigma_1 \) is the unique bounded forward or backward restriction of \( \Sigma \) to \( \mathcal{X}_1 \). The evolution group \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the restriction to \( \mathcal{X}_1 \) of the evolution group \( \mathfrak{A} \) of \( \Sigma \), and \( \Sigma_1 \) and \( \Sigma \) are both forward and backward externally equivalent.

Proof. (i) ⇔ (iii): That (iii) ⇒ (i) follows from Lemma \( 2.5.36 \). Since bounded i/s/o systems are have the continuation property the converse implication follows from Lemma \( 2.5.48 \).

(ii) ⇔ (iv): This equivalence is proved in the same way with \( \Sigma \) replace by the time-reflection \( \Sigma_r \) of \( \Sigma \).

(i) ⇔ (ii): See Lemma \( 3.2.2 \).
We have now proved the equivalence of (i)–(iv), and it remains to prove the additional claims about the uniqueness and boundedness of the restriction. The (forward) restriction \( \Sigma_1 = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} ; X_1, U, Y \right) \) defined in Lemma 2.5.48 coincides with the system \( \Sigma_1 \) above, and the uniqueness claim for the (forward) restriction is part of Lemma 2.5.48. The same argument applied to the time-reflected system shows that \( \Sigma_1 \) is also a backward restriction of \( \Sigma \), and that it is unique among all solvable backward restrictions of \( \Sigma \) to \( X_1 \). By Lemma 3.2.2, \( X_1 \) is \( A \)-invariant, and by Theorem 3.1.9 the evolution group \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the restriction of the evolution group \( \mathfrak{A} \) of \( \Sigma \) to \( X_1 \). That \( \Sigma_1 \) and \( \Sigma \) are both forward and backward externally equivalent follows from Lemmas 2.5.29 and 2.5.36 and Theorem 3.2.7.

**3.2.12. Theorem.** Let \( \Sigma = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} ; X, U, Y \right) \) be a bounded i/s/o system, and let \( X = X_1 \oplus Z_1 \) be a direct sum decomposition of \( X \). Then the following conditions are equivalent:

(i) \( Z_1 \) is a (forward) unobservably invariant subspace for \( \Sigma \);
(ii) \( Z_1 \) is a backward unobservably invariant subspace for \( \Sigma \);
(iii) \( \Sigma \) has a (forward) projection onto \( X_1 \) along \( Z_1 \);
(iv) \( \Sigma \) has a backward projection onto \( X_1 \) along \( Z_1 \).

Suppose that these equivalent conditions hold, and define \( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \in B(\left[ \begin{bmatrix} U_1 \\ Y \end{bmatrix} \right]) \)

by

\[
\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} P^Z_{X_1} A|_{X_1} \& P^Z_{X_1} B \\ C|_{X_1} \& D \end{bmatrix}.
\]

Then the bounded i/s/o system \( \Sigma_1 = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} ; X_1, U, Y \right) \) is both a forward and a backward projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \), and \( \Sigma_1 \) is the unique bounded forward or backward projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \). The evolution group \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the projection onto \( X_1 \) along \( Z_1 \) of the evolution group \( \mathfrak{A} \) of \( \Sigma \), and \( \Sigma_1 \) and \( \Sigma \) are both forward and backward externally equivalent.

**Proof.** (iii) \( \Rightarrow \) (i): This follows from Lemma 2.5.39.

(i) \( \Rightarrow \) (iii): Suppose that (i) holds. Then by Lemma 3.2.3, \( Z_1 \subset \ker (C) \), \( AZ_1 \subset Z_1 \) and \( \mathfrak{A}^t Z_1 \subset Z \) for all \( t \in \mathbb{R} \). Let \( \Sigma_1 \) be the bounded i/s/o system whose system operator \( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \) is given by (3.2.11). Then by Theorem 3.1.15, \( A_1 \) is the projection of \( A \) in \( B(\mathcal{X}_1) \) along \( Z_1 \), and the evolution continuous group \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the projection of the the evolution group \( \mathfrak{A} \) of \( \Sigma \) onto \( X_1 \) along \( Z_1 \). In particular, \( P^Z_{X_1} \mathfrak{A}^t = \mathfrak{A}_1^t P^Z_{X_1} \) for all \( t \in \mathbb{R} \).

Since bounded i/s/o systems are have the continuation property, to prove that \( \Sigma_1 \) is a projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \) it suffices to show that conditions (i) and (ii) in Definition 2.5.28 hold with \( I = \mathbb{R}^+ \) (see Lemma 2.5.46).

Let \( \begin{bmatrix} x \\ u \end{bmatrix} \) be a generalized future trajectory of \( \Sigma \). By Theorem 2.1.14, \( x \) and \( y \) are given by (2.1.19) with \( t_0 = 0 \) and \( I = \mathbb{R}^+ \), i.e.,

\[
x(t) = \mathfrak{A}^t x^0 + \int_0^t \mathfrak{A}^{t-s} Bu(s) \, ds, \quad y(t) = Cx(t) + Du(t), \quad t \in \mathbb{R}^+.
\]
Define \( x_1(t) = P_{X_1}^{Z_1} x(t), \ t \in \mathbb{R}^+ \). Then, for all \( t \in \mathbb{R} \),

\[
    x_1(t) = P_{X_1}^{Z_1} x(t) = P_{X_1}^{Z_1} x^0 + \int_0^t P_{X_1}^{Z_1} Q t^{-s} B u(s) \, ds
\]

\[
    = \mathfrak{A}_1 P_{X_1}^{Z_1} x^0 + \int_0^t \mathfrak{A}_1^{-s} P_{X_1}^{Z_1} B u(s) \, ds
\]

\[
    = \mathfrak{A}_1 P_{X_1}^{Z_1} x^0 + \int_0^t \mathfrak{A}_1^{-s} B_1 u(s) \, ds,
\]

\[
y(t) = C x(t) + D u(t) = C (P_{X_1}^{Z_1} + P_{X_1}^{Z_1}) x(t) + D u(t)
\]

\[
    = C P_{X_1}^{Z_1} x(t) + D u(t) = C x_1(t) + D_1 u(t).
\]

By Theorem 2.1.14 \( \begin{bmatrix} x_1 \\ u \end{bmatrix} \) is a generalized trajectory of \( \Sigma_1 \). This means that condition (i) in Definition 2.5.28 holds with \( I = \mathbb{R}^+ \).

Conversely, let \( \begin{bmatrix} x_2 \\ u \end{bmatrix} \) be a generalized future trajectory of \( \Sigma_1 \) on \( \mathbb{R}^+ \) and define

\[
x(t) := \mathfrak{A}_1^t x^0 + \int_0^t \mathfrak{A}_1^{-s} B u(s) \, ds, \quad t \in \mathbb{R}^+.
\]

Then the same computation that we carried out above shows that if we define \( x_2(t) = P_{X_1}^{Z_1} x(t), \ t \in \mathbb{R}^+ \), then \( \begin{bmatrix} x_2 \\ u \end{bmatrix} \) is a generalized future trajectory of \( \Sigma_1 \). This trajectory has the same initial state \( x_0 \) as \( x_1 \), and by the uniqueness part of Theorem 2.1.14 \( x_1(t) = x_2(t) \) for all \( t \in \mathbb{R}^+ \). Thus also condition (ii) in Definition 2.5.28 holds with \( I = \mathbb{R}^+ \). This proves that \( \Sigma_1 \) is a projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \).

(i) \( \iff \) (ii): See Lemma 3.2.2.

We have now proved the equivalence of (i)–(iv). We have also shown that the bounded i/s/o system \( \Sigma_1 \) whose system operator \( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \) is given by (3.2.11) is a (forward) projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \), and that the evolution group \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the projection of the evolution group \( \mathfrak{A} \) of \( \Sigma \) onto \( X_1 \) along \( Z_1 \). By replacing \( \Sigma \) by the time-reflected system and repeating the same argument we find that \( \Sigma_1 \) is also a backward projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \). By Lemmas 2.5.31 and 2.5.39 solvable forward or backward projections of \( \Sigma \) onto \( X_1 \) along \( Z_1 \) are unique. Thus, \( \Sigma_1 \) is both the unique solvable forward projection and the unique solvable backward projection onto \( X_1 \) along \( Z_1 \). That \( \Sigma_1 \) and \( \Sigma \) are both forward and backward externally equivalent follows from Lemmas 2.5.29 and 2.5.39 and Theorem 3.2.7. \( \Box \)

3.2.13. Remark. It is possible to give alternative proofs of Theorems 3.2.11 and 3.2.12 which are based on Lemma 3.2.16 below and the fact that \( \Sigma_1 \) is a restriction of \( \Sigma \) to \( X_1 \) if and only if \( \Sigma_1 \) and \( \Sigma \) are intertwined by the embedding operator \( X_1 \to X \), and that \( \Sigma_1 \) is a projection of \( \Sigma \) onto \( X_1 \) along \( Z_1 \) if and only if \( \Sigma \) and \( \Sigma_1 \) are intertwined by the projection operator \( P_{X_1}^{Z_1} \) (see Lemmas 2.5.36 and 2.5.39).

3.2.4. Interconnections of bounded i/s/o systems. In Section 2.3 we introduced the notions of the cross product and the parallel, the difference, and the cascade connection of two i/s/o systems \( \Sigma_1 \) and \( \Sigma_2 \). As we noticed there, if \( \Sigma_1 \) and \( \Sigma_2 \) are both bounded, then all these connections are bounded as well. Here we take a closer look at these connections in the bounded case.
3.2.14. **Lemma.** Let $\Sigma_i = ([A_i, B_i]: X_i, U_i, Y_i), i = 1, 2$ be two bounded i/s/o systems with i/o maps $D_i$ and i/o resolvents $\tilde{D}_i$, $i = 1, 2$.

(i) The cross product $\Sigma_x := \Sigma_1 \times \Sigma_2$ of $\Sigma_1$ and $\Sigma_2$ has the following properties:

(a) the i/o map $\Sigma_x$ is $\tilde{D}_1 \times \tilde{D}_2 = [ \tilde{D}_1, 0 ]$, $[ 0, \tilde{D}_2 ]$;

(b) the i/o resolvent $\tilde{D}_x$ of $\Sigma_x$ satisfies $\tilde{D}_x(\lambda) = \tilde{D}_1(\lambda) \times \tilde{D}_1(\lambda) = [ \tilde{D}_1(\lambda), 0 ]$, $0, \tilde{D}_2(\lambda) ]$, $\lambda \in \rho(A_1) \cap \rho(A_2)$.

(ii) If $U_1 = U_2$ and $Y_2 = Y_2$, so that the parallel and difference connections $\Sigma_{||} := \Sigma_1 \parallel \Sigma_2$ and $\Sigma_{\oplus} := \Sigma_1 \oplus \Sigma_2$ of $\Sigma_1$ and $\Sigma_2$ are defined, then

(a) the i/o map of $\Sigma_{||}$ is $\tilde{D}_1 + \tilde{D}_2$, and the i/o map of $\Sigma_{\oplus}$ is $\tilde{D}_1 - \tilde{D}_2$;

(b) the i/o resolvent $\tilde{D}_{||}$ of $\Sigma_{||}$ satisfies $\tilde{D}_{||}(\lambda) = \tilde{D}_1(\lambda) + \tilde{D}_2(\lambda)$ and the i/o resolvent $\tilde{D}_{\oplus}$ of $\Sigma_{\oplus}$ satisfies $\tilde{D}_{\oplus}(\lambda) = \tilde{D}_1(\lambda) - \tilde{D}_2(\lambda)$, $\lambda \in \rho(A_1) \cap \rho(A_2)$.

(iii) If $Y_1 = U_2$, so that the cascade connection $\Sigma_c := \Sigma_2 \circ \Sigma_1$ of $\Sigma_1$ and $\Sigma_2$ is defined, then

(a) the i/o map of $\Sigma_c$ is $\tilde{D}_2 \tilde{D}_1$;

(b) the i/o resolvent $\tilde{D}_c$ of $\Sigma_c$ satisfies $\tilde{D}_c(\lambda) = \tilde{D}_2(\lambda) \tilde{D}_1(\lambda)$, $\rho(A_1) \cap \rho(A_2)$.

**Proof.** This follows from Examples 2.3.33, 2.3.39, and 2.3.44, Lemma 3.1.20, and Definition 3.2.8.

3.2.15. **Lemma.** If two bounded i/s/o systems $\Sigma_i = ([A_i, B_i]: X_i, U_i, Y_i), i = 1, 2$, are intertwined by $P \in ML(X_1; X_2)$, then $\Sigma_1$ and $\Sigma_2$ are also intertwined by the closure of $P$.

**Proof.** Let $[x_1, u, y]$ be a generalized trajectory $\Sigma_1$ on some interval $I$ of the type $I = [0, T]$ or $I = \mathbb{R}^+$, suppose that $x_1^0 := x_1(0) \in \text{dom}(R)$, and let $x_2^0 \in \mathbb{R}x_1^0$. Then there exists a sequence of vectors $[ x_2^n, x_1^n ] \in \text{gph}(R)$ such that $[ x_2^n, x_1^n ] \to [ x_2, x_1 ]$ as $n \to \infty$. Let $[ x_{1,n}, u_n, y_n ]$ be the generalized trajectory of $\Sigma_1$ on $I$ with initial state $x_{1,n}(0) = x_1^n$ given by Theorem 2.1.14. Then $x_{1,n} \to x_1$ in $C(I; X)$ and $y_n \to y$ in $L^1_{\text{loc}}(I; Y)$. Since $\Sigma_1$ and $\Sigma_2$ are intertwined by $R$, for each $n \in \mathbb{N}$ there exists a generalized trajectory $[ x_{2,n}, u_n ]$ of $\Sigma_2$ on $I$ satisfying $x_{2,n}(0) = x_2^n$ and $[ x_{2,n}(t), x_{1,n}(t) ] \in \text{gph}(P)$ for all $t \in I$. As $n \to \infty$, it follows from Theorem 2.1.14 that this sequence of trajectories tends to a generalized trajectory $[ x_2, y ]$ of $\Sigma_2$ satisfying $x_2(0) = x_1^0$ and $[ x_2(t), x_1(t) ] \in \text{gph}(P)$ for all $t \in I$. Thus condition (i) in Definition 2.5.22 holds with $P$ replaced by $P$. That also condition (ii) holds is proved in the same way, interchanging $\Sigma_1$ and $\Sigma_2$ and replacing $P$ by $P^{-1}$. Thus $\Sigma_1$ and $\Sigma_2$ are intertwined by $\overline{P}$.

3.2.16. **Lemma.** Let $\Sigma_i = ([A_i, B_i]: X_i, U_i, Y_i), i = 1, 2$, be two bounded i/s/o systems, let $P \in ML(X_1; X_2)$ be closed, and let $\Sigma = ([A, B]: X, U, Y) = \Sigma_2 \parallel \Sigma_1$
be the difference of $\Sigma_2$ and $\Sigma_1$ (see Example 2.3.39). Then the following conditions are equivalent:

(i) $\Sigma_1$ and $\Sigma_2$ are (forward) intertwined by $P$;
(ii) $\Sigma_1$ and $\Sigma_2$ are backward intertwined by $P$;
(iii) $\text{gph}(P)$ is an invariant subspace for $A_2 \times A_1$, $\text{rng} \left( \left[ \begin{array}{c} B_2 \\ -C_1 \end{array} \right] \right) \subseteq \text{gph}(P) \subseteq \ker \left( \left[ \begin{array}{cc} C_2 & -C_1 \end{array} \right] \right)$, and $D_1 = D_2$;
(iv) $\Sigma_1$ and $\Sigma_2$ are externally equivalent and $\text{gph}(P)$ is both a strongly invariant and an unobservably invariant subspace for $\Sigma_2$;
(v) $\text{gph}(P)$ is both a strongly invariant and an unobservably invariant subspace for $\Sigma_2$, and the i/o resolvent of $\Sigma_2$ vanishes on $\rho_{\infty}(\Sigma)$.

**Proof.** (i) $\Rightarrow$ (iv): Suppose that (i) holds. The strong invariance of $\text{gph}(P)$ follows directly from Definitions 2.5.8 and 2.5.22 combined with Lemma 2.3.41. To see that $\text{gph}(P)$ is also an unobservably invariant for $\Sigma$ we take any $x^0 = \left[ \begin{array}{c} x_1^0 \\ x_2^0 \end{array} \right] \in \text{gph}(P)$, and let $\left[ \begin{array}{c} \dot{x}_i(t) \\ 0 \end{array} \right]$ be the trajectories of $\Sigma$ with $x_i(0) = x_i^0$ (and with $u = 0$), $i = 1, 2$, given by Theorem 2.1.14. Then $\left[ \begin{array}{c} x_2(t) \\ x_1(t) \end{array} \right] \in \text{gph}(P)$ and $y_1(t) = y_2(t)$ for all $t \in \mathbb{R}^+$. By Lemma 2.3.41 if we define $x = \left[ \begin{array}{c} x_2^0 \\ 0 \end{array} \right]$, then $\left[ \begin{array}{c} 0 \\ 0 \end{array} \right]$ is a trajectory of $\Sigma$. Thus $\text{gph}(P)$ is an unobservably invariant subspace for $\Sigma$. By Lemma 2.5.27, $\Sigma_1$ and $\Sigma_2$ are externally equivalent.

(iv) $\Rightarrow$ (v): This follows Theorem 3.2.7, Lemma 2.3.41, and Remark 3.2.9.

(v) $\Rightarrow$ (iii): This follows from Lemmas 3.2.2 and 3.2.3.

(iii) $\Rightarrow$ (i): It follows from Definition 2.5.22 and Theorem 2.1.14 that (i) holds if and only if

\[
(3.2.12) \quad \left[ A_1^t x_1^0 + \int_0^t A_1^{t-s} B_1 u(s) \, ds \right] \quad \left[ A_2^t x_2^0 + \int_0^t A_2^{t-s} B_2 u(s) \, ds \right] \subseteq \text{gph}(P)
\]

and

\[
(3.2.13) \quad C_1 \left[ A_1^t x_1^0 + \int_0^t A_1^{t-s} B_1 u(s) \, ds \right] + D_1 u(t) = C_2 \left[ A_2^t x_2^0 + \int_0^t A_2^{t-s} B_2 u(s) \, ds \right] + D_2 u(t)
\]

for all $\left[ \begin{array}{c} x_2^0 \\ x_2^0 \end{array} \right] \in \text{gph}(P)$, all $u \in L_{\text{loc}}^1(\mathbb{R}^+; U)$, and all $t \in \mathbb{R}^+$. If (iii) holds, then by Lemma 3.1.7, $\text{gph}(P)$ is an invariant subspace for $\mathfrak{A}$ and this combined with the other condition in (iii) implies that (3.2.12) and (3.2.13) hold.

(ii) $\Leftrightarrow$ (iii): This follows from the equivalence of (i) and (iii), applied to the time reflected system we get by replacing $A_i$ by $-A_i$ and $B_i$ by $-B_i$, $i = 1, 2$. $\square$

In Lemma 3.2.16, we used the difference connection $\Sigma_2 \dashv \Sigma_1$ of $\Sigma_2$ and $\Sigma_2$ to test if the i/s/o systems $\Sigma_1$ and $\Sigma_2$ were intertwined by some closed $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$. Another alternative way is to use the $\text{gph}(P)$-short circuit connection as follows.

**3.2.17. Lemma.** Let $\Sigma_i = \left( \left[ \begin{array}{c} A_i \\ C_i \\ D_i \end{array} \right] ; \mathcal{X}_i, \mathcal{U}, \mathcal{Y} \right)$, $i = 1, 2$, be two bounded i/s/o systems, let $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ be closed, and let $\Sigma = (S; \text{gph}(P), \mathcal{U}, \mathcal{Y})$ be the $\text{gph}(P)$-short circuit connection of $\Sigma_2$ and $\Sigma_1$ (see Definition 2.3.37).

(i) $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$ if an only if $\Sigma$ is a bounded i/s/o node.
(ii) Suppose that $\Sigma$ is a bounded i/s/o node. Then the (bounded) system operator $S = \left[ \begin{array}{c} A \\ C \end{array} \right]$ of $\Sigma$ is given by

$$A = \begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix}_{\text{gph}(P)}, \quad B = \begin{bmatrix} B_2 \\ B_1 \end{bmatrix},$$

(3.2.14)

Moreover, $\Sigma$ and $\Sigma_1$ are intertwined by the bounded operator $P_{X_1}^e|_{\text{gph}(P)}$, $\Sigma$ and $\Sigma_2$ are intertwined by the bounded operator $P_{X_2}^e|_{\text{gph}(P)}$, and $\Sigma$, $\Sigma_1$, and $\Sigma_2$ are both forward and backward externally equivalent.

**Proof.** Suppose first that $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$. Then condition (iii) in Lemma 3.2.16 holds, and we may define $\left[ \begin{array}{c} A \\ C \end{array} \right]$ by (3.2.14). A direct inspection then shows that $\Sigma = (\left[ \begin{array}{c} A \\ C \end{array} \right]; \text{gph}(P), U, Y)$ is the $\text{gph}(P)$-short circuit of $\Sigma_2$ and $\Sigma_1$.

Conversely, suppose that $\Sigma$ is a bounded i/s/o node. Then $S$ is single-valued and $\text{dom}(S) = \left[ \begin{array}{c} \text{gph}(P) \\ \text{ut} \end{array} \right]$. This is equivalent to the condition that for every $\left[ \begin{array}{c} \bar{x}_1 \\ \bar{x}_2 \end{array} \right] \in \text{gph}(P)$ and $u \in U$ there exists unique vectors $\left[ \begin{array}{c} \bar{x}_1 \\ \bar{x}_2 \end{array} \right] \in \text{gph}(P)$ and $y \in Y$ such that $\left[ \begin{array}{c} \bar{y} \\ \bar{y}_1 \end{array} \right] = \left[ \begin{array}{c} A_i B_i \\ C_i D_i \end{array} \right] \left[ \begin{array}{c} \bar{x}_i \\ u \end{array} \right]$, $i = 1, 2$. This implies that condition (iii) in Lemma 3.2.16 holds, and hence $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$.

The final claims follow from Lemma 3.2.16. \hfill \Box

### 3.2.18. Theorem

Let $\Sigma_i = (\left[ \begin{array}{c} A_i \\ C_i \end{array} \right]; X_i, U, Y)$, $i = 1, 2$, be two bounded i/s/o systems (with the same input and output spaces). Then the following claims are true.

(i) $\Sigma_1$ and $\Sigma_2$ are intertwined by some $P \in \mathcal{ML}(X_1; X_2)$ if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent.

(ii) If $\Sigma_1$ and $\Sigma_2$ are externally equivalent, then the following claims are true.

(a) There exists a unique minimal closed $P_{\min} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique closed $P_{\min} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P_{\min}) \subset \text{gph}(P)$ for any other closed $P \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\min}$ is equal to the reachable subspace of $\Sigma_{\text{ir}}$. The graph of $P_{\min}$ is closed.

(b) There exists a unique maximal operator $P_{\max} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique $P_{\max} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P_{\max}) \subset \text{gph}(P_{\max})$ for any other $P \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\max}$ is equal to the unobservable subspace of $\Sigma_{\text{ir}}$. In particular, $P_{\max}$ is closed.

(c) Let $\Sigma_{\text{ir}} = \Sigma_2 \parallel \Sigma_1$ be the difference of $\Sigma_1$ and $\Sigma_2$ (see Example 2.3.39). Then the graph of $P_{\min}$ is equal to the reachable subspace of $\Sigma_{\text{ir}}$, and the graph of $P_{\max}$ is equal to the unobservable subspace of $\Sigma_{\text{ir}}$. In particular, $P_{\max}$ is closed.

Thus, if $P$ is an arbitrary closed operator in $\mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, then

$$\text{gph}(P_{\min}) \subset \text{gph}(P) \subset \text{gph}(P_{\max}).$$

**Proof.** (i) By Lemma 2.5.27 if $\Sigma_1$ and $\Sigma_2$ are intertwined by some multi-valued linear operator $P$, then they are externally equivalent. The converse part


of (i) follows from (ii), which we shall prove next. (The proof of (ii) does not use (i).

(ii) Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. Let \( P_{\min} \in \mathcal{ML}(X_1; X_2) \) be the closed multi-valued operator whose graph is the reachable subspace of \( \Sigma_{\#}^1 \), and let \( P_{\max} \in \mathcal{ML}(X_1; X_2) \) be the closed multi-valued operator whose graph is the unobservable subspace of \( \Sigma_{\#}^2 \). By Lemma 3.2.4 \( \text{gph}(P_{\min}) \) is the minimal closed strongly invariant for \( \Sigma_{\#}^1 \), and \( \text{gph}(P_{\max}) \) is the maximal unobservably invariant subspace for \( \Sigma_{\#}^2 \). In particular, by Lemma 3.2.16, if \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some closed \( P \in \mathcal{ML}(X_1; X_2) \), then \( \text{gph}(P_{\min}) \subset \text{gph}(P) \subset \text{gph}(P_{\max}) \).

We claim that \( \text{gph}(P_{\min}) \) is unobservably invariant for \( \Sigma_{\#}^2 \), and that \( \text{gph}(P_{\max}) \) is strongly invariant for \( \Sigma_{\#}^1 \). We know from Lemmas 3.2.2 and 3.2.3 that both of these subspaces are invariant for the evolution group \( \mathfrak{A} \) of \( \Sigma_{\#}^1 \), so by the same two lemmas, to prove this claim we only have to check that \( \text{gph}(P_{\min}) \subset \text{ker}(C) = \text{ker}([C_2 \ -C_1]) \) and that \( \text{rng}(B) = \text{rng}([B_1 \ B_2]) \subset \text{gph}(P_{\max}) \). By Lemma 3.2.4 both of these conditions are equivalent to the condition that

\[
C(\lambda - A)^{-1}B = 0, \quad \lambda \in \rho_{\infty}(A).
\]

That this condition, indeed, holds follows from Theorem 3.2.7 and Lemma 2.3.41 since we assume \( \Sigma_1 \) and \( \Sigma_1 \) to be externally equivalent. Thus, both \( \text{gph}(P_{\min}) \) and \( \text{gph}(P_{\max}) \) are both strongly invariant and unobservably invariant for \( \Sigma_{\#}^2 \). In addition, again by Theorem 3.2.7 and Lemma 2.3.41 \( D = D_2 - D_1 = 0 \). By Lemma 3.2.16 both \( P_{\min} \) and \( P_{\max} \) intertwine \( \Sigma_1 \) and \( \Sigma_2 \).

3.2.19. Corollary. Let \( \Sigma_i = ([A_i \ B_i \ C_i \ D_i]; X_i; U; Y) \), \( i = 1, 2 \), be two bounded i/s/o systems (with the same input and output spaces). Moreover, suppose that both \( \Sigma_1 \) and \( \Sigma_2 \) are controllable and observable. Then \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. Among all the pseudo-similarities between \( \Sigma_1 \) and \( \Sigma_2 \) there is a (unique) minimal one \( P_{\min} \) and a (unique) maximal one \( P_{\max} \), namely those defined in Theorem 3.2.18 (both of which in this case are single-valued densely defined injective operators with dense range).

Proof. If \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar, then it follows from Theorem 3.2.18(i) that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

Conversely, suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. By Theorem 3.2.18 \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some \( P \in \mathcal{ML}(X_1; X_2) \). By Lemma 2.5.27 and the controllability assumption, \( \text{dom}(P) \) is dense in \( X_1 \) and \( \text{rng}(P) \) is dense in \( X_2 \). Furthermore, by Lemma 2.5.27 and the observability assumption, both \( \text{ker}(P) = \{0\} \) and \( \text{mul}(P) = \{0\} \). Thus, \( P \) is both injective and single-valued.

3.2.6. Compressions of bounded i/s/o systems. At this point the reader may want to recall Definition 2.5.28 of what we mean by a compression of an i/s/o system.

3.2.20. Theorem. Let \( \Sigma = ([A \ B \ C \ D]; X; U; Y) \) be a bounded i/s/o system with evolution group \( \mathfrak{A} \), and let \( X = X_1 \oplus Z_1 \) be a direct sum decomposition of \( X \). Then the following conditions are equivalent:

(i) \( \Sigma \) has a (forward) compression onto \( X_1 \) along \( Z_1 \).

(ii) \( \Sigma \) has a backward compression onto \( X_1 \) along \( Z_1 \).

(iii) The following four conditions hold:

(a) \( \mathfrak{A}^t \) has a compression onto \( X_1 \) along \( Z_1 \) for all \( t \in \mathbb{R} \).
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Σ forward and a backward compression of

by

group

A

e, s,
the generalized trajectory of Σ on
[0, ∞)

x

of Σ on
[0, ∞)

X

1

1

Proof.

We claim that

P_{X_1} \mathcal{X} B = P_{X_1}^R \mathcal{X} P_{X_1}^Z : B for all t ∈ R,

for all t ∈ R,

d) \mathcal{X} B = C P_{X_1} \mathcal{X} P_{X_1}^Z : B for all t ∈ R.

(iv) The following four conditions hold:

(a) A has a compression onto X_1 along Z,
(b) P_{X_1}^Z : A^n B = P_{X_1}^Z : A^n P_{X_1}^Z : B for all n ∈ Z^+,
(c) CA^\| X = C P_{X_1}^Z : A^n X = C P_{X_1}^Z : A^n P_{X_1}^Z : X for all n ∈ Z^+,
(d) CA^\| B = C P_{X_1}^Z : A^n P_{X_1}^Z : B for all n ∈ Z^+.

(v) The following four conditions hold:

(a) (λ - A)^{-1} has a compression onto X_1 along Z for all λ ∈ ρ_∞(A).
(b) P_{X_1}^Z : (λ - A)^{-1} B = P_{X_1}^Z : (λ - A)^{-1} P_{X_1}^Z : B for all λ ∈ ρ_∞(A),
(c) C(λ - A)^{-1} X = C P_{X_1}^Z : (λ - A)^{-1} X for all λ ∈ ρ_∞(A),
(d) C(λ - A)^{-1} B = C P_{X_1}^Z : (λ - A)^{-1} P_{X_1}^Z : B for all λ ∈ ρ_∞(A).

(vi) Conditions (a)-(d) in (iv) hold with ρ_∞(A) replaced by an arbitrary subset of ρ_∞(A) which has a cluster point in ρ_∞(A).

Suppose that these equivalent conditions hold, and define

\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \in \mathcal{B}(\Phi, \Psi)

by (3.2.11). Then the bounded i/s/o system Σ_1 = ([A_1, B_1]; X_1, U, Y) is both a forward and a backward compression of Σ onto X_1 along Z_1, and Σ_1 is the unique bounded forward or backward compression of Σ onto X_1 along Z_1. The main operator A_1 of Σ_1 is the compression onto X_1 along Z_1 of the main operator A of Σ, the evolution group \mathcal{X}_1 of Σ_1 is the compression onto X_1 along Z_1 of the evolution group \mathcal{X} of Σ, ρ_∞(A) ⊂ ρ(A_1), and (λ - A_1)^{-1} is the compression in \mathcal{B}(X_1) along Z_1 of (λ - A)^{-1} for all λ ∈ ρ_∞(A).

Proof. (i) ⇒ (iii): Let x^0 ∈ X_1 and u ∈ L^1_{loc}(R^+; U), let \begin{bmatrix} x \\ u \end{bmatrix}

be the future generalized trajectory of Σ with x(0) = x^0 given by Theorem 2.1.14, and let \begin{bmatrix} x \\ u \end{bmatrix}

de the generalized trajectory of Σ on [s, ∞) with x_s(s) = P_{X_1}^Z : x(s) given by Theorem 2.1.14 (with the same input function u).

We claim that P_{X_1}^Z : x_s(t) = \begin{bmatrix} x_u \\ u \end{bmatrix}

term and y_s(t) = y(t) for all t ∈ [s, ∞). To prove this we first notice that since Σ_1 is a compression of Σ onto X_1 along Z_1, the triple \begin{bmatrix} x_u \\ y \end{bmatrix}

is a generalized future trajectory of Σ_1 on R^+, and hence also on the interval [s, ∞). Since Σ is the dilation of Σ_1, this implies that there exists a trajectory \begin{bmatrix} x_s \\ u \end{bmatrix}

of Σ on [s, ∞) with initial state x_2(s) = P_{X_1}^Z : x(s) satisfying P_{X_1}^Z : x_2(t) = P_{X_1}^Z : x(t) for all t ∈ [s, ∞). Observe that this trajectory has the same initial state at time s and the same input function as the trajectory \begin{bmatrix} x_u \\ y \end{bmatrix}

. Since a trajectory of Σ is determined uniquely by its initial state and input function (see Theorem 2.1.14 we find that x_2(t) = x_s(t) and y(t) = y_s(t) for all t ∈ [s, ∞). This implies, in particular, that P_{X_1}^Z : x(t) = P_{X_1}^Z : x_2(t) = P_{X_1}^Z : x_s(t), t ∈ [s, ∞). After a change of variable these identities can alternatively be written in the form P_{X_1}^Z : x(s + t) = P_{X_1}^Z : x_s(s + t) and y(s + t) = y_s(s + t) for all s, t ∈ R^+.
To simplify the computations we first investigate what happens when \( u = 0 \). Then the above computation combined with Theorem 2.1.14 gives for all \( s, t \in \mathbb{R}^+ \),

\[
P_{X_1}^{s+t}x_0 = P_{X_1}^s x(s+t) = P_{X_1}^s x_s + P_{X_1}^{s+t}x_0,
\]

\[
C^s x_0 = y(s+t) = y_s + (s+t) = C^t P_{X_1}^{s+t}x_0.
\]

By Lemma 3.1.30, the first of these identities is equivalent to condition (a) in (iii).

From the second identity we get condition (c) in (iii) by taking \( t = 0 \).

We next look at the case where \( x^0 = 0 \), but \( u \) is allowed to be nonzero. By again applying Theorem 2.1.14, we get for all \( s, t \in \mathbb{R}^+ \),

\[
P_{X_1}^{s+t} \int_0^{s+t} \mathfrak{A}^s vBu(v)dv = P_{X_1}^s x(s+t) = P_{X_1}^s x_s + P_{X_1}^{s+t}x_0,
\]

\[
C \int_0^{s+t} \mathfrak{A}^s vBu(v)dv = y(s+t) - D(s+t) = y_s(s+t) - D(s+t)
\]

\[
= C \left( \mathfrak{A}^t P_{X_1}^s \int_0^s \mathfrak{A}^s vBu(v)dv + \int_s^{s+t} \mathfrak{A}^s vBu(v)dv \right).
\]

If we here take \( u(v) = 0 \) for \( v \geq s \), then the integrals over the interval \([s, s + t]\) vanish, and we get

\[
P_{X_1}^t \int_0^s \mathfrak{A}^s vBu(v)dv = P_{X_1}^s \mathfrak{A}^t P_{X_1}^s \int_0^s \mathfrak{A}^s vBu(v)dv,
\]

\[
C \mathfrak{A}^t \int_0^s \mathfrak{A}^s vBu(v)dv = C \mathfrak{A}^t P_{X_1}^s \int_0^s \mathfrak{A}^s vBu(v)dv.
\]

Here we may, e.g., take \( u(v) = u^0 \) for \( v \in [0, s] \) where \( u^0 \) is an arbitrary vector in \( \mathcal{U} \). After dividing the above identities by \( s \) and letting \( s \downarrow 0 \) we then get

\[
P_{X_1}^t \mathfrak{A}^t Bu_0 = P_{X_1}^s \mathfrak{A}^t P_{X_1}^t Bu_0, \quad C \mathfrak{A}^t Bu_0 = C \mathfrak{A}^t P_{X_1}^s Bu_0.
\]

The first of these conditions coincides with condition (b) in (iii), and the second condition combined with condition (c) in (iii) gives

\[
C \mathfrak{A}^t Bu_0 = C \mathfrak{A}^t P_{X_1}^s Bu_0 = C \mathfrak{A}^t P_{X_1}^t P_{X_1}^s Bu_0.
\]

This gives us condition (d) in (iii). This completes the proof of the implication (i) \( \Rightarrow \) (iii).

(iii) \( \Rightarrow \) (i): Let \( \Sigma_1 = \left( \left( A_1, B_1 \right); X_1, \mathcal{U}, \mathcal{Y} \right) \) be the bounded i/s/o system whose system operator \( \left[ \begin{array}{c} A_1 \\ B_1 \end{array} \right] \in \mathcal{B} \left( \left[ X_1, Y_1 \right] \right) \) is given in \( \left( 3.2.11 \right) \). It follows from condition (a) in (iii) and Lemma 3.1.30 that the evolution group \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the compression of the evolution group \( \mathfrak{A} \) of \( \Sigma \). In particular, \( \mathfrak{A}_1^t = P_{X_1}^t \mathfrak{A}_1^t \mathfrak{A}_1^s \) for all \( t \in \mathbb{R} \). By comparing formula \( \left( 2.1.19 \right) \) for the two systems \( \Sigma \) and \( \Sigma_1 \) to each other it is easy to see that \( \Sigma_1 \) is the compression of \( \Sigma \) onto \( X_1 \) along \( X \).

(iii) \( \Rightarrow \) (ii): This proof is the same as the proof above with \( \Sigma \) replaced by the time reflected system, i.e., \( A \) is replaced by \(-A\), \( B \) is replaced by \(-B\), and \( \mathfrak{A} \) is replaced by \( \mathfrak{A}^{-t} \). The bounded system that we get as a part of the proof of the implication (iii) \( \Rightarrow \) (ii) coincides with the bounded system \( \Sigma_1 \) that we obtained in the proof of the implication (iii) \( \Rightarrow \) (i) above.
We have now proved that (i), (ii), and (iii) are equivalent. Before continuing to prove that these conditions are equivalent to (iv) and (v), let us observe that we have also shown that the bounded i/s/o system \( \Sigma \) defined above is both a forward and a backward compression of \( \Sigma \) onto \( X_1 \) along \( Z_1 \), and that the evolution group of \( \Sigma_1 \) is the compression of the evolution group \( \mathfrak{A} \) of \( \Sigma \). By Lemma 2.5.31 solvable forward or backward compressions of \( \Sigma \) onto \( X_1 \) along \( Z_1 \) are unique. Thus, \( \Sigma_1 \) is both the unique solvable forward compression and the unique solvable backward projection onto \( X_1 \) along \( Z_1 \). Thus, it only remains to prove that (iv) and (v) are equivalent to conditions (i)–(iii).

(iii) \( \Leftrightarrow \) (iv): That condition (a) in (iii) is equivalent to condition (a) in (iv) follows from Lemma 3.1.30. That also the other conditions (b), (c), and (d) in (iii) is equivalent to the corresponding condition in (iv) can be proved in an analogous way: Both the right and the left sides of the equations in (b), (c), and (d) in (iii) are entire functions of \( \lambda \), and hence they coincide for all \( \lambda \in \mathbb{R} \) if and only if they have the same Taylor expansions at zero. The coefficients in these Taylor expansions are exactly those which are given in the corresponding conditions in (iv).

(iv) \( \Leftrightarrow \) (v): That condition (a) in (iv) is equivalent to condition (a) in (v) follows from Lemma 3.1.33. That also the other conditions (b), (c), and (d) in (iv) is equivalent to the corresponding condition in (v) can be proved in an analogous way: Both the right and the left sides of the equations in (b), (c), and (d) in (iii) are entire functions of \( \lambda \) in a neighborhood of infinity, and hence they coincide for all \( \lambda \in \rho_\infty(A) \) if and only if they have the same Taylor expansions at infinity. The coefficients in these Taylor expansions are exactly those which are given in the corresponding conditions in (v) (multiplied by \( n! \)).

(v) \( \Leftrightarrow \) (vi): This follows from the analyticity of all the functions appearing in (vi) and the fact that \( \rho_\infty(A) \) is connected.

\[ \square \]

Theorem 3.2.20 can be reformulated in many different ways. One such reformulation which puts equal emphasis on the compression \( \Sigma_1 \) and the dilation \( \Sigma \) is the following:

3.2.21. \textbf{Lemma.} Let \( \Sigma = ([A \ B] ; X, U, Y) \) be a bounded i/s/o system with evolution group \( \mathfrak{A} \), let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \), and let \( \Sigma_1 = ([A_1 \ B_1] ; X_1, U, Y) \) be another bounded i/s/o system with evolution group \( \mathfrak{A}_1 \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) is the (unique) bounded compression of \( \Sigma \);

(ii) \textbf{The following four conditions hold:}

(a) \( \mathfrak{A}_1^t = P_{X_1}^t \mathfrak{A}^t|_{X_1} \) for all \( t \in \mathbb{R} \) (i.e., \( \mathfrak{A}_1 \) is the compression of \( \mathfrak{A} \) onto \( X_1 \) along \( Z_1 \));

(b) \( \mathfrak{A}_1^t B_1 = P_{X_1}^t \mathfrak{A} B \) for all \( t \in \mathbb{R} \);

(c) \( C_1 \mathfrak{A}_1^t = CA^t|_{X_1} \) for all \( t \in \mathbb{R} \);

(d) \( D_1 = D \) and \( C_1 \mathfrak{A}_1^t B_1 = CA^t B \) for all \( t \in \mathbb{R} \).

(iii) \textbf{The following four conditions hold:}

(a) \( A_1^n = P_{X_1}^n A^n|_{X_1} \) for all \( n \in \mathbb{Z}^+ \) (i.e., \( A_1 \) is a compression of \( A \) onto \( X_1 \) along \( Z_1 \));

(b) \( A_1^n B_1 = P_{X_1}^n A^n B \) for all \( n \in \mathbb{Z}^+ \);

(c) \( C_1 A_1^n = CA^n|_{X_1} \) for all \( n \in \mathbb{Z}^+ \);

(d) \( D_1 = D \) and \( C_1 A_1^n B_1 = CA^n B \) for all \( n \in \mathbb{Z}^+ \).

(iv) \textbf{The following four conditions hold:}
The following conditions are equivalent:

(a) $\rho_\infty(A) \subset \rho(A_1)$ and $(\lambda - A_1)^{-1} = P_{X_1}^\omega(\lambda - A)^{-1}|_{X_1}$ for all $\lambda \in \rho_\infty(A)$ (or equivalently, $(\lambda - A_1)^{-1}$ is a compression in $X_1$ along $Z_1$ of $(\lambda - A)^{-1}$ for all $\lambda \in \rho_\infty(A)$);

(b) $(\lambda - A_1)^{-1}B_1 = P_{X_1}^\omega(\lambda - A)^{-1}B$ for all $\lambda \in \rho_\infty(A)$,

(c) $C(\lambda - A_1)^{-1} = C(\lambda - A)^{-1}|_{X_1}$ for all $\lambda \in \rho_\infty(A)$,

(d) $D_1 = D$ and $C(\lambda - A_1)^{-1}B_1 = C(\lambda - A)^{-1}B$ for all $\lambda \in \rho_\infty(A)$.

(v) Conditions (a)–(d) in (iv) hold with $\rho_\infty(A)$ replaced by an arbitrary subset $\Omega'$ of $\rho_\infty(A)$ which has a cluster point in $\rho_\infty(A)$.

**Proof.** This follows immediately from Theorem 3.2.20

Another reformulation of Theorem 3.2.20 is the following.

**Lemma.** Let $\Sigma = \{[A_{\mathcal{C}} B_{\mathcal{D}}]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\}$ be a bounded i/s/o system with evolution group $\mathfrak{A}$, and let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Then the following conditions are equivalent:

(i) Conditions (a)–(d) in part (iii) of Theorem 3.2.20 are equivalent to the following four conditions:

(a) $P_{X_1}^\omega A^n P_{X_1}^\omega A^n|_{X_1} = 0$ all $s, t \in \mathbb{R}$,

(b) $P_{X_1}^\omega A^n P_{X_1}^\omega A^n B = 0$ for all $s, t \in \mathbb{R}$,

(c) $CA^n P_{X_1}^\omega A^n|_{X_1} = 0$ for all $s, t \in \mathbb{R}$,

(d) $CA^n P_{X_1}^\omega A^n B = 0$ for all $s, t \in \mathbb{R}$.

(ii) Conditions (a)–(d) in part (iv) of Theorem 3.2.20 are equivalent to the following four conditions:

(a) $P_{X_1}^\omega A^n P_{X_1}^\omega A^n|_{X_1} = 0$ all $m, n \in \mathbb{Z}^+$,

(b) $P_{X_1}^\omega A^n P_{X_1}^\omega A^n B = 0$ for all $m, n \in \mathbb{Z}^+$,

(c) $CA^n P_{X_1}^\omega A^n|_{X_1} = 0$ for all $m, n \in \mathbb{Z}^+$,

(d) $CA^n P_{X_1}^\omega A^n B = 0$ for all $m, n \in \mathbb{Z}^+$.

(iii) Conditions (a)–(d) in part (v) of Theorem 3.2.20 are equivalent to the following four conditions:

(a) $P_{X_1}^\omega (\lambda - A)^{-1} P_{X_1}^\omega (\mu - A)^{-1}|_{X_1} = 0$ all $\lambda, \mu \in \rho_\infty(A)$,

(b) $P_{X_1}^\omega (\lambda - A)^{-1} P_{X_1}^\omega (\mu - A)^{-1} B = 0$ for all $\lambda, \mu \in \rho_\infty(A)$,

(c) $C(\lambda - A)^{-1} P_{X_1}^\omega (\mu - A)^{-1}|_{X_1} = 0$ for all $\lambda, \mu \in \rho_\infty(A)$,

(d) $C(\lambda - A)^{-1} P_{X_1}^\omega (\mu - A)^{-1} B = 0$ for all $\lambda, \mu \in \rho_\infty(A)$.

(iv) Conditions (a)–(d) in part (vi) of Theorem 3.2.20 are equivalent to conditions (a)–(d) in (iii) above with $\rho_\infty(A)$ replaced by $\Omega'$.

**Proof.** That the different versions of condition (a) in (i)–(iv) above are equivalent to each other and also equivalent to the different versions of condition (a) in Theorem 3.2.20 follows from Lemmas 3.1.30 and 3.1.33. If we assume the equivalent conditions (a) to hold, then it is easy to show that the different versions of condition (b) in (i)–(iv) are equivalent to the corresponding condition (b) in Theorem 3.2.20 (and that actually all of the different versions of condition (b) are equivalent to each other whenever (a) holds). Likewise, if we assume the equivalent conditions (a) to hold, then it is easy to show that the different versions of condition (c) in (i)–(iv) are equivalent to the corresponding condition (c) in Theorem 3.2.20 (and actually all of the different versions of condition (c) are equivalent to each other whenever (a) holds). Finally, if we assume that both conditions (b) and conditions (c) hold,
then it is easy to show that the different versions of condition (d) in (i)–(iv) are equivalent to the corresponding condition (d) in Theorem 3.2.20 (and that actually all of the different versions of condition (d) are equivalent to each other whenever (b) and (c) hold).

\[ \square \]

3.2.7. The general structure of a bounded i/s/o compression.

3.2.23. Lemma. Let \( \Sigma = ([A B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o system, let \( \mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1 \) be a direct sum decomposition of \( \mathcal{X} \), and let \( \Sigma_{\text{ext}} = ([A_{\text{ext}} B_{\text{ext}}]; \mathcal{X}', [\mathcal{U}], [\mathcal{Y}']) \) be the bounded i/s/o system with

\[
\begin{bmatrix}
A_{\text{ext}} & B_{\text{ext}} \\
C_{\text{ext}} & D_{\text{ext}}
\end{bmatrix}
= 
\begin{bmatrix}
A & B \mathcal{I}_X \\
C & D \\
P_{X_1} & 0
\end{bmatrix}.
\]

(i) There exists a (unique) minimal closed strongly invariant subspace \( \mathcal{X}_{\text{min}} \) for \( \Sigma \) which contains \( \mathcal{X}_1 \) (i.e., \( \mathcal{X}_{\text{min}} \) is closed and strongly invariant for \( \Sigma \), and \( \mathcal{X}_{\text{min}} \) is contained in every other closed strongly invariant subspace of \( \Sigma \) which contains \( \mathcal{X}_1 \)). This subspace has the following alternative descriptions:

(a) \( \mathcal{X}_{\text{min}} \) is the reachable subspace of \( \Sigma_{\text{ext}} \);
(b) \( \mathcal{X}_{\text{min}} \) is the closed linear span of the reachable subspace of \( \Sigma \) and the subspace \( \mathcal{X}_{\text{min}} \) defined in Lemma 3.1.36.

(ii) The space \( \mathcal{X}_{\text{min}} \) has the direct sum decomposition \( \mathcal{X}_{\text{min}} = \mathcal{X}_1 + \mathcal{Z}_{\text{min}} \), where

\[
\mathcal{Z}_{\text{min}} = \mathcal{X}_{\text{min}} \cap \mathcal{Z}_1 = P_{X_1}^\perp \mathcal{X}_{\text{min}}.
\]

(iii) There exists a (unique) maximal unobservably invariant subspace \( \mathcal{Z}_{\text{max}} \) for \( \Sigma \) which is contained in \( \mathcal{Z}_1 \) (i.e., \( \mathcal{Z}_{\text{max}} \) is observably invariant for \( \Sigma \), and \( \mathcal{Z}_{\text{max}} \) contains every other unobservably invariant subspace for \( \Sigma \) which is contained in \( \mathcal{Z}_1 \)). This subspace has the following alternative descriptions:

(a) \( \mathcal{Z}_{\text{max}} \) is the unobservable subspace of \( \Sigma_{\text{ext}} \);
(b) \( \mathcal{Z}_{\text{max}} \) is the intersection of the unobservable subspace of \( \Sigma \) and the subspace \( \mathcal{Z}_{\text{max}} \) defined in Lemma 3.1.36.

In particular, \( \mathcal{Z}_{\text{max}} \) is closed.

Note that the space \( \mathcal{X}_{\text{min}} \) above depends only on \( \mathcal{X}_1 \) (and not on \( \mathcal{Z}_1 \)), and that the space \( \mathcal{Z}_{\text{max}} \) above depends only on \( \mathcal{Z}_1 \) (and not on \( \mathcal{X}_1 \)).

**Proof of Lemma 3.2.23.** Proof of (i): Denote the reachable subspace of \( \Sigma_{\text{ext}} \) by \( \mathcal{R}_{\Sigma_{\text{ext}}} \). Then by Lemma 3.2.4 \( \mathcal{R}_{\Sigma_{\text{ext}}} \) is the minimal closed \( A \)-invariant subspace which contains \( \text{rng}(B_{\text{ext}}) = \text{rng}(B X_1') = \text{rng}(B) + X_1 \). This is equivalent to the statement that \( \mathcal{R}_{\Sigma_{\text{ext}}} \) is the minimal closed \( A \)-invariant subspace which contains both \( X_1 \) and \( \text{rng}(B) \), and also to the statement that \( \mathcal{R}_{\Sigma_{\text{ext}}} \) is the minimal strongly invariant subspace for \( \Sigma \) which contains \( X_1 \).

Denote the subspace \( \mathcal{X}_{\text{min}} \) in Lemma 3.1.36 by \( \mathcal{X}_{\text{min}}^A \), and the reachable subspace of \( \Sigma \) by \( \mathcal{R}_\Sigma \). By Lemmas 3.1.36 and 3.2.4 \( \mathcal{X}_{\text{min}}^A \) is the minimal \( A \)-invariant subspace which contains \( X_1 \), and \( \mathcal{R}_\Sigma \) is the minimal \( A \)-invariant subspace which contains \( \text{rng}(B) \). This implies that the span of \( \mathcal{X}_{\text{min}}^A \) and \( \mathcal{R}_\Sigma \) is the minimal \( A \)-invariant subspace which contains both \( X_1 \) and \( \text{rng}(B) \), and that \( \mathcal{X}_{\text{min}}^A \vee \mathcal{R}_\Sigma \) is the minimal
closed subspace which contains both $X_1$ and $\text{rng}(B)$. By Lemma 3.1.6, $X^A_{\text{min}} \vee R_{\Sigma}$ is $A$-invariant. Thus, $X^A_{\text{min}} \vee R_{\Sigma}$ is the minimal closed $A$-invariant subspace which contains both $X_1$ and $\text{rng}(B)$.

**Proof of (ii):** Define $Z_{\text{min}}$ by $Z_{\text{min}} = X_{\text{min}} \cap Z_1$. Then by Lemma 3.1.35, $X_{\text{min}} = X_1 + Z_{\text{min}}$ and $Z_{\text{min}} = P_{Z_1} X_{\text{min}}$. The proof of (iii) is analogous to the proof of (i) given above. □

3.2.24. **Lemma.** Let $\Sigma = ([A_B] ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a bounded i/s/o system with evolution semigroup $\mathcal{A}$, and let $X = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\rho_{\infty}(A)$, and let $\lambda_0 \in \rho_{\infty}(A)$. Denote the injection operator $X_1 \hookrightarrow \mathcal{X}$ by $I_{X_1}$.

(i) The subspace $X_{\text{min}}$ in Lemma 3.2.23 can be computed in the following alternative ways:

\[
X_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng}(\mathcal{A}^t [B \ I_{X_1}]) = \bigvee_{t \in \mathbb{R}^-} \text{rng}(\mathcal{A}^t [B \ I_{X_1}])
\]

\[
= \bigvee_{n \in \mathbb{Z}^+} \text{rng}(A^n [B \ I_{X_1}])
\]

\[
= \bigvee_{\lambda \in \rho_{\infty}(A)} \text{rng}((\lambda - A)^{-1} [B \ I_{X_1}]) = \bigvee_{\lambda \in \Omega'} \text{rng}((\lambda - A)^{-1} [B \ I_{X_1}])
\]

\[
= \bigvee_{n \in \mathbb{Z}^+} \text{rng}((\lambda_0 - A)^{-n+1} [B \ I_{X_1}]).
\]

(ii) The subspace $Z_{\text{min}}$ in Lemma 3.2.23 can be computed in the following alternative ways:

\[
Z_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng}(P_{Z_1} \mathcal{A}^t [B \ I_{X_1}]) = \bigvee_{t \in \mathbb{R}^-} \text{rng}(P_{Z_1} \mathcal{A}^t [B \ I_{X_1}])
\]

\[
= \bigvee_{n \in \mathbb{Z}^+} \text{rng}(P_{Z_1} A^n [B \ I_{X_1}])
\]

\[
= \bigvee_{\lambda \in \rho_{\infty}(A)} \text{rng}(P_{Z_1} (\lambda - A)^{-1} [B \ I_{X_1}]) = \bigvee_{\lambda \in \Omega'} \text{rng}(P_{Z_1} (\lambda - A)^{-1} [B \ I_{X_1}])
\]

\[
= \bigvee_{n \in \mathbb{Z}^+} \text{rng}(P_{Z_1} (\lambda_0 - A)^{-n+1} [B \ I_{X_1}]).
\]
3.2. INTERTWINEMENTS AND COMPRESSIONS OF BOUNDED I/S/O SYSTEMS

(3.2.19) The subspace \( Z_{\max} \) in Lemma 3.2.23 can be computed in the following alternative ways:

\[
Z_{\max} = \bigcap_{t \in \mathbb{R}^+} \{ x \in \mathcal{X} \mid A^t x \in Z_1 \text{ and } CA^t x = 0 \}
\]

\[
= \bigcap_{t \in \mathbb{R}^-} \{ x \in \mathcal{X} \mid A^t x \in Z_1 \text{ and } CA^t x = 0 \}
\]

\[
= \bigcap_{n \in \mathbb{Z}^+} \{ x \in \mathcal{X} \mid A^n x \in Z_1 \text{ and } CA^n x = 0 \}
\]

\[
= \bigcup_{\lambda \in \rho(A)} \{ x \in \mathcal{X} \mid (\lambda - A)^{-1} x \in Z_1 \text{ and } C(\lambda - A)^{-1} x = 0 \}
\]

\[
= \bigcup_{\lambda \in \rho(A)} \{ x \in \mathcal{X} \mid (\lambda - A)^{-1} x \in Z_1 \text{ and } C(\lambda - A)^{-1} x = 0 \}
\]

\[
= \bigcap_{n \in \mathbb{Z}^+} \{ x \in \mathcal{X} \mid (\lambda_0 - A)^{-(n+1)} x \in Z_1 \text{ and } C(\lambda_0 - A)^{-(n+1)} x = 0 \}.
\]

PROOF. This follows from Lemmas 3.2.4 and 3.2.23.

3.2.25. THEOREM. Let \( \Sigma = ([A \ B] ; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o system, and let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \). Let \( \mathcal{X}_\text{min} \) be the minimal closed strongly invariant subspace of \( \Sigma \) which contains \( \mathcal{X}_1 \), let \( Z_{\max} \) be the maximal unobservably invariant subspace of \( \Sigma \) which is contained in \( Z_1 \), and let \( Z_{\min} = \mathcal{X}_\text{min} \cap Z_1 \) (cf. Lemma 3.2.23). Then the following conditions are equivalent:

(i) \( \Sigma \) has a compression onto \( \mathcal{X}_1 \) along \( Z_1 \);
(ii) \( \mathcal{X}_1 \) contains some closed unobservably invariant subspace \( Z \) for \( \Sigma \) such that \( \mathcal{X}_1 + Z \) is strongly invariant for \( \Sigma \);
(iii) \( Z_{\min} \) in an unobservably invariant subspaces for \( \Sigma \);
(iv) \( \mathcal{X} + Z_{\max} \) is a strongly invariant subspace for \( \Sigma \);
(v) \( Z_{\min} \subset Z_{\max} \).

Two possible choices of the subspace \( Z \) in (ii) are \( Z = Z_{\min} \) and \( Z = Z_{\max} \), and every possible subspace \( Z \) in (ii) satisfies \( Z_{\min} \subset Z \subset Z_{\max} \).

Suppose that the equivalent conditions (i)–(v) hold, and define \( [A_1 \ B_1] \in B\left( [\mathcal{X}_1] \left[ \begin{array}{c} \mathcal{U} \end{array} \right] \right) \) by \( 3.2.11 \). Then the bounded i/s/o system \( \Sigma_1 = ([A_1 \ B_1] ; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) is a compression of \( \Sigma \) onto \( \mathcal{X}_1 \) along \( Z_1 \), and \( \Sigma_1 \) is the unique bounded forward or backward compression of \( \Sigma \) onto \( \mathcal{X}_1 \) along \( Z_1 \).

It is possible to prove Theorem 3.2.25 in at least three different ways, depending on which of the different characterizations (iii), (iv), or (v) in Theorem 3.2.20 of compressions we decide to work with. This theorem is extended to semi-bounded systems in Theorem 4.2.29 and to a frequency domain setting in Theorem 3.2.25. In the latter setting it is natural to use the resolvent characterization (v) of compressions. Theorem 3.2.25 is also extended to the well-posed setting in Theorem 8.2.20 and there it is natural to work with the (semi)group characterization (iii). For completeness we here give a proof which is based on (iv), i.e., in this proof the main operator \( A \) of \( \Sigma \) plays the main role. This proof has the additional advantage that the same proof applies in the case where the bounded i/s/o system \( \Sigma \) is replaced by a (well-posed) discrete time system.
Proof of Theorem 3.2.25

(i) $\Rightarrow$ (iii): Suppose that (i) holds, and define $[A_1, B_1] \in \mathcal{B} \left( \mathcal{X}_1 | \mathcal{U} \right)$ by (3.2.11), and let $\Sigma_1$ be the bounded i/s/o system $\Sigma_1 = (\left[ A_1, B_1 \right]; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$. By Theorem 3.2.20 $\Sigma_1$ is a (bounded) compression of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, and $A_1$ is a compression onto $\mathcal{X}_1$ of $A$ along $\mathcal{Z}_1$. By Lemma 3.2.22 for all $x \in \mathcal{X}_1$, $u \in \mathcal{U}$, and $m, n \in \mathbb{Z}^+$, we have

$$P_{\mathcal{X}_1} Z_{z_1} A^m Z_{\mathcal{X}_1} A^n (x + Bu) = 0, \quad CA^m Z_{\mathcal{X}_1} A^n (x + Bu) = 0. \tag{3.2.20}$$

By taking the closed linear span over all $n, x,$ and $u$ and using the fact that $X_{\text{min}} = \bigvee_{n \in \mathbb{Z}^+} A^n(\mathcal{X}_1 + BU)$ we get

$$P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^m X_{\text{min}} = 0, \quad CA^m Z_{\mathcal{X}_1} X_{\text{min}} = 0. \tag{3.2.21}$$

Taking $m = 1$ in the first identity and $m = 0$ in the second identity we get (recall that $P_{\mathcal{X}_1} X_{\text{min}} = Z_{\text{min}}$)

$$AZ_{\text{min}} \subset Z_1, \quad Z_{\text{min}} \subset \ker(C). \tag{3.2.22}$$

By Lemma 3.2.23, $X_{\text{min}}$ is a strongly invariant subspace for $\Sigma$, and hence $X_{\text{min}}$ is $A$-invariant. Since $Z_{\text{min}} = X_{\text{min}} \cap Z_1 \subset X_{\text{min}}$ we get

$$AZ_{\text{min}} \subset AX_{\text{min}} \subset X_{\text{min}}. \tag{3.2.23}$$

Thus $AZ_{\text{min}} \subset X_{\text{min}} \cap Z_1 = Z_{\text{min}}$. This shows that $Z_{\text{min}}$ is $A$-invariant. Since, in addition, $Z_{\text{min}} \subset \ker(C)$, it follows from Lemma 3.2.3 that $Z_{\text{min}}$ is unobservably invariant for $\Sigma$.

(iii) $\Rightarrow$ (ii): This follows from Lemma 3.2.23(i) (take $Z = Z_{\text{min}}$).

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. It follows from (ii) and Lemmas 3.2.2 and 3.2.3 that that for all $n \in \mathbb{Z}^+$,

$$A^n(\mathcal{X}_1 + Z) \subset \mathcal{X}_1 + Z, \quad A^nZ \subset Z, \quad \ker(B) \subset \mathcal{X}_1 + Z, \quad Z \subset \ker(C). \tag{3.2.24}$$

From these conditions follow that for all $x \in \mathcal{X}_1$, $u \in \mathcal{U}$, and $m, n \in \mathbb{Z}^+$ we have

$$A^n(x + Bu) \in \mathcal{X}_1 + Z \Rightarrow P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^n(x + Bu) \in Z \Rightarrow A^m P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^n(x + Bu) \in Z \Rightarrow P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^m P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^n(x + Bu) = 0 \Rightarrow CA^m P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^n(x + Bu) = 0.$$ 

By taking $u = 0$ we get conditions (a) and (c) in part (ii) of Lemma 3.2.22, and by taking $x = 0$ we get conditions (b) and (d) in part (ii) of Lemma 3.2.22. Consequently, by Theorem 3.2.20 and Lemma 3.2.22 $\Sigma$ has a compression onto $\mathcal{X}_1$ along $Z_1$.

(i) $\Rightarrow$ (iv): Suppose that (i) holds. By taking the intersection over all $m$ in $\left(3.2.20\right)$ we find that for all $x \in \mathcal{X}_1$, $u \in \mathcal{U}$, and $n \in \mathbb{Z}^+$ we have $P_{\mathcal{X}_1} Z_{\mathcal{X}_1} A^n(x + Bu) \in Z_{\text{max}}$. Therefore $A^n(x + Bu) = (P_{\mathcal{X}_1} Z_{\mathcal{X}_1}) A^n(x + Bu) \in \mathcal{X}_1 + Z_{\text{max}}$. In particular, $\ker(B) \subset \mathcal{X}_1 + Z_{\text{max}}$ and $AX_1 \subset \mathcal{X}_1 + Z_{\text{max}}$. By Lemma 3.2.23, $AZ_{\text{max}} \subset Z_{\text{max}}$. Thus $A(\mathcal{X}_1 + Z_{\text{max}}) \subset \mathcal{X}_1 + Z_{\text{max}}$. By Lemma 3.2.22, $\mathcal{X}_1 + Z_{\text{max}}$ is a strongly invariant subspace for $\Sigma$.

(iv) $\Rightarrow$ (ii): This follows from Lemma 3.2.23(ii) (take $Z = Z_{\text{max}}$).

(iii) $\Rightarrow$ (v): This follows from Lemma 3.2.23(ii).

(v) $\Rightarrow$ (iii): By Lemma 3.2.23, $\mathcal{X}_1 + Z_{\text{min}}$ is strongly invariant for $\Sigma$ and $Z_{\text{max}}$ is unobservably invariant for $\Sigma$. By Lemma 3.2.3, $Z_{\text{max}} \in \ker(C)$, so if (v) holds, then also $Z_{\text{min}} \in \ker(C)$. In additional the $A$-invariance of $\mathcal{X}_1 + Z_{\text{min}}$
implies that \(AZ_{\min} \subseteq X_1 + Z_{\min}\), and the \(A\)-invariance of \(Z_{\max}\) together with the condition \(Z_{\min} \subseteq Z_{\max}\) implies that \(AZ_{\min} \subseteq AZ_{\max} \subseteq Z_{\max} \subseteq Z_1\). Thus \(AZ_{\min} \subseteq (X_1 + Z_{\min}) \cap Z_1 = Z_{\min}\). This combined with Lemma 3.2.3 shows that \(Z_{\min}\) is unobservably invariant for \(\Sigma\).

3.2.26. COROLLARY. Let \(\Sigma = \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] : \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a bounded i/s/o system, let \(X = X_1 + Z_1\) be a direct sum decomposition of \(X\), and let \(\Sigma_1 = \left[ \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right] : \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be the bounded i/s/o system whose system operator is given by (3.2.11). Then \(\Sigma_1\) is the compression of \(\Sigma\) onto \(X_1\) along \(Z_1\) if and only if \(X_1\) has a direct sum decomposition \(X_1 = Z + Z_c\) such that \(\left[ \begin{array}{cc} A & B \\ C & D \end{array} \right]\) has the following structure with respect to the decomposition \(X = Z + X_1 + Z_c\) of \(X\) (where we use “\(\ast\)” to mark irrelevant entries):

\[
(A \ B) = \left[
\begin{array}{ccc}
A_Z & \ast & \ast \\
0 & A_1 & B_1 \\
0 & 0 & A_{Zc} \\
0 & C_1 & D_1 \\
\end{array}
\right].
\]

From this structure follows that \(Z + X_1\) is strongly invariant for \(\Sigma\), that \(Z\) is unobservably invariant for \(\Sigma\), that \(A_{Zc}\) is the restriction of \(A\) in \(\mathcal{B}(Z)\), and that \(A_{Zc}\) is the projection in \(\mathcal{B}(Z_c)\) of \(A_1 + Z\). The subspace \(Z\) in this decomposition can be chosen to be the same as the subspace \(Z\) in condition (ii) in Theorem 3.2.25, and the subspace \(Z_c\) can be chosen to be an arbitrary direct complement to \(Z\) in \(Z_1\). In particular, two possible choices of \(Z\) are \(Z = Z_{\min}\) and \(Z = Z_{\max}\), where \(Z_{\min}\) and \(Z_{\max}\) are the subspaces defined in Lemma 3.2.23.

PROOF. This follows from the equivalence of (i) and (ii) in Theorem 3.2.25 (take \(Z_c\) to be an arbitrary direct complement to \(Z\) in \(Z_1\)).

3.2.27. THEOREM. Let \(\Sigma = \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] : \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a bounded i/s/o system, let \(X = X_1 + Z_1\) be a direct sum decomposition of \(X\), and suppose that \(\Sigma_1 = \left[ \begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array} \right] : \mathcal{X}_1, \mathcal{U}, \mathcal{Y} \) is a bounded compression of \(\Sigma\) onto \(X_1\) along \(Z_1\). Let \(Z\) satisfy the conditions listed in (ii) in Theorem 3.2.25, and let \(Z_c\) be an arbitrary direct complement to \(Z\) in \(Z_1\).

(i) Let \(\Sigma_2\) be the unique bounded restriction of \(\Sigma\) to the strongly invariant subspace \(X_1 + Z\) for \(\Sigma\) given by Theorem 3.2.17. Then \(Z\) is unobservably invariant for \(\Sigma_2\), and \(\Sigma_1\) is the unique bounded projection onto \(X_1\) along \(Z\) of \(\Sigma_2\) given by Theorem 3.2.12.

(ii) Let \(\Sigma_3\) be the unique bounded restriction of \(\Sigma\) onto \(X_1 + Z_c\) along \(Z\) given by Theorem 3.2.12. Then \(X_1\) is strongly invariant for \(\Sigma_3\), and \(\Sigma_1\) is the unique bounded restriction of \(\Sigma_3\) to \(X_1\) given by Theorem 3.2.11.

(iii) The main operator \(A_1\) of \(\Sigma_1\) is the projection in \(\mathcal{B}(X_1)\) along \(Z\) of the restriction in \(\mathcal{B}(X_1 + Z)\) of \(A\), and it is also the restriction in \(\mathcal{B}(X_1)\) of the projection in \(\mathcal{B}(X_1 + Z_c)\) along \(Z\) of \(A\).

PROOF. (i) With the notation of 3.2.21, the system operator \(\left[ \begin{array}{cc} A_2 & B_2 \\ C_2 & D_2 \end{array} \right]\) of \(\Sigma_2\) has the structure (with respect to the decomposition \(Z + X_1\) of the state space of \(\Sigma_2\))

\[
\left[
\begin{array}{ccc}
A_2 & B_2 \\
0 & A_1 & B_1 \\
0 & C_1 & D_1 \\
\end{array}
\right].
\]
Thus by Lemma 3.2.3, $\mathcal{Z}$ is unobservably invariant for $\Sigma_2$. By Theorem 3.2.12, the bounded projection of $\Sigma_2$ onto $X_1$ along $\mathcal{Z}$ is equal to $\Sigma_1$.

(ii) With the notation of 3.2.21, the system operator $[A_3, B_3]$ of $\Sigma_3$ has the structure (with respect to the decomposition $X_1 + Z_c$ of the state space of $\Sigma_3$

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = \begin{bmatrix} A_1 & * & B_1 \\ 0 & * & 0 \\ C_1 & * & D_1 \end{bmatrix}.$$ 

Thus by Lemma 3.2.2, the restriction of $\Sigma_3$ to $X_1$ is strongly invariant for $\Sigma_3$. By Theorem 3.2.11, the restriction of $\Sigma_3$ to $X_1$ is equal to $\Sigma_1$.

(iii) That (iii) holds follows from (i) and (ii) together with Theorems 3.2.20 and Lemmas 3.2.2 and 3.2.3.

3.2.28. Lemma. Let $\Sigma = ([A B]_c; X, U, Y)$ and $\Sigma_1 = ([A_1 B_1]_c; X_1, U, Y)$ be two bounded i/s/o systems with $X = X_1 + Z_1$. Then the following two conditions are equivalent.

(i) $\Sigma_1$ is the compression of $\Sigma$ onto $X_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z$ such that $\Sigma$ and $\Sigma_1$ are intertwined by the operator $P_{X_1}^{Z_1}|_{X_1+Z}$.

Condition (ii) above holds for some particular subspace $Z$ if and only condition (ii) in Theorem 3.2.25 holds for the same subspace $Z$. Thus, in particular, two possible choices of the subspace $Z$ in (ii) are the subspaces $Z = Z_{\min}$ and $Z = Z_{\max}$ defined in Lemma 3.2.23, and every possible subspace $Z$ satisfies $Z_{\min} \subset Z \subset Z_{\max}$.

Proof. (i) ⇒ (ii): Let $\mathcal{Z}$ satisfy condition (ii) in Theorem 3.2.25 and let $\Sigma_2$ be i/s/o system in Theorem 3.2.27(i). Then by Lemma 2.5.36, $\Sigma$ and $\Sigma_2$ are intertwined by the embedding operator $X_1 + Z \hookrightarrow X_2$ and by Lemma 2.5.39, $\Sigma_2$ and $\Sigma_1$ are intertwined by the projection operator $P_{X_1}^Z$. Thus by Lemma 2.5.25, $\Sigma$ and $\Sigma_1$ are intertwined by the composition of these two operators, which is the operator $P_{X_1}^Z = P_{X_1}^{Z_1}|_{X_1+Z}$.

(ii) ⇒ (i): If (ii) holds, then by Lemma 2.5.27, $X_1 + Z$ is strongly invariant for $\Sigma$, and $Z$ is unobservably invariant for $\Sigma$. By Theorem 3.2.20, $\Sigma$ has a bounded compression $\Sigma_3$ onto $X_1$ along $Z$. By comparing the conditions (iii) in Lemmas 3.2.16 and 3.2.21 to each other and using the strong invariance of $X_1 + Z$ and the unobservable invariance of $Z$ we find that $\Sigma_3 = \Sigma_1$.

3.2.29. Theorem. A bounded i/s/o system $\Sigma = ([A B]_c; X, U, Y)$ is minimal if and only if $\Sigma$ is both controllable and observable.

Proof. If $\Sigma$ is not observable, then $U_\Sigma \neq \{0\}$ where $U_\Sigma$ is the unobservable subspace of $\Sigma$. By Lemma 3.2.4(v), $U_\Sigma$ is unobservably invariant for $\Sigma$, and it follows from Theorem 3.2.12 that $\Sigma$ has a bounded projection along $U_\Sigma$ onto any direct complement to $U_\Sigma$. This projection is a nontrivial compression of $\Sigma$.

If $\Sigma$ is not controllable, then $R_\Sigma \neq X$ where $R_\Sigma$ is the reachable subspace of $\Sigma$. By Lemma 3.2.4(iv), $R_\Sigma$ is strongly invariant for $\Sigma$, and it follows from Theorem 3.2.11 that the restriction of $\Sigma$ to $R_\Sigma$ is a nontrivial compression of $\Sigma$.

Thus, minimality of $\Sigma$ implies both controllability and observability.
Conversely, suppose that $\Sigma_1 = (S_1; X_1, U, Y)$ is a compression of $\Sigma$. If $\Sigma$ is observable, then $Z_{\text{max}} = \{0\}$ where $Z_{\text{max}}$ is the subspace defined in Lemma 3.2.23 and hence by Theorem 3.2.25, $X_1$ is strongly invariant for $\Sigma$. If furthermore $\Sigma$ is controllable, then by Lemma 3.2.4(iv), $X_1 = X$. Thus, if $\Sigma$ is both controllable and observable, then $\Sigma_1 = \Sigma$, and hence $\Sigma$ does not have any nontrivial compression. □

3.2.30. Corollary. Let $\Sigma_i = (A_i C_i D_i ; X_i, U, Y)$, $i = 1, 2$, be two minimal bounded i/s/o systems (with the same input and output spaces). Then $\Sigma_1$ and $\Sigma_2$ are pseudo-similar if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent. Among all the pseudo-similarities between $\Sigma_1$ and $\Sigma_2$ there is a (unique) minimal one $P_{\text{min}}$ and a (unique) maximal one $P_{\text{max}}$, namely those defined in Theorem 3.2.18 (both of which in this case are single-valued densely defined injective operators with dense range).

Proof. This follows from Corollary 3.2.19 and Theorem 3.2.29. □

As the following theorem shows, every bounded i/s/o system can be compressed into a minimal bounded i/s/o system. This result can be used, e.g., to prove the existence of minimal bounded i/s/o realizations of a given function which is analytic at infinity.

3.2.31. Theorem. Every bounded i/s/o system $\Sigma$ has a bounded minimal compression.

Two families of such compressions are described below, where we have denoted the reachable and unobservable subspaces of $\Sigma$ by $\mathcal{R}_\Sigma$ respectively $\mathcal{U}_\Sigma$:

(i) Let $X_1$ be a direct complement to $\mathcal{U}_\Sigma$ in $X$, and let $X_0 = P^\mathcal{R}_\Sigma X_1$. Then the formula

$$
(3.2.22) \quad \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} = \begin{bmatrix} P^\mathcal{R}_\Sigma A |_{X_0} & P^\mathcal{R}_\Sigma B \\ C |_{X_0} & D \end{bmatrix}
$$

defines an operator $[A_0 \ B_0] \in \mathcal{B}([X_0 \ U] ; [X_0 \ Y])$, and $\Sigma_0 = ([A_0 \ B_0] ; X_0, U, Y)$ is a bounded minimal compression of $\Sigma$. This compression is the bounded i/s/o system that one gets by first projecting $\Sigma$ onto $X_1$ along its unobservable subspace $\mathcal{U}_\Sigma$, and then restricting the resulting system to its reachable subspace $X_0$. The main operator $A_0$ of $\Sigma_0$ is restriction in $\mathcal{B}(X_0)$ of the projection of $A$ in $\mathcal{B}(X_1)$ along $\mathcal{U}_\Sigma$.

(ii) By Lemma 3.2.4, $A \mathcal{R}_\Sigma \subset \mathcal{R}_\Sigma$ and $\text{rng} (B) \subset \mathcal{R}_\Sigma$, so that we may interpret $A|_{\mathcal{R}_\Sigma}$ and $B$ as operators with range space $\mathcal{R}_\Sigma$. Let $X_1$ be a direct complement to $\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma$ in $\mathcal{R}_\Sigma$, and define (with the above interpretation of $A|_{X_1} = (A|_{\mathcal{R}_\Sigma})|_{X_1}$ and $B$)

$$
(3.2.23) \quad \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} P^{\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma} A |_{X_1} & P^{\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma} B \\ C |_{X_1} & D \end{bmatrix}.
$$

Then $\Sigma_1 = ([A_1 \ B_1] ; X_1, U, Y)$ is a minimal bounded compression of $\Sigma$. This compression is the system that one gets by first restricting $\Sigma$ to its reachable subspace $\mathcal{R}_\Sigma$, and then projecting the resulting system onto $X_1$ along its unobservable subspace $\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma$. The main operator $A_1$ of $\Sigma_1$ is projection in $\mathcal{B}(X_1)$ along $\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma$ of the restriction of $A$ in $\mathcal{B}(\mathcal{R}_\Sigma)$.

Proof. Proof of (i): By Lemma 3.2.4(v) $\mathcal{U}_\Sigma$ is a closed unobservably invariant subspace for $\Sigma$, and by Theorem 3.2.12 $\Sigma$ has a bounded projection $\Sigma_1$ onto $X_1$. The main operator $A_0$ of $\Sigma_0$ is restriction in $\mathcal{B}(X_0)$ of the projection of $A$ in $\mathcal{B}(X_1)$ along $\mathcal{U}_\Sigma$.
along $\mathcal{U}_\Sigma$. By Lemmas 2.5.32 and 2.5.39 the reachable and unobservable subspaces $\mathcal{R}_{\Sigma_1}$ and $\mathcal{U}_{\Sigma_1}$ of $\Sigma_1$ are

$$\mathcal{X}_1 := \mathcal{R}_{\Sigma_1} = \frac{P_{\Sigma_1}^*}{\mathcal{A}_1} \mathcal{R}_{\Sigma}, \quad \mathcal{U}_{\Sigma_1} = \mathcal{X}_1 \cap \mathcal{U}_\Sigma = \{0\}.$$ 

In particular, $\Sigma_1$ is observable. By Lemma 3.2.24 (v) $\mathcal{X}_1$ is a closed strongly invariant subspace for $\Sigma_1$. By Theorem 3.2.11 $\Sigma_1$ has a bounded restriction $\Sigma_{\circ}$ to $\mathcal{X}_1$. Let $\mathcal{Z}_1$ be an arbitrary direct complement to $\mathcal{X}_1$ in $\mathcal{X}_1$. By Lemmas 2.5.32 and 2.5.36 the reachable and unobservable subspaces $\mathcal{R}_{\Sigma_{\circ}}$ and $\mathcal{U}_{\Sigma_{\circ}}$ of $\Sigma_{\circ}$ are

$$\mathcal{R}_{\Sigma_{\circ}} = \frac{P_{\Sigma_{\circ}}^*}{\Sigma_{\circ}} X_{\circ} = \mathcal{X}_0, \quad \mathcal{U}_{\Sigma_{\circ}} = \mathcal{X}_0 \cap \mathcal{U}_{\Sigma_{\circ}} = \{0\}.$$ 

Thus $\Sigma_{\circ}$ is both controllable and observable, and hence by Theorem 3.2.31, it is minimal. Moreover, it follows from Lemmas 2.5.30, 2.5.36, and 2.5.39 that $\Sigma_{\circ}$ is a compression of $\Sigma$ onto $\mathcal{X}_0$ along $\mathcal{U}_{\Sigma_{\circ}} \cap \mathcal{Z}_1$. Formula (3.2.22) follows from Theorem 3.2.25.

Note that the only part of the above construction where we are allowed to make a non-unique choice is when we fix the direct complement $\mathcal{X}_1$ to $\mathcal{U}_\Sigma$. This complement is unique if and only if $\mathcal{U}_\Sigma = \mathcal{X}$ or $\mathcal{U}_\Sigma = \{0\}$.

**Proof of (ii):** By Lemma 3.2.24 (iv) $\mathcal{R}_\Sigma$ is a closed strongly invariant subspace for $\Sigma$, and by Theorem 3.2.11 $\Sigma$ has a bounded restriction $\Sigma_{\circ}$ to $\mathcal{R}_\Sigma$. Let $\mathcal{Z}_2$ be an arbitrary direct complement to $\mathcal{R}_\Sigma$ in $\mathcal{X}$ (the final result will be independent of the choice of $\mathcal{Z}_2$). By Lemmas 2.5.32 and 2.5.36 the reachable and unobservable subspaces $\mathcal{R}_{\Sigma_1}$ and $\mathcal{U}_{\Sigma_1}$ of $\Sigma_1$ are

$$\mathcal{R}_{\Sigma_1} = \frac{P_{\Sigma_{\circ}}^*}{\Sigma_{\circ}} \mathcal{R}_\Sigma = \mathcal{R}_\Sigma, \quad \mathcal{U}_{\Sigma_1} = \mathcal{R}_\Sigma \cap \mathcal{U}_{\Sigma_1}.$$ 

In particular, $\Sigma_1$ is controllable. By Lemma 3.2.24 (v) $\mathcal{U}_{\Sigma_1}$ is a closed unobservably invariant subspace for $\Sigma_1$, and by Theorem 3.2.12 $\Sigma_1$ has a bounded projection $\Sigma_{\circ}$ to $\mathcal{X}$ along $\mathcal{U}_{\Sigma_1}$. By Lemmas 2.5.32 and 2.5.39 the reachable and unobservable subspaces $\mathcal{R}_{\Sigma_{\circ}}$ and $\mathcal{U}_{\Sigma_{\circ}}$ of $\Sigma_{\circ}$ are

$$\mathcal{R}_{\Sigma_{\circ}} = \frac{P_{\Sigma_{\circ}}^*}{\Sigma_{\circ}} \mathcal{R}_{\Sigma_{\circ}} = \mathcal{X}, \quad \mathcal{U}_{\Sigma_{\circ}} = \mathcal{X} \cap \mathcal{U}_{\Sigma_{\circ}} = \{0\}.$$ 

Thus $\Sigma_{\circ}$ is both controllable and observable, and hence by Theorem 3.2.31 it is minimal. Moreover, it follows from Lemmas 2.5.30, 2.5.36, and 2.5.39 that $\Sigma_{\circ}$ is a compression of $\Sigma$ onto $\mathcal{X}_{\circ}$ along $\mathcal{U}_{\Sigma_{\circ}} \cap \mathcal{Z}_2$. Formula (3.2.23) follows from Theorem 3.2.25.

Note that the above construction does not depend on the (arbitrary) choice of the direct complement $\mathcal{Z}_2$ to $\mathcal{U}_\Sigma$ in $\mathcal{X}$, but it does depend on the choice of the direct complement $\mathcal{Z}_1$ to $\mathcal{U}_\Sigma$ in $\mathcal{R}_{\Sigma_1}$. This complement is unique if and only if either $\mathcal{R}_\Sigma \subset \mathcal{U}_\Sigma$ (in which case $\mathcal{X}_0 = \{0\}$) or $\mathcal{U}_\Sigma \cap \mathcal{R}_\Sigma = \{0\}$ (in which case $\mathcal{X}_{\circ} = \mathcal{R}_\Sigma$).

**3.2.32. Lemma.** The minimal bounded compression of a bounded i/s/o system $\Sigma = ([A,B,D];\mathcal{X},\mathcal{U},\mathcal{Y})$ is unique if and only at least one of conditions (i) and (ii) below holds:

(i) $\Sigma$ is observable, i.e., $\mathcal{U}_\Sigma = \{0\}$, where $\mathcal{U}_\Sigma$ is the unobservable subspace of $\Sigma$,

(ii) the following equivalent conditions hold:

(a) $\Sigma$ has a compression with state space $\{0\}$,
(b) the i/o resolvent of $\Sigma$ is a constant in $\rho_\Sigma(\Sigma)$,
(c) $\mathcal{R}_\Sigma \subset \mathcal{U}_\Sigma$, where $\mathcal{R}_\Sigma$ is the reachable subspace of $\Sigma$. 

In case (i) the unique minimal compression \( \Sigma_{\text{min}} \) is the restriction of \( \Sigma \) to \( \mathcal{R}_\Sigma \), i.e., \( \Sigma_{\text{min}} = (S_{\text{min}}; \mathcal{R}_\Sigma, \mathcal{U}, \mathcal{V}) \) where

\[
S_{\text{min}} = \begin{bmatrix} A_{\text{min}} & B_{\text{min}} \\ C_{\text{min}} & D_{\text{min}} \end{bmatrix} = \begin{bmatrix} A_{|\mathcal{R}_\Sigma} & B \\ C_{|\mathcal{R}_\Sigma} & D \end{bmatrix}.
\]

In case (ii) the unique minimal compression is \( \Sigma_{\text{min}} = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \{0\}, \mathcal{U}, \mathcal{V}) \). If neither (i) nor (ii) holds, then \( \Sigma \) has an infinite number of minimal compressions.

**Proof.** Suppose first that \( \Sigma \) is observable, and let \( \Sigma_1 = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}_1, \mathcal{U}, \mathcal{V}) \) be a bounded minimal compression of \( \Sigma \). Since \( \Sigma \) is observable, it follows from Lemmas 3.2.4(v) and 3.2.23 and Theorem 3.2.25 that the two subspaces \( Z_{\text{min}} \) and \( Z_{\text{max}} \) in Lemma 3.2.29 satisfy \( Z_{\text{min}} = Z_{\text{max}} = 0 \). Consequently \( \mathcal{X}_1 \) is a strongly invariant subspace for \( \Sigma \), and by Lemma 3.2.4(iv), \( \mathcal{R}_\Sigma \subset \mathcal{X}_1 \). This implies that \( \mathcal{R}_\Sigma \) is also the reachable subspace of \( \Sigma_1 \). Since \( \mathcal{X}_1 \) is minimal, it follows that \( \mathcal{X}_1 = \mathcal{R}_\Sigma \), and that \( \Sigma_1 \) is the (unique) bounded restriction of \( \Sigma \) to \( \mathcal{R}_\Sigma \).

Next we show that conditions (a)–(c) in (ii) are equivalent.

(a) \( \Rightarrow \) (b): Suppose that \( \Sigma_1 \) is a compression of \( \Sigma \) with state space \( \{0\} \). Then the i/o resolvent map of \( \Sigma_1 \) is a constant, and since \( \Sigma \) and \( \Sigma_1 \) are externally equivalent, it follows from Theorem 3.2.27 that this constant is equal to \( D \). This implies that the system operator of \( \Sigma_1 \) is \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). In particular, \( \Sigma_1 \) is determined uniquely by \( \Sigma \).

(b) \( \Rightarrow \) (c): Suppose that (b) holds. Then by 3.2.9, the i/o resolvent \( \mathcal{D} \) of \( \Sigma \) satisfies \( \mathcal{D}(\lambda) = D \) and \( C(\lambda - A)^{-1}B = 0 \) for all \( \lambda \in \rho_\infty(A) \). This together with the resolvent identity (3.1.5) implies that \( C(\mu - A)^{-1}(\lambda - A)^{-1}B = 0 \) for all \( \mu, \lambda \in \rho_\infty(A) \), and hence by Lemma 3.2.4(iv), \( \mathcal{R}_\Sigma \subset \mathcal{X}_1 \).

(c) \( \Rightarrow \) (a): If (c) holds, then the two realizations presented in Theorem 3.2.31 coincide, and their common state space is \( \{0\} \).

We have now showed that (a)–(c) are equivalent, and that the minimal bounded compression of \( \Sigma \) is unique if (i) holds. That it is also unique if (ii) holds can be seen as follows. If \( \Sigma \) has a compression with state space \( \{0\} \), then it follows from Corollary 3.2.19 and Theorem 3.2.29 that every minimal compression of \( \Sigma \) has state space zero. As we saw in the proof of the implication (a) \( \Rightarrow \) (b), the compression with state space \( \{0\} \) is unique.

It remains to show that if neither (i) nor (ii) holds, then \( \Sigma \) has infinitely many minimal compressions. Suppose that \( \{0\} \neq \mathcal{U}_\Sigma \neq \mathcal{X} \) (if \( \mathcal{U}_\Sigma = \{0\} \), then (i) holds, and if \( \mathcal{U}_\Sigma = \mathcal{X} \) then (ii) holds). Let \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) be two direct complements to \( \mathcal{U}_\Sigma \) satisfying \( \mathcal{X}_1 \cap \mathcal{X}_2 = \{0\} \). We claim that then the two minimal compressions constructed in part (i) of Theorem 3.2.31 corresponding to these different choices of direct complement to \( \mathcal{U}_\Sigma \) differ from each other unless \( \Sigma \) has a minimal compression with state space \( \{0\} \). This can be seen as follows. If these two compressions coincide, then their common state space \( \mathcal{X}_c \) must be contained in both \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), and hence \( \mathcal{X}_c = \{0\} \). This proves the above claim. Since \( \mathcal{U}_\Sigma \) has an infinite number of direct complements whose mutual intersections are zero, this implies that there exists an infinite number of minimal compressions of \( \Sigma \) if neither (i) nor (ii) holds.

\( \square \)
3.3. Discrete Time Input/State/Output Systems (Jan 02, 2016)

This book is primarily devoted to the theory of i/s/o and s/s systems in continuous time, but there also exists an analogous theory for i/s/o systems in discrete time. Up to now we have interpreted a bounded i/s/o node \( \Sigma = (A, B, C, D; X, U, Y) \) as the generator of a continuous time system, but the same node also generates the discrete time system

\[
\Sigma: \begin{cases} 
  x(n + 1) = Ax(n) + Bu(n), \\
  y(n) = Cx(n) + Du(n), 
\end{cases} \quad n \in I,
\]

where \( I \) is some discrete time interval. Here \( u \) and \( y \) are sequences defined on \( I \) with values in \( U \) respectively \( Y \), and \( x \) is a sequence with values in \( X \) which is defined on \( I \) if \( I \) is unbounded to the right, and on the interval \( I \cup \{ \ell + 1 \} \) if \( I \) has a finite end-point \( \ell \).

3.3.1. Introduction to discrete time i/s/o systems.

3.3.1. Notation. Let \( I \) be a discrete time interval. By the notation \( I_{\text{ext}} \) we mean the interval \( I_{\text{ext}} = I \) if \( I \) is unbounded to the right, and \( I_{\text{ext}} = I \cup \{ \ell + 1 \} \) if \( I \) has a finite end-point \( \ell \).

Thus, for example, if \( I = \mathbb{Z}^+ \) or \( I = \mathbb{Z} \), then \( I_{\text{ext}} = I \), but if \( I = \mathbb{Z}^- \) then \( I_{\text{ext}} = \mathbb{Z}^- \cap \{ 0 \} \).

3.3.2. Definition. Let \( \Sigma = (A, B, C, D; X, U, Y) \) be a bounded i/s/o node.

(i) By a discrete time trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) generated by \( \Sigma \) on the discrete time interval \( I \) we mean a triple of sequences \( x = \{ x(n) \}_{n \in I_{\text{ext}}} \), \( u = \{ u(n) \}_{n \in I_{\text{ext}}} \), and \( y = \{ y(n) \}_{n \in I} \) with values in \( X, U, \) and \( Y \), respectively, which satisfy (3.3.1).

(ii) By the (bounded) discrete time i/s/o (input/state/output) system induced by \( \Sigma \) we mean the node \( \Sigma \) itself together with sets of all discrete time trajectories generated by \( \Sigma \). We use the same notation \( \Sigma = (A, B, C, D; X, U, Y) \) for the discrete time i/s/o system as for bounded i/s/o node defining this system.

3.3.3. Definition. By a future, past, or two-sided trajectory of a discrete time i/s/o system \( \Sigma = (A, B, C, D; X, U, Y) \) we mean a trajectory defined on \( \mathbb{Z}^+, \mathbb{Z}^- \), or \( \mathbb{Z} \), respectively.

Note that in discrete time there is no need to distinguish between “classical” and “generalized” trajectories: all trajectories are automatically “classical” in the sense that they all satisfy (3.3.1).

3.3.4. Lemma. Let \( \Sigma = (A, B, C, D; X, U, Y) \) be a discrete time i/s/o system, and let \( I \) be an arbitrary discrete time interval with finite left end-point \( m \). Then for each \( x^0 \in X \) and each \( U \)-valued sequence \( \{ u(n) \}_{n \in I} \) \( \Sigma \) has a unique trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) (with input sequence \( u \)) on the interval \( I \) satisfying \( x(m) = x_0 \). The state and
output components $x$ and $y$ of this trajectory are given by

\[
x(n) = A^{n-m}x^0 + \sum_{k=m}^{n-1} A^{n-k}Bu(k), \quad n \in \mathbb{I}_{\text{ext}},
\]

\[
y(n) = CA^{n-m}x^0 + \sum_{k=m}^{n-1} CA^{n-k-1}Bu(k) + Du(n), \quad n \in \mathbb{I},
\]

where we interpret $\sum_{k=m}^{m-1} = 0$.

**Proof.** By solving the equation (3.3.1) recursively we get (3.3.2). $\square$

If $I = \mathbb{Z}^+$ and $x^0 = 0$, then the second formula in (3.3.2) simplifies into

\[
y(n) = \sum_{k=0}^{n} D_{n-k}u(k), \quad n \in \mathbb{Z}^+, \tag{3.3.3}
\]

where

\[
D_k := \begin{cases} D, & k = 0, \\ CA^{k-1}B, & k \geq 1. \end{cases} \tag{3.3.4}
\]

**3.3.5. Definition.** Let $\Sigma = ([\begin{array}{cc} A & B \\ C & D \end{array}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a discrete i/s/o time system. The map from the (input) sequence $\{u(n)\}_{n \in \mathbb{Z}^+}$ to the (output) sequence $\{y(n)\}_{n \in \mathbb{Z}^+}$ defined by (3.3.3)–(3.3.4) is called the (discrete time future) i/o map (input/output map) of $\Sigma$.

**3.3.6. Remark.** In this book we shall only discuss discrete time i/s/o systems which are bounded, systems induced by bounded i/s/o nodes. This condition can be interpreted as a natural “well-posedness” condition in discrete time, since it follows from Lemma 3.3.4 that it is equivalent to the condition that a discrete time trajectory $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ on any finite discrete time interval of the type $[0, m]$ is determined uniquely by and depends continuously on the initial state $x(0)$ and the input sequence $u$. In particular, every discrete time system is “uniquely solvable” and has the “continuation property” in the sense of the (forward) discrete time version of Definition 2.1.17.

**3.3.7. Properties of discrete time i/s/o systems.**

**3.3.8. Definition.** Let $\Sigma = ([\begin{array}{cc} A & B \\ C & D \end{array}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a discrete time system. Then the following discrete time notions are defined analogously to the corresponding continuous time notions (in the forward time direction):

(i) $(P, R, Q)$-similarity, $P$-similarity, and similarity of discrete time i/s/o systems (see Definition 2.3.11);
(ii) the $(P, Q, R)$-image of a discrete time i/s/o system (see Definition 2.3.16);
(iii) the part and static projection of a discrete time i/s/o system (see Definition 2.3.20);
(iv) static output feedback and dynamic feedback for discrete time i/s/o systems (see Definitions 2.3.23 and 2.3.47);
(v) bounded i/o extensions of a discrete time i/s/o system (see Definition 2.3.27);
(vi) the cross product and the parallel, difference, and cascade connections of two discrete time i/s/o systems (see Examples 2.3.33, 2.3.39 and 2.3.44);
(vii) exactly reachable states, reachable states, and unobservable trajectories and states of a discrete time i/s/o system (see Definition 2.5.1);  
(viii) the exactly reachable subspace, the reachable subspace, and the unobservable subspace of a discrete time i/s/o system (see Definition 2.5.3);  
(ix) strongly and unobservable invariance subspaces of a discrete time i/s/o system (see Definition 2.5.8);  
(x) controllability and observability of a discrete time i/s/o system (see Definition 2.5.11);  
(xi) external equivalence of two discrete time systems (see Definition 2.5.21);  
(xii) intertwinements and pseudo-similarity of two discrete time systems (see Definitions 2.5.22 and 2.5.23);  
(xiii) compressions, dilations, restrictions, and projections of discrete time systems (see Definitions 2.5.28, 2.5.33, and 2.5.37);  
(xiv) minimality of discrete time i/s/o systems (see Definition 2.5.41).

As the following lemma says, most of the results that we have proved for general (bounded or non-bounded) i/s/o systems in continuous time remain valid for i/s/o systems in discrete time in the forward time direction. (See the next section for a discussion of how the forward and backward time directions differ from each other in the discrete time case.)

3.3.8. Lemma. The analogues of the following results about i/s/o systems remain valid for discrete time i/s/o systems:

(i) Lemmas 2.3.13, 2.3.19, 2.3.25, 2.3.30, 2.3.31, 2.3.34, 2.3.41, and 2.3.45 (about trajectories of similarity transformed systems, images, static output and dynamic feedbacks, i/o extended systems, cross products, and parallel, difference, and cascade connections);

(ii) Lemmas 2.5.4, 2.5.9, 2.5.10, and Corollary 2.5.12 (about the reachable and unobservable subspaces);

(iii) Lemma 2.5.24, 2.5.26, 2.5.27, 2.5.29, 2.5.30, 2.5.32, 2.5.36, and 2.5.39 (about intertwinements, pseudo-similarity, compressions, restrictions, and (dynamic) projections);

(iv) Lemma 2.5.44, 2.5.45, 2.5.46, 2.5.47, and 2.5.48 (about results which are true for i/s/o systems which have the continuation property).

3.3.3. Time reflection of discrete time i/s/o systems. There is one particular notion which is not mentioned in Definition 3.3.7 above, namely the notion of the time reflection of a discrete time i/s/o system. The reason for this omission is that time reflection is a more complicated operation in discrete time than in continuous time, due to the fact that we (in this book) throughout require the time reflection of a bounded system to be bounded.

The idea behind a time reflection (in discrete or continuous time) is to create a new system with the property that to each past trajectory of the original system corresponds a future trajectory of the time-reflected system and vice versa. In the case of the continuous time system (2.1.12) we simply replace the interval I by the interval \(-I = \{-t \mid t \in I\}\), and map a (classical or generalized) trajectory \([x(t) \ y(t)]\) on the interval I onto the time reflected trajectory \([x(t) \ y(t)]\) on \(-I\), where

\[
\begin{bmatrix}
  x \\
  u \\
  y \\
  f(t)
\end{bmatrix}
= \begin{bmatrix}
  x(t) \\
  u(t) \\
  y(t) \\
  f(t)
\end{bmatrix},
\]
t \in I. The new system is still induced by a bounded i/s/o node,
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The discrete time case is more complicated. In particular, a discrete time reflection should map the past time interval $\mathbb{Z}^-$ into the future time interval $\mathbb{Z}^+$ and vice versa, and a past time trajectory $[\begin{pmatrix} x \\ y \end{pmatrix}]$ of the discrete time system should be mapped into the time reflected trajectory

$$
(3.3.5)\quad \begin{bmatrix} x(n) \\ u(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} x(-m) \\ u(-(m+1)) \\ y(-(m+1)) \end{bmatrix}
$$

of the new time reflected system $\Sigma \mathcal{R}$. The same transformation can be used to define an arbitrary discrete time interval $I$ into the (discrete) reflected time interval $I \mathcal{R} = \{- (n+1) \mid n \in I \}$. If (and only if) $A$ has a bounded inverse, then it is possible to rewrite (3.3.1) in the form

$$
(3.3.6)\quad \Sigma: \quad \begin{cases}
    x(n) = A^{-1}x(n+1) - A^{-1}Bu(n), \\
y(n) = CA^{-1}x(n+1) + (D - CA^{-1}B)u(n),
\end{cases} \quad n \in I.
$$

After replacing $\begin{bmatrix} x \\ u \end{bmatrix}$ by $\begin{bmatrix} x \mathcal{R} \\ u \mathcal{R} \end{bmatrix}$ and $I$ we get the equivalent equation

$$
(3.3.7)\quad \Sigma: \quad \begin{cases}
    x(n+1) = A^{-1}x(n) - A^{-1}Bu(n), \\
y(n+1) = CA^{-1}x(n) + (D - CA^{-1}B)u(n),
\end{cases} \quad m \in I \mathcal{R}.
$$

3.3.9. Definition. Let $\Sigma = ([A \ B] \colon \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a discrete time system.

(i) $\Sigma$ is time-invertible if $A$ has a bounded inverse.

(ii) If $\Sigma$ is time-invertible, then by the time reflection of $\Sigma$ we mean the discrete time system $\Sigma \mathcal{R} = ([A \mathcal{R} \ B \mathcal{R}] \colon \mathcal{X}, \mathcal{U}, \mathcal{Y})$ where

$$
(3.3.8)\quad \begin{bmatrix} A \mathcal{R} & B \mathcal{R} \\ C \mathcal{R} & D \mathcal{R} \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{bmatrix}.
$$

There is an analogue of Lemma 3.3.4 in the backard time direction which is valid for time-invertible i/s/o systems in discrete time.

3.3.10. Lemma. Let $\Sigma = ([A \ B] \colon \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a time-invertible discrete time system. Let $I$ is an arbitrary discrete time interval with finite right end-point $m$. Then for each $x^0 \in \mathcal{X}$ and each $\mathcal{U}$-valued sequence $\{u(n)\}_{n \in I}$ $\Sigma$ has a unique trajectory $[\begin{pmatrix} x \\ y \end{pmatrix}]$ (with input sequence $u$) satisfying $x(m + 1) = x_0$. The state and output components $x$ and $y$ of this trajectory are given by

$$
(3.3.9)\quad \begin{align*}
x(n) &= A^{n-k}x^0 - \sum_{k=n}^\ell A^{n-k}Bu(k), & n \in I_{\text{ext}}, \\
y(n) &= CA^{n-k}x^0 + \sum_{k=m}^\ell CA^{n-k}Bu(k) + Du(n), & n \in I,
\end{align*}
$$

where we interpret $\sum_{k=\ell+1}^\ell = 0$.

Proof. If $A$ has a bounded inverse, then we get (3.3.9) by solving (3.3.6) recursively (backward in time).
3.3.4. Connections between continuous and discrete time i/s/o properties. Earlier in this chapter we have proved a number of results which are true for bounded i/s/o systems in continuous time. It turns out that most of these remain valid for i/s/o systems in discrete time in a somewhat unexpected sense: Most of the properties that we have discussed in this chapter are of the following type:

3.3.11. Principle. If \( \Sigma = ([A B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is a bounded i/s/o node, then the continuous time system generated by \( \Sigma \) has a certain property if and only if the discrete time system generated by \( \Sigma \) has the same property (in the forward time direction).

Below we shall list several instances where this principle is valid.

3.3.12. Lemma. Let \( \Sigma = ([A B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) and \( \Sigma_i = ([A_i B_i]; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i) \) be bounded i/s/o nodes. Then the following claims are true: (in the list below, when we say “in continuous time” or “in discrete time” we refer to the continuous respectively discrete time system i/s/o generated by \( \Sigma \), and we only include the forward time direction).

(i) A closed subspace \( Z \) is strongly or unobservably invariant in continuous time if and only if \( Z \) is strongly respectively unobservably invariant in discrete time.

(ii) The continuous time reachable and unobservable subspaces of \( \Sigma \) coincide with the discrete time reachable respectively unobservable subspaces of \( \Sigma \).

(iii) \( \Sigma \) is controllable or observable in continuous time if and only if \( \Sigma \) is controllable or observable in discrete time.

(iv) \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent in continuous time if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent in discrete time.

(v) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some closed \( P \in \mathcal{M}(X_1; X_2) \) in continuous time if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \) in discrete time.

(vi) \( \Sigma_1 \) is a continuous time restriction, or projection, or compression of \( \Sigma_2 \) if and only if \( \Sigma_1 \) is a discrete time restriction, or projection, or compression, respectively, of \( \Sigma_2 \).

(vii) The continuous time versions of the subspaces \( X_{\min}, Z_{\min}, \) and \( Z_{\max} \) in Lemma 3.2.23 coincide with the corresponding discrete time versions of these subspaces.

(viii) Discrete time compressions of \( \Sigma \) have the same structure as the continuous time compressions of \( \Sigma \) described in Theorem 3.2.25.

(ix) \( \Sigma \) is minimal in continuous time if and only if \( \Sigma \) is minimal in discrete time.

(x) \( \Sigma \) is minimal in discrete time if and only if \( \Sigma \) is both controllable and observable in discrete time.

(xi) The families of continuous time minimal compressions of \( \Sigma \) described in Theorem 3.2.31 can also be interpreted as discrete time minimal compressions of \( \Sigma \).

(xii) A minimal compression of \( \Sigma \) is unique in continuous time if and only if it is unique in discrete time.

PROOF. The proofs of these claims are straightforward. In addition to the representation formulas (2.1.19) and (3.3.2) for trajectories in continuous respectively discrete time they use the following results from this chapter:

(i) See Lemmas 3.2.2 and 3.2.3.
(ii) See Lemma 3.2.4.
(iii) This follows from (ii).
(iv) See Theorem 3.2.7.
(v) See Lemma 3.2.16.
(vi) See Theorems 3.2.11 and 3.2.12.
(vii) See Lemma 3.2.24.
(viii)–(xii) These claims follow from (i)–(vii).

3.3.13. Remark. The above lemma applies only in the forward time direction, i.e., the results listed in that lemma are not in general true in the backward time direction without any additional assumptions. One such additional assumption is that \( \Sigma \) is time-invertible, and that spectrum of \( \Sigma \) does not separate zero from the point at infinity, i.e., that \( 0 \in \rho_\infty(\Sigma) \).

The results presented in Lemma 3.3.12 make it possible to transfer the dynamical notions that we have defined separately for the continuous time and the discrete time i/s/o systems generated by a bounded i/s/o into a notion which applies to the node itself.

3.3.14. Definition. Let \( \Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} ; \mathcal{X}, \mathcal{U}, \mathcal{Y} \) and \( \Sigma_i = \left( \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} ; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i \right) \) be bounded i/s/o nodes.

(i) A closed subspace \( \mathcal{Z} \) of \( \mathcal{X} \) is strongly invariant for \( \Sigma \) if \( A \mathcal{Z} \subset \mathcal{Z} \) and \( \text{rng} (B) \subset \mathcal{Z} \), i.e., \( \mathcal{Z} \) is strongly invariant for the continuous time i/s/o system induced by \( \Sigma \), or equivalently, \( \mathcal{Z} \) is strongly invariant for the discrete time i/s/o system induced by \( \Sigma \) (see Lemma 3.2.2(iv)–(ix) for additional equivalent conditions).

(ii) A closed subspace \( \mathcal{Z} \) of \( \mathcal{X} \) is unobservably invariant for \( \Sigma \) if \( A \mathcal{Z} \subset \mathcal{Z} \) and \( \mathcal{Z} \subset \ker (C) \), i.e., \( \mathcal{Z} \) is unobservably invariant for the continuous time i/s/o system induced by \( \Sigma \), or equivalently, \( \mathcal{Z} \) is unobservably invariant for the discrete time i/s/o system induced by \( \Sigma \) (see Lemma 3.2.3(iii)–(viii) for additional equivalent conditions).

(iii) The subspace \( \mathcal{R}_{\Sigma} \) defined in (3.2.4) is called the reachable subspace of \( \Sigma \) (this is the common reachable subspace of both the continuous time and the discrete time i/s/o system induced by \( \Sigma \)).

(iv) \( \Sigma \) is controllable if the continuous time i/s/o system induced by \( \Sigma \) is controllable, or equivalently, if the discrete time i/s/o system induced by \( \Sigma \) is controllable (i.e., the common reachable subspace of the continuous time and the discrete time system induced by \( \Sigma \) is the full state space).

(v) The subspace \( \mathcal{U}_{\Sigma} \) defined in (3.2.5) is called the unobservable subspace of \( \Sigma \) (this is the common unobservable subspace of both the continuous time and the discrete time i/s/o system induced by \( \Sigma \)).

(vi) \( \Sigma \) is observable if the continuous time i/s/o system induced by \( \Sigma \) is observable, or equivalently, if the discrete time i/s/o system induced by \( \Sigma \) is observable (i.e., the common unobservable subspace of the continuous time and the discrete time system induced by \( \Sigma \) is \( \{0\} \)).

(vii) \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent if the two continuous time i/s/o systems induced by \( \Sigma_i, i = 1, 2 \), are externally equivalent, or equivalently, if the two continuous time i/s/o systems induced by \( \Sigma_i, i = 1, 2 \), are externally equivalent (see Theorem 3.2.7(iii)–(v) for additional equivalent conditions).
(viii) $\Sigma_1$ and $\Sigma_2$ are intertwined by a closed $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ if the two continuous time i/s/o systems induced by $\Sigma_i$, $i = 1, 2$, are intertwined by $P$, or equivalently, if the two continuous time i/s/o systems induced by $\Sigma_i$, $i = 1, 2$, are intertwined by $P$ (see Lemma 3.2.16(iii)-(v) for additional equivalent conditions).

(ix) $\Sigma_1$ is the restriction to $\mathcal{X}_1$, or projection or compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ of $\Sigma_2$ if the continuous time i/s/o system induced by $\Sigma_1$ is the restriction to $\mathcal{X}_1$, or projection or compression onto $\mathcal{X}_1$, respectively, along $\mathcal{Z}_1$ of the continuous time i/s/o system induced by $\Sigma_2$, or equivalently, if the continuous time i/s/o system induced by $\Sigma_1$ is the restriction to $\mathcal{X}_1$, or projection or compression onto $\mathcal{X}_1$, respectively, along $\mathcal{Z}_1$ of the continuous time i/s/o system induced by $\Sigma_2$ (see Theorems 3.2.11 and 3.2.12 and Lemma 3.2.21 for additional equivalent conditions).

(x) $\Sigma$ is minimal if the continuous time i/s/o system induced by $\Sigma$ is minimal, or equivalently, if the discrete time i/s/o system induced by $\Sigma$ is minimal.
3.4. Intertwinements and Compressions of Bounded S/S Systems (Jan 02, 2016)

In this section we study bounded s/s systems in the same spirit as in the previous section on bounded i/s/o system. Many of the results are natural extensions of the corresponding i/s/o results, but some of the natural s/s formulations are less obvious. In some cases we are able to prove results directly for s/s system by using arguments analogous to those that we gave for i/s/o system, and in other cases the s/s results are reduced to the corresponding i/s/o result by use of i/s/o bounded i/s/o representations of the bounded s/s system. Recall that by Theorem 2.2.27 a s/s system is bounded if and only if it has a bounded i/s/o representation.

3.4.1. The i/s/o-bounded resolvent set of a bounded s/s system. At this point the reader may want to recall the definitions of the characteristic node, signal/state, and signal bundles of a s/s system that were introduced in Section 1.6.

3.4.1. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a bounded s/s node, and let \( \Sigma_{i/o} = ([\mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D}]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o representation of \( \Sigma \).

(i) For each \( \lambda \in \rho(A) \) the fiber \( \hat{\mathcal{E}}(\lambda) \) of the characteristic node bundle \( \hat{\mathcal{E}} \) of \( \Sigma \) has the equivalent representations

\[
\hat{\mathcal{E}}(\lambda) = \left\{ \begin{bmatrix} x^0 \\ x_\lambda \\ w_\lambda \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \mid x_\lambda = (\lambda - A)^{-1}(x^0 + BP^\mathcal{Y}_\mathcal{U} w_\lambda) \right\},
\]

\[
\hat{\mathcal{E}}(\lambda) = \text{rng} \left[ \begin{bmatrix} 1_\mathcal{X} \\ 0 \\ (\lambda - A)^{-1}B \end{bmatrix} \begin{bmatrix} 0 \\ I_{\mathcal{Y}}C(\lambda - A)^{-1} \\ I_{\mathcal{Y}}(C(\lambda - A)^{-1}B + D) + I_{\mathcal{U}} \end{bmatrix} \right],
\]

\[
\hat{\mathcal{E}}(\lambda) = \text{ker} \left[ \begin{bmatrix} 1_\mathcal{X} \\ 0 \\ (\lambda - A)^{-1}B \end{bmatrix} \begin{bmatrix} 0 \\ I_{\mathcal{Y}}C(\lambda - A)^{-1} \\ I_{\mathcal{Y}}(C(\lambda - A)^{-1}B + D) + I_{\mathcal{U}} \end{bmatrix} \right].
\]

(ii) For each \( \lambda \in \rho(A) \) the fiber \( \hat{\mathcal{G}}(\lambda) \) of the characteristic signal/state bundle \( \hat{\mathcal{G}} \) of \( \Sigma \) has the equivalent representations

\[
\hat{\mathcal{G}}(\lambda) = \left\{ \begin{bmatrix} x_\lambda \\ w_\lambda \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \mid x_\lambda = (\lambda - A)^{-1}BP^\mathcal{Y}_\mathcal{U} w_\lambda \right\},
\]

\[
\hat{\mathcal{G}}(\lambda) = \text{rng} \left[ \begin{bmatrix} (\lambda - A)^{-1}B \\ (\lambda - A)^{-1}BP^\mathcal{Y}_\mathcal{U} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{\mathcal{Y}}(C(\lambda - A)^{-1}B + D) + I_{\mathcal{U}} \end{bmatrix} \right],
\]

\[
\hat{\mathcal{G}}(\lambda) = \text{ker} \left[ \begin{bmatrix} (\lambda - A)^{-1}B \\ (\lambda - A)^{-1}BP^\mathcal{Y}_\mathcal{U} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{\mathcal{Y}}(C(\lambda - A)^{-1}B + D) + I_{\mathcal{U}} \end{bmatrix} \right].
\]

(iii) For each \( \lambda \in \rho(A) \) the fiber \( \hat{\mathcal{H}}(\lambda) \) of the characteristic signal bundle \( \hat{\mathcal{H}} \) of \( \Sigma \) has the equivalent representations

\[
\hat{\mathcal{H}}(\lambda) = \{ w \in \mathcal{W} \mid P^\mathcal{Y}_\mathcal{U} w = (C(\lambda - A)^{-1}B + D)P^\mathcal{Y}_\mathcal{U} w \},
\]

\[
\hat{\mathcal{H}}(\lambda) = \text{rng} \left[ I_{\mathcal{Y}}(C(\lambda - A)^{-1}B + D) + I_{\mathcal{U}} \right],
\]

\[
\hat{\mathcal{H}}(\lambda) = \text{ker} \left( (C(\lambda - A)^{-1}B + D)P^\mathcal{Y}_\mathcal{U} - P^\mathcal{Y}_\mathcal{U} \right).
\]

Proof. Proof of (i): It is easy to see that the three different versions of (3.4.1) are equivalent to each other, so it suffices to prove, e.g., (3.4.1a).
By Definition 1.6.1 \( \begin{bmatrix} x_0^w \ \\ w_\lambda \end{bmatrix} \in \tilde{E}(\lambda) \) if and only if \( \begin{bmatrix} \lambda x_\lambda - x_0^w \\ w_\lambda \end{bmatrix} \in V \). By 2.2.33a, this is equivalent to the two conditions \( \lambda x_\lambda - x_0^w = Ax_\lambda + Bu_\lambda \) and \( y_\lambda = Cx_\lambda + Du_\lambda \), where \( u_\lambda = P_\lambda^w w_\lambda \) and \( y_\lambda = q_\lambda^w w_\lambda \). The former of these two equations is equivalent to the equation \( x_\lambda = (\lambda - A)^{-1}(x_0^w + Bu_\lambda) \). This is equivalent to (3.4.1a).

**Proof of (i):** This proof is analogous to the proof of (i) given above. \( \Box \)

3.4.2. **Definition.** Let \((V; \mathcal{X}, \mathcal{W})\) be a bounded s/s node.

(i) The \(i/s/o\)-bounded resolvent set of \( \Sigma \) is the union of the resolvent sets of the main operators of all bounded \(i/s/o\) representations of \( \Sigma \). This set is denoted by \( \rho^{bnd}(\Sigma) \).

(ii) The unbounded component of \( \rho^{bnd}(\Sigma) \) is denoted by \( \rho^{bnd}_\infty(\Sigma) \).

Note that by Theorem 2.2.27 and Lemma 3.1.2 that the \(i/s/o\)-bounded resolvent set of \( \Sigma \) is analytic in \( \rho^{bnd}(\Sigma) \), and it is also analytic at infinity.

3.4.3. **Lemma.** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a bounded s/s node, let \( \mathcal{X}_1 \) be a closed subspace of \( \mathcal{X} \), and let \( E \in B(\mathcal{X}; \mathcal{Z}) \) for some H-space \( \mathcal{Z} \).

(i) The characteristic node bundle \( \hat{\mathcal{E}} \) of \( \Sigma \) is analytic in the full complex plane, and it is also analytic at infinity.

(ii) The bundle \( \mathcal{G}_E \) in \( \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{Z} \\ \mathcal{W} \end{bmatrix} \) whose fibers are given by

\[
\mathcal{G}_E(\lambda) := \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left( \hat{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{Z} \end{bmatrix} \right), \quad \lambda \in \mathbb{C},
\]

is analytic in \( \rho^{bnd}(\Sigma) \), and it is also analytic at infinity.

(iii) The characteristic signal/state bundle \( \hat{\mathcal{G}} \) of \( \Sigma \) is analytic in \( \rho^{bnd}(\Sigma) \), and it is also analytic at infinity.

(iv) The characteristic signal bundle \( \hat{\mathcal{S}} \) of \( \Sigma \) is analytic in \( \rho^{bnd}(\Sigma) \), and it is also analytic at infinity.

(v) \( \hat{\mathcal{S}}(\infty) = \mathcal{W}_0 \), where \( \mathcal{W}_0 \) is the canonical input space of \( \Sigma \) (see Definition 2.2.30), and the values of the other bundles at infinity can be obtained from \( \hat{\mathcal{S}}(\infty) \) as follows:

\[
\hat{\mathcal{E}}(\infty) = \begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix}, \quad \mathcal{G}_E(\infty) = \begin{bmatrix} \mathcal{X}_1 \\ \{0\} \end{bmatrix}, \quad \hat{\mathcal{S}}(\infty) = \begin{bmatrix} \{0\} \\ \hat{\mathcal{S}}(\infty) \end{bmatrix}.
\]

**Proof.** **Proof of (i):** According to Lemma 1.6.4 the characteristic signal bundle \( \hat{\mathcal{E}} \) is analytic in the full complex plane. Analyticity of \( \hat{\mathcal{E}} \) at infinity is proved in the same way as the analyticity of \( \mathcal{G}_E \) at infinity with \( \mathcal{X}_1 = \mathcal{X} \) and \( E = 1\mathcal{X} \) (see the proof of (ii) below).

**Proof of (ii):** Let \( \lambda \in \rho^{bnd}(\Sigma) \), and let \( \Sigma_{i/s/o} = \{ [A \ be] \colon \mathcal{X}, \mathcal{U}, \mathcal{Y} \} \) be a bounded \(i/s/o\) representation of \( \Sigma \). Then it follows from the representation (3.4.1) of \( \hat{\mathcal{E}}(\lambda) \) that \( \mathcal{G}_E(\lambda) \) has the representation

\[
\mathcal{G}_E(\lambda) = \text{rng} \left( \begin{bmatrix} 1 \mathcal{X}_1 & 0 \\ E(\lambda - A)^{-1} | \mathcal{X}_1 & E(\lambda - A)^{-1} B \\ I_{\mathcal{X}} C(\lambda - A)^{-1} | \mathcal{X}_1 & I_{\mathcal{X}}(C(\lambda - A)^{-1} B + D) + I_{\mathcal{U}} \end{bmatrix} \right).
\]

Thus, the restriction of \( \mathcal{G}_E \) to \( \rho^{bnd}(\Sigma) \) is locally the graph of an analytic function, and hence by Lemma 3.3.3 \( \mathcal{G}_E \) is analytic in \( \rho^{bnd}(\Sigma) \). The same representation shows that \( \mathcal{G}_E \) is also analytic at infinity.
3.4. INTERTWINEMENTS AND COMPRESSIONS OF BOUNDED S/S SYSTEMS

Proof of (iii): This is essentially a special case of (ii) (take $X_1 = \{0\}$ and $E = 1$).

Proof of (iv): This is essentially a special case of (ii) (take $X_1 = \{0\}$ and $E = 0$).

To get the formulas for the values at infinity we let $\lambda \to \infty$ in the representations formulas (3.4.1), (3.4.6), and (3.4.3).

3.4.2. Strongly invariant and unobservably invariant subspaces. At this point the reader may want to recall the notions of strongly invariant and unobservably invariant subspaces of a s/s system introduced in Definition 1.5.8.

3.4.4. Lemma. Let $\Sigma = (V; X, W)$ be a bounded s/s system.

(i) If $Z$ is a strongly invariant or unobservable invariant subspace for $\Sigma$, then the closure of $Z$ is also strongly invariant respectively unobservably invariant for $\Sigma$.

(ii) If both $Z_1$ and $Z_2$ are strongly invariant for $\Sigma$, then $Z_1 + Z_2$ and $Z_1 \lor Z_2$ are strongly invariant for $\Sigma$.

(iii) If both $Z_1$ and $Z_2$ are unobservably invariant for $\Sigma$, then $Z_1 \cap Z_2$ is unobservably invariant for $\Sigma$.

Proof. By Theorem 2.2.27 $\Sigma$ has a bounded i/s/o representation $\Sigma_{i/s/o}$, and by Proposition 2.5.49 $Z$ is strongly invariant or unobservable invariant for $\Sigma$ if and only if $Z$ is strongly invariant or unobservable invariant for $\Sigma_{i/s/o}$. This combined Lemma 3.2.1 gives Lemma 3.4.4. □

3.4.5. Lemma. Let $\Sigma = (V, X, W)$ be a bounded s/s system, and let $Z$ be a closed subspace of $X$. Then the following conditions are equivalent:

(i) $Z$ is a (forward) strongly invariant subspace for $\Sigma$;

(ii) $Z$ is a backward strongly invariant subspace for $\Sigma$;

(iii) $[1_\varphi \ 0 \ 0 \ 0] \left( V \cap \begin{bmatrix} X \\ Z \end{bmatrix} \right) \subset Z$;

(iv) $V \cap \begin{bmatrix} X \\ Z \end{bmatrix} = V \cap \begin{bmatrix} Z \\ W \end{bmatrix}$;

(v) $[1_\varphi \ 0 \ 0 \ 0] \left( V \cap \begin{bmatrix} X \\ Z \end{bmatrix} \right) \subset Z$ and $[0 \ 1_\varphi \ 0 \ 0] \left( V \cap \begin{bmatrix} Z \\ W \end{bmatrix} \right) = Z$.

Proof. (i) ⇔ (ii) ⇔ (iii) ⇔ (iv): By Theorem 2.2.27 $\Sigma$ has a bounded i/s/o representation $\Sigma_{i/s/o} = ([A \ B] \colon (X, U, Y)$, and by Proposition 2.5.49 $Z$ is forward or backward strongly invariant for $\Sigma$ if and only if $Z$ is forward or backward strongly invariant for $\Sigma_{i/s/o}$. By Lemma 3.2.2 $Z$ is forward strongly invariant for $\Sigma_{i/s/o}$ if and only if $\text{rng}(B) \subset Z$ and $AZ \subset Z$. It follows from the representation formula (2.2.33b) for $V$ that $\text{rng}(B) \subset Z$ and $AZ \subset Z$ if and only if (iii) holds. Finally, it is easy to see that (iii) ⇔ (iv).

(iii) ⇔ (v): Let again $\Sigma_{i/s/o} = ([A \ B] \colon (X, U, Y)$ be a bounded i/s/o representation of $\Sigma$. It follows from the representation (2.2.33b) of $V$ that the first of the two conditions in (iv) is equivalent to the condition $\text{rng}(B) \subset Z$, and that the second of the two conditions in (iv) is equivalent to the following condition: For every $x \in Z$ there exists some $u \in U$ such that $Ax + Bu \in Z$. Thus, (v) holds if and only if $\text{rng}(B) \subset Z$ and $AZ \subset Z$. As we noticed above, this is equivalent to (iii). □
To the equivalent conditions listed in Lemma 3.4.5, it is possible to add some further equivalent conditions which are formulated in terms of the node bundle \( \mathfrak{E} \) of \( \Sigma \). We begin with a preliminary result.

3.4.6. Lemma. Let \( \Sigma = (V, \mathcal{X}, \mathcal{Y}) \) be a bounded s/s system with characteristic node bundle \( \mathfrak{E} \); let \( \mathcal{Z} \) be a closed subspace of \( \mathcal{X} \), and let \( \lambda \in \rho^{\text{bnd}}(\Sigma) \). Then conditions (3.4.7a)–(3.4.7c) below are equivalent:

\[
\begin{align*}
(3.4.7a) & \quad \left[ 0 \ 1_{\mathcal{X}} \ 0 \right] \left( \mathfrak{E}(\lambda) \cap \left[ \frac{0}{\mathcal{W}} \right] \right) \subset \mathcal{Z}, \\
(3.4.7b) & \quad \mathfrak{E}(\lambda) \cap \left[ \frac{\mathcal{Y}}{\mathcal{W}} \right] = \mathfrak{E}(\lambda) \cap \left[ \frac{0}{\mathcal{W}} \right], \\
(3.4.7c) & \quad \left[ 0 \ 1_{\mathcal{X}} \ 0 \right] \left( \mathfrak{E}(\lambda) \cap \left[ \frac{0}{\mathcal{W}} \right] \right) \subset \mathcal{Z} \text{ and } \left[ 1_{\mathcal{X}} \ 0 \ 0 \right] \left( \mathfrak{E}(\lambda) \cap \left[ \frac{\mathcal{Y}}{\mathcal{W}} \right] \right) = \mathcal{Z}.
\end{align*}
\]

Proof. The proof of this lemma is essentially the same as the proof of the equivalences (iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v) in Lemma 3.4.5, with \( V \) replaced by \( \mathfrak{E}(\lambda) \) and the representation (2.2.33b) of \( V \) replaced by the representation (3.4.1b) of \( \mathfrak{E}(\lambda) \). \( \square \)

3.4.7. Lemma. Let \( \Sigma = (V, \mathcal{X}, \mathcal{Y}) \) be a bounded s/s system, and let \( \mathcal{Z} \) be a closed subspace of \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( \mathcal{Z} \) is a strongly invariant subspace for \( \Sigma \);

(ii) at least one of conditions (3.4.7a)–(3.4.7c) holds for some \( \lambda \in \rho^{\text{bnd}}(\Sigma) \);

(iii) conditions (3.4.7a)–(3.4.7c) holds for all \( \lambda \in \rho^{\text{bnd}}(\Sigma) \).

Proof. (i) \( \Rightarrow \) (ii): By Theorem 2.2.27 \( \Sigma \) has a bounded i/s/o representation \( \Sigma_{i/s/o} = ([A \ B \ C \ D] ; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), and by Proposition 2.5.49 \( \mathcal{Z} \) is strongly invariant for \( \Sigma \) if and only if \( \mathcal{Z} \) is strongly invariant for \( \Sigma_{i/s/o} \). By Lemma 3.2.2, \( \mathcal{Z} \) is strongly invariant for \( \Sigma_{i/s/o} \) if and only if \( \text{rng}((\lambda - A)^{-1}B) \subset \mathcal{Z} \) and \((\lambda - A)^{-1}\mathcal{Z} \subset \mathcal{Z} \) for all \( \lambda \in \rho_{\infty}(A) \). It follows from the representation formula (3.4.1b) for \( \mathfrak{E}(\lambda) \) that this is true if and only if (3.4.7a) holds for all \( \lambda \in \rho_{\infty}(A) \). Since \( \rho_{\infty}(A) \subset \rho_{\infty}(\Sigma) \) this implies (ii).

(ii) \( \Rightarrow \) (iii): If (3.4.7a)–(3.4.7c) holds some \( \lambda_0 \in \rho^{\text{bnd}}(\Sigma) \), then also (3.4.7a) holds for the same \( \lambda_0 \). Let \( \Sigma_{i/s/o} = ([A \ B \ C \ D] ; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o representation of \( \Sigma \) satisfying \( \lambda_0 \in \rho(\Sigma) \). Then it follows from the representation (3.4.1b) for \( \mathfrak{E}(\lambda) \) that \( \text{rng}((\lambda_0 - A)^{-1}B) \subset \mathcal{Z} \) and \((\lambda_0 - A)^{-1}\mathcal{Z} \subset \mathcal{Z} \). Consequently \( \text{rng}((\lambda_0 - A)^{-n}B) \subset \mathcal{Z} \) and \((\lambda_0 - A)^{-n}\mathcal{Z} \subset \mathcal{Z} \) for all \( n \in \mathbb{N} \). Since \( \frac{1}{\lambda_0}(\lambda - A)^{-1} = (-1)^n n!(\lambda - A)^{-(n+1)} \) for all \( \lambda \in \rho(A) \) and all \( n \in \mathbb{Z}^+ \), it follows from Lemma A.3.6 applied to the analytic functions \( \lambda \mapsto (\lambda - A)^{-1}B \) and \( \lambda \mapsto (\lambda - A)^{-1}_Z \) that \( \text{rng}((\lambda - A)^{-1}B) \subset \mathcal{Z} \) and \((\lambda - A)^{-1}\mathcal{Z} \subset \mathcal{Z} \) for all \( \lambda \in \Omega \), where \( \Omega \) is the component of \( \rho(A) \) which contains \( \lambda_0 \). Note that \( \Omega \subset \rho_{\infty}(\Sigma) \) (although we do not claim that \( \Omega = \rho_{\infty}(A) \)). By using the representation (3.4.1b) once more we find that (3.4.7a) holds for all \( \lambda \in \Omega \). Therefore by Lemma 3.4.6 also conditions (3.4.7b) and (3.4.7c) hold for all \( \lambda \in \Omega \). By Lemma A.3.10 condition (3.4.7b) holds for all \( \lambda \in \rho_{\infty}(\Sigma) \), and consequently, by Lemma 3.4.6 also conditions (3.4.7a) and (3.4.7c) hold for all \( \lambda \in \rho_{\infty}(\Sigma) \).

(iii) \( \Rightarrow \) (i): Let \( \Sigma_{i/s/o} = ([A \ B \ C \ D] ; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o representation of \( \Sigma \). Arguing as above we find that if (iii) holds, then condition (vii) in Lemma 3.2.2 holds. Consequently, by Lemma 3.2.2 \( \mathcal{Z} \) is strongly invariant for \( \Sigma_{i/s/o} \), and by Proposition 2.5.49 \( \mathcal{Z} \) is also strongly invariant for \( \Sigma \). \( \square \)
Theorem 3.4.8. Lemma. Let $\Sigma = (V, X, \mathcal{W})$ be a bounded s/s system with characteristic node bundle $E$, and let $Z$ be a closed subspace of $X$ with a direct complement $X_1$. Then the following conditions are equivalent:

(i) $Z$ is a (forward) unobservably invariant subspace for $\Sigma$;
(ii) $Z$ is a backward unobservably invariant subspace for $\Sigma$;
(iii) $[0 \; 1_Z \; 0] \left(V \cap \left[ \frac{X}{(0)} \right] \right) = Z$;
(iv) $P_{X_1}^z 0 \; 0 \; P_{X_1}^z V = \begin{bmatrix} P_{X_1}^z & 0 & 0 \\ 0 & 1_{X_1} & 0 \\ 0 & 0 & 1_W \end{bmatrix} \left(V \cap \left[ \frac{X}{(0)} \right] \right)$;
(v) $[0 \; 1_Z \; 0] \left(V \cap \left[ \frac{X}{(0)} \right] \right) = Z$ and $[1_X \; 0 \; 0] \left(V \cap \left[ \frac{X}{(0)} \right] \right) \subseteq Z$.

Proof. (i) $\iff$ (ii): By Theorem 2.2.27 $\Sigma$ has a bounded i/s/o representation $\Sigma_{i/s/o} = (\begin{bmatrix} A & B \\ C & D \end{bmatrix} ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, and by Proposition 2.5.49 $Z$ is forward or backward unobservably invariant for $\Sigma$ if and only if $Z$ is forward or backward unobservably invariant for $\Sigma_{i/s/o}$. By Lemma 3.2.3 $Z$ is forward unobservably invariant for $\Sigma_{i/s/o}$ if and only if $Z$ is backward unobservably invariant for $\Sigma_{i/s/o}$, and this is true if and only if $Z \subset \ker(C)$ and $AZ \subset Z$. It follows from the representation formula (2.2.33b) for $V$ that $Z \subset \ker(C)$ and $AZ \subset Z$ if and only if (iii) holds.

(iii) $\Rightarrow$ (iv): Let again $\Sigma_{i/s/o} = (\begin{bmatrix} A & B \\ C & D \end{bmatrix} ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a bounded i/s/o representation of $\Sigma$. It follows from the representation formula (2.2.33b) for $V$ that (iv) is equivalent to the condition

$\text{rng} \left( \begin{bmatrix} P_{X_1}^z A & P_{X_1}^z B \\ P_{X_1}^z C & P_{X_1}^z D + P_{X_1}^z I \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} P_{X_1}^z A_{X_1} & P_{X_1}^z B \\ 1_{X_1} & 0 \\ I_3 C_{X_1} & I_3 D + I_3 I \end{bmatrix} \right)$.

If (iii) holds, then as we saw above, $AZ \subset Z$ and $Z \subset \ker(C)$, and this clearly (3.4.8). Thus (iv) holds.

(iv) $\Rightarrow$ (iii): Let $\begin{bmatrix} z \\ x_1 \\ w \end{bmatrix}$ be an arbitrary vector in $\begin{bmatrix} X \\ U \end{bmatrix}$, and define $\begin{bmatrix} z_1 \\ w \end{bmatrix}$ by

$$\begin{bmatrix} z \\ x_1 \\ w \end{bmatrix} = \begin{bmatrix} P_{X_1}^z A x + P_{X_1}^z B u \\ P_{X_1}^z x \\ I_3 C x + I_3 D u + I_3 u \end{bmatrix}.$$ 

Then $\begin{bmatrix} z_1 \\ x_1 \\ w \end{bmatrix}$ belongs to the left-hand side of (3.4.8), and if (iv) holds, then it also belongs to the right-hand side of (3.4.8). This means that the same vector has a representation of the form

$$\begin{bmatrix} z \\ x_1 \\ w \end{bmatrix} = \begin{bmatrix} P_{X_1}^z A x + P_{X_1}^z B u_1 \\ x_1 \\ I_3 C x + I_3 D u + I_3 u \end{bmatrix}.$$ 

Here $x_1 = P_{X_1}^z x$, and $u_1 = P_{U_1}^z w = u$. After subtracting $\begin{bmatrix} P_{X_1}^z B u \\ I_3 D u + I_3 u \end{bmatrix}$ from both of the representations of $\begin{bmatrix} z \\ x_1 \\ w \end{bmatrix}$ we find that $P_{X_1}^z A x = P_{X_1}^z A P_{X_1}^z x$ and $C x = CP_{X_1}^z x$. Since this is true for all $x \in X$ it follows that $AZ \subset Z$ and $Z \subset \ker(C)$. As we saw above, this is equivalent to (iii).

(iii) $\iff$ (v): Let again $\Sigma_{i/s/o} = (\begin{bmatrix} A & B \\ C & D \end{bmatrix} ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a bounded i/s/o representation of $\Sigma$. It follows from the representation (2.2.33b) of $V$ that the first of the two
conditions in (iv) is equivalent to the condition for all \( x \in \mathcal{Z} \) it is true that \( Cx = 0 \), i.e., \( \mathcal{Z} \subset \ker(C) \), and that the second of the two conditions in (v) is equivalent to the condition that if \( x \in \mathcal{Z} \cap \ker(C) \), then \( Ax \in \mathcal{Z} \). Thus, (v) holds if and only if \( \mathcal{Z} \subset \ker(C) \) and \( A\mathcal{Z} \subset \mathcal{Z} \). As we noticed above, this is equivalent to (iii).

3.4.9. Lemma. Let \( \Sigma = (V, \mathcal{X}, \mathcal{W}) \) be a bounded s/s system with characteristic node bundle \( \hat{\mathcal{E}} \), let \( \mathcal{Z} \) be a closed subspace of \( \mathcal{X} \), and let \( \lambda \in \rho_{\text{bnd}}(\Sigma) \). Then conditions (3.4.9a)–(3.4.9c) below are equivalent:

(3.4.9a) \[ [1_Z \ 0 \ 0] \left( \hat{\mathcal{E}}(\lambda) \cap \left[ \begin{bmatrix} Z \\ \{0\} \end{bmatrix} \right] \right) = \mathcal{Z}, \]

(3.4.9b) \[ \begin{bmatrix} P_{x_{i}}^{z} & 0 & 0 \\ 0 & P_{z_{i}}^{x} & 0 \\ 0 & 0 & 1_w \end{bmatrix} \hat{\mathcal{E}}(\lambda) = \begin{bmatrix} 1_{X_{i}} & 0 & 0 \\ 0 & P_{x_{i}}^{z} & 0 \\ 0 & 0 & 1_w \end{bmatrix} \left( \hat{\mathcal{E}}(\lambda) \cap \left[ \begin{bmatrix} X_{i} \\ Z_{i} \\ W \end{bmatrix} \right] \right), \]

(3.4.9c) \[ [1_Z \ 0 \ 0] \left( \hat{\mathcal{E}}(\lambda) \cap \left[ \begin{bmatrix} Z_{i} \\ \{0\} \end{bmatrix} \right] \right) \subset \mathcal{Z} \text{ and } [0 \ 1_{X} \ 0] \left( \hat{\mathcal{E}}(\lambda) \cap \left[ \begin{bmatrix} X_{i} \\ \{0\} \end{bmatrix} \right] \right) \subset \mathcal{Z}; \]

Proof. The proof of this lemma is essentially the same as the proof of the equivalences (iii) \( \equiv \) (iv) \( \equiv \) (v) in Lemma 3.4.8 with \( \hat{\mathcal{E}}(\lambda) \) replaced by \( \mathcal{E}(\lambda) \) and the representation (2.2.33b) of \( V \) replaced by the representation (3.4.1b) of \( \mathcal{E}(\lambda) \). □

3.4.10. Lemma. Let \( \Sigma = (V, \mathcal{X}, \mathcal{W}) \) be a bounded s/s system with characteristic node bundle \( \hat{\mathcal{E}} \), and let \( \mathcal{Z} \) be a closed subspace of \( \mathcal{X} \) with a direct complement \( \mathcal{X}_{1} \). Then the following conditions are equivalent:

(i) \( \mathcal{Z} \) is a unobservably invariant subspace for \( \Sigma \);

(ii) at least one of conditions (3.4.9a)–(3.4.9c) holds for some \( \lambda \in \rho_{\text{bnd}}(\Sigma) \);

(iii) conditions (3.4.9a)–(3.4.9c) holds for all \( \lambda \in \rho_{\text{bnd}}(\Sigma) \).

Proof. (i) \( \Rightarrow \) (ii): Let \( \Sigma_{i/s/o} = (\mathcal{E}_{i/s/o} \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o representation of \( \Sigma \). Then by (3.4.1b), for all \( \lambda \in \rho(A) \) we have

(3.4.10) \[ \hat{\mathcal{E}}(\lambda) \cap \left[ \begin{bmatrix} \mathcal{Z} \\ \mathcal{X} \\ \{0\} \end{bmatrix} \right] = \text{rng} \left( \begin{bmatrix} 1_{Z \cap \ker(C(\lambda - A)^{-1})} \\ (\lambda - A)^{-1} |_{Z \cap \ker(C(\lambda - A)^{-1})} \\ 0 \end{bmatrix} \right). \]

Thus, the first condition in (3.4.9c) is equivalent to the condition that \( \mathcal{Z} \subset \ker(C(\lambda - A)^{-1}) \), and if this condition holds, then the second condition in (3.4.9c) is equivalent to the condition that \( (\lambda - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \). This together with the equivalence of conditions (i) and (vii) in Lemma 3.4.8 implies that condition (3.4.9c) holds for all \( \lambda \in \rho_{\text{bnd}}(\Sigma) \). Since \( \rho_{\text{bnd}}(\Sigma) \subset \rho_{\infty}(\Sigma) \) this implies (ii).

(ii) \( \Rightarrow \) (iii): By Lemma 3.4.9 if (3.4.9a) or (3.4.9b) holds some \( \lambda_{0} \in \rho_{\text{bnd}}(\Sigma) \), then also (3.4.9c) holds for the same \( \lambda_{0} \). Let \( \Sigma_{i/s/o} = (\mathcal{E}_{i/s/o} \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a bounded i/s/o representation of \( \Sigma \) satisfying \( \lambda_{0} \in \rho(\Sigma) \). Then it follows from (3.4.10) that \( \mathcal{Z} \subset \ker(C(\lambda - A)^{-1}) \) and that \( (\lambda - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \). Therefore \( \mathcal{Z} \subset \ker(C(\lambda - A)^{-n}) \) and \( (\lambda - A)^{-n} \mathcal{Z} \subset \mathcal{Z} \) for all \( n \in \mathbb{N} \). Since \( \frac{d^{n}}{d\lambda^{n}}(\lambda - A)^{-1} = (-1)^{n}n!(\lambda - A)^{-n+1} \) it follows from Lemma A.3.6 that \( \mathcal{Z} \subset \ker(C(\lambda - A)^{-1}) \) and \( (\lambda - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \) for all \( \lambda \in \Omega \), where \( \Omega \) is the component of \( \rho(A) \) which contains \( \lambda_{0} \). Note that \( \Omega \subset \rho_{\infty}(\Sigma) \) (although we do not claim that \( \Omega = \rho_{\infty}(\Sigma) \)). This combined with (3.4.10) implies that condition (3.4.9c) holds for all \( \lambda \in \Omega \). Therefore by Lemma 3.4.9 also conditions (3.4.9b) and (3.4.9c) hold for all \( \lambda \in \Omega \). By Lemma 3.4.3(ii), the bundle \( \hat{\mathcal{E}} \cap \left[ \begin{bmatrix} X_{i} \\ Z_{i} \\ W \end{bmatrix} \right] \) is analytic in \( \rho_{\infty}(\Sigma) \), and therefore by Lemma A.3.10 condition
(3.4.9b) holds for all $\lambda \in \rho_\infty(\Sigma)$. By Lemma 3.4.9 also (3.4.9a) and (3.4.9c) hold for all $\lambda \in \rho_\infty(\Sigma)$.

(iii) $\Rightarrow$ (i): Let $\Sigma_{i/s/o} = ([A \ B] : X, U, Y)$ be a bounded i/s/o representation of $\Sigma$. Then the representation (3.4.10) is valid for all $\lambda \in \rho(A)$. If, in addition, $\lambda \in \rho^\text{bnd}_\infty(\Sigma)$, then this implies that $Z \subset \ker(C(\lambda - A)^{-1})$ and $(\lambda - A)^{-1}Z \subset Z$. Consequently, by Lemma 3.2.3, $Z$ is unobservably invariant for $\Sigma_{i/s/o}$, and by Proposition 2.5.49 $Z$ is also unobservably invariant for $\Sigma$. □

At this point the reader may want to recall the notions of the reachable subspace and the unobservable subspace of an s/s system introduced in Definition 1.5.3.

3.4.11. Lemma. Let $\Sigma = (V; X, W)$ be a bounded s/s system with characteristic node bundle $\hat{E}$. Let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\rho^\text{bnd}_\infty(\Sigma)$. Then the following claims are true:

(i) The forward and backward reachable subspaces of $\Sigma$ are the same, and the forward and backward unobservable subspaces of $\Sigma$ are also the same.

(ii) The (forward and backward) reachable subspace $R_{\Sigma}$ of $\Sigma$ is the minimal closed strongly invariant subspace for $\Sigma$.

(iii) The (forward and backward) unobservable subspace $U_{\Sigma}$ of $\Sigma$ coincides with the (forward and backward) classically unobservable subspace of $\Sigma$, and it is the maximal unobservably invariant subspace of $\Sigma$.

(iv) $R_{\Sigma}$ can be computed in the following ways:

$$R_{\Sigma} = \bigvee_{\lambda \in \rho^\text{bnd}_\infty(\Sigma)} \begin{bmatrix} 0 & 1 \ X & 0 \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} \{0\} \ X \end{bmatrix} \right)$$  (3.4.11)

$$= \bigvee_{\lambda \in \Omega'} \begin{bmatrix} 0 & 1 \ X & 0 \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} \{0\} \ X \end{bmatrix} \right).$$

(v) $U_{\Sigma}$ can be computed in the following ways:

$$U_{\Sigma} = \bigcap_{\lambda \in \rho^\text{bnd}_\infty(\Sigma)} \begin{bmatrix} 1 \ X & 0 \ 0 \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X \ X \end{bmatrix} \{0\} \right)$$  (3.4.12)

$$= \bigcap_{\lambda \in \Omega'} \begin{bmatrix} 1 \ X & 0 \ 0 \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X \ X \end{bmatrix} \{0\} \right).$$

Proof. Throughout this proof we let $\Sigma_{i/s/o} = ([A \ B] : X, U, Y)$ be a bounded i/s/o representation of $\Sigma$ (that such a representation exists follows from Lemma 2.2.27). Recall that by Proposition 2.5.49 the forward and backward reachable and unobservable subspaces of $\Sigma$ coincide with the corresponding subspaces for $\Sigma_{i/s/o}$.

Proofs of (i), (ii), and (iii): These claims follows from Lemma 3.2.4 (Claims (ii) and (iii) may alternatively be proved with the help of Lemmas 1.5.9 and 3.4.4)

Proof of (iv): It follows from Lemma 3.2.4 and the representation formula (3.4.1b) that (3.4.11) with $\Omega' = \rho_\infty(A)$. By Lemma 3.4.3 the bundle $\hat{E}(\lambda) \cap \begin{bmatrix} \{0\} \ X \end{bmatrix}$ is analytic in $\rho^\text{bnd}_\infty(\Sigma)$. Since $\begin{bmatrix} 0 & 1 \ X & 0 \end{bmatrix}$ is the projection onto $\begin{bmatrix} \{0\} \ X \end{bmatrix}$ along $\begin{bmatrix} \{0\} \ W \end{bmatrix}$ in $\begin{bmatrix} X \ X \end{bmatrix}$, this combined with Lemmas 3.4.3 and A.3.10 gives (iv).
Proof of (v): It follows from Lemma 3.2.4 and the representation formula (3.4.1b) that

\[
\mathcal{U}_\Sigma = \bigcap_{\lambda \in \rho_\infty(A)} \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \left( \mathcal{E}(\lambda) \cap \left[ \begin{bmatrix} X' \\ \{0\} \end{bmatrix} \right] \right) = \bigcap_{\lambda \in \rho_\infty(A)} \left( \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{E}(\lambda) \right]\cap \left[ \begin{bmatrix} X' \\ \{0\} \end{bmatrix} \right] \right),
\]

where \(\rho_\infty(A) \subset \rho_\infty^\text{bnd}()\) trivially contains a cluster point in \(\rho_\infty^\text{bnd}()\). By Lemma 3.4.3 the bundle \(\left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{E}(\lambda) \right]\) is analytic in \(\rho_\infty^\text{bnd}()\). By combining this fact with Lemmas 3.4.3 and A.3.10 we get (v). \(\square\)

3.4.3. External equivalence of bounded s/s system. At this point the reader may want to recall Definitions 1.5.21 and 1.5.43.

3.4.12. Lemma. Let \(\Sigma_i = \left( \mathcal{V}_i; \mathcal{X}_i; \mathcal{W}_i \right), i = 1, 2\), be two bounded s/s systems (with the same signal space \(\mathcal{W}\)) with characteristic signal bundles \(\hat{\mathcal{X}}_i\). Denote the unbounded component of \(\rho^\text{bnd}(\Sigma_1) \cap \rho^\text{bnd}(\Sigma_2)\) by \(\Omega\), and let \(\Omega'\) be a subset of \(\Omega\) which contains a cluster point in \(\Omega\). Then the following conditions are equivalent:

(i) \(\Sigma_1\) and \(\Sigma_2\) are (forward) externally equivalent;
(ii) \(\Sigma_1\) and \(\Sigma_2\) are backward externally equivalent;
(iii) \(\Sigma_1\) and \(\Sigma_2\) have the same future behavior;
(iv) \(\hat{F}_1(\lambda) = \hat{F}_2(\lambda)\) for all \(\lambda \in \Omega\);
(v) \(\hat{F}_1(\lambda) = \hat{F}_2(\lambda)\) for all \(\lambda \in \Omega'\).

Proof. (i) \(\Leftrightarrow\) (ii): By Theorem 2.2.27, \(\Sigma_j\) has a bounded i/o representation \(\Sigma_j^i/s/o\), \(j = 1, 2\), and by Proposition 2.5.50, \(\Sigma_1\) and \(\Sigma_2\) are forward or backward externally equivalent if and only if \(\Sigma_1^i/s/o\) and \(\Sigma_2^i/s/o\) are forward or backward externally equivalent. This together with the equivalence of conditions (i) and (ii) in Theorem 3.2.7 and the representation formula (3.4.3b) for \(V\) implies that (i) \(\Leftrightarrow\) (ii).

(i) \(\Leftrightarrow\) (iii): See Lemma 1.5.44 and Corollary 2.2.28.

(i) \(\Rightarrow\) (v): Arguing as above and also using the implication (i) \(\Rightarrow\) (v) in Theorem 3.2.7 and the representation formula (3.4.3b) for \(\mathcal{E}(\lambda)\) we find that \(\hat{F}_1(\lambda) = \hat{F}_2(\lambda)\) for all \(\lambda\) in the unbounded component of \(\rho(A_1) \cap \rho(A_2)\), where \(A_j\) is the main operator of the i/o representation \(\Sigma_j^i/s/o\) of \(\Sigma_j, j = 1, 2\).

(v) \(\Rightarrow\) (iv): This follows from Lemmas 3.4.3 and A.3.9.

(iv) \(\Rightarrow\) (i): Let \(\Sigma_j^i/s/o\) be a bounded i/o representation of \(\Sigma_j, j = 1, 2\). Arguing as above we find that if (iii) holds, then condition (v) in Theorem 3.2.7 holds. Consequently, by Theorem 3.2.7, \(\Sigma_1^i/s/o\) and \(\Sigma_2^i/s/o\) are externally equivalent, and hence also \(\Sigma_1\) and \(\Sigma_2\) are externally equivalent. \(\square\)

3.4.13. Lemma. Let \(\Sigma_i = \left( \mathcal{V}_i; \mathcal{X}_i; \mathcal{W}_i \right), i = 1, 2\), be two bounded externally equivalent s/s systems. Then the i/o representation \((\mathcal{U}, \mathcal{Y})\) of \(\mathcal{W}\) is boundedly i/o-admissible for \(\Sigma_1\) if and only if it is boundedly i/o-admissible for \(\Sigma_2\).

Proof. By Lemma 3.4.12 \(\hat{F}_1(\lambda) = \hat{F}_2(\lambda)\) for all \(\lambda \in \Omega_\infty\), and hence by Lemma 3.4.3 the canonical input spaces \(W_0^i\) and \(W_0^j\) or \(\Sigma_1\) respectively \(\Sigma_2\) coincide. It therefore follows from Theorem 2.2.27 that \(\Sigma_1\) and \(\Sigma_2\) have the same boundedly i/o-admissible i/o representations. \(\square\)
3.4.4. Restrictions and projections of bounded s/s systems. At this point the reader may want to recall Definitions [1.5.33] and [1.5.37] of what we mean by a restriction and a projection of an s/s system.

3.4.14. Theorem. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a bounded s/s system, let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}$, and let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be the part of $\Sigma$ in $[\mathcal{X}_1/\mathcal{W}]$, i.e.,

$$V_1 = V \cap \left[ \mathcal{X}_1 / \mathcal{W} \right].$$

Then the following conditions are equivalent:

(i) $\mathcal{X}_1$ is a (forward) strongly invariant subspace for $\Sigma$;
(ii) $\mathcal{X}_1$ is a backward strongly invariant subspace for $\Sigma$;
(iii) $\Sigma$ has a (forward) restriction to $\mathcal{X}_1$;
(iv) $\Sigma$ has a backward restriction to $\mathcal{X}_1$;
(v) $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a bounded s/s system, and the canonical input spaces of $\Sigma$ and $\Sigma_1$ coincide.

If these equivalent conditions hold then $\Sigma_1$ is the unique bounded forward and backward restriction of $\Sigma$ to $\mathcal{X}_1$, and $\Sigma$ and $\Sigma_1$ are both forward and backward externally equivalent.

Proof. (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv): By Theorem [2.2.27], $\Sigma$ has a bounded i/o representation $\Sigma_{i/o}$. By Propositions [2.5.49] and [2.5.50], each one of conditions (i)–(iv) above is equivalent to the corresponding condition (i)–(iv) in Theorem [3.2.11]. Since conditions (i)–(iv) in Theorem [3.2.11] are equivalent to each other, it follows that also conditions (i)–(iv) above are equivalent to each other.

(i) $\implies$ (v): Let $\Sigma_{i/o}$ be a bounded i/o representation of $\Sigma$, let $\Sigma_{i/o}^1 = ([A_1; B_1]: \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ be the bounded i/o system in Theorem [3.2.11] and define $V_1$ by [2.2.33] with $[C; D]$ replaced by $[A_1; B_1]$. Then $\Sigma_{i/o}^1$ is an i/o representation of $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$, and it follows from [2.2.33] and [3.2.10] that $V_1$ is given by [3.4.13]. That $\Sigma$ and $\Sigma_1$ have the same canonical input space follows from [3.2.10] and [2.2.20]. Finally, that $\Sigma$ is both a forward and backward restriction of $\Sigma_1$ and that $\Sigma$ and $\Sigma_1$ are both forward and backward externally equivalent follows from Theorem [3.2.11] and Proposition [2.5.50].

(v) $\implies$ (i): Since $\Sigma$ and $\Sigma_1$ have the same canonical input space $\mathcal{W}_0$ it follows from Theorem [2.2.29] that an i/o representation $(\mathcal{U}, \mathcal{Y})$ be such an i/o representation, and denote the corresponding i/o systems by $\Sigma_{i/o} = ([A; B]: \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\Sigma_{i/o}^1 = ([A_1; B_1]: \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$. Then it follows from [3.4.13] and [2.2.33] that $\mathcal{X}_1 \subset \mathcal{X}$ and $\text{rng}(B) \subset \mathcal{X}_1$, and that $[A_1; B_1]$ is given by [3.2.10]. By Lemma [3.2.2], $\mathcal{X}_1$ is strongly invariant for $\Sigma_{i/o}^1$, and by Proposition [2.5.49], $\mathcal{X}_1$ is also strongly invariant for $\Sigma$.

The final uniqueness claim follows from Lemmas [1.5.31] and [1.5.36] and Corollary [2.2.28].

3.4.15. Theorem. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a bounded s/s system, and let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}_1$ be a direct sum decomposition of $\mathcal{X}$. Let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be the static
projection of $\Sigma$ onto $[\mathcal{X}_1 \mid \mathcal{W}]$, along $[\mathcal{Z}_1 \mid 0]$, i.e.,

$$
V_1 = \begin{bmatrix}
P_{\mathcal{X}_1} & 0 & 0 \\
0 & P_{\mathcal{X}_1} & 0 \\
0 & 0 & 1_{\mathcal{W}}
\end{bmatrix} V.
$$

Then the following conditions are equivalent:

(i) $\mathcal{Z}_1$ is a (forward) unobservably invariant subspace for $\Sigma$;
(ii) $\mathcal{Z}_1$ is a backward unobservably invariant subspace for $\Sigma$;
(iii) $\Sigma$ has a (forward) projection onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(iv) $\Sigma$ has a backward projection onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(v) $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ is a bounded s/s system, and the canonical input spaces of $\Sigma$ and $\Sigma_1$ coincide.

If these equivalent conditions hold then $\Sigma_1$ is the unique bounded forward and backward projection of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, and $\Sigma$ and $\Sigma_1$ are both forward and backward externally equivalent.

**Proof.** (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv): By Theorem 2.2.27 $\Sigma$ has a bounded i/s/o representation $\Sigma_{i/s/o}$. By Proposition 2.5.50 each one of conditions (i)–(iv) above is equivalent to the corresponding condition (i)–(iv) in Theorem 3.2.12. Since conditions (i)–(iv) in Theorem 3.2.12 are equivalent to each other, it follows that also conditions (i)–(iv) above are equivalent to each other.

(i) $\implies$ (v): Let $\Sigma_{i/s/o}$ be a bounded i/s/o representation of $\Sigma$, let $\Sigma^1_{i/s/o} = \left( [A^1, B^1] ; \mathcal{X}_1, \mathcal{U}, \mathcal{Y} \right)$ be the bounded i/s/o system in Theorem 3.2.12 and define $V_1$ by (3.4.13) with $[A \ B]$ replaced by $[A^1 \ B^1]$. Then $\Sigma^1_{i/s/o}$ is an i/s/o representation of $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$, and it follows from (3.4.14) and (3.2.11) and condition (iv) in Lemma 3.4.8 that $V_1$ is given by (3.4.14). That $\Sigma$ and $\Sigma_1$ have the same canonical input space follows from (3.2.11) and (3.2.20). Finally, that $\Sigma_1$ is both a forward and backward restriction of $\Sigma_1$, and that $\Sigma$ and $\Sigma_1$ are both forward and backward externally equivalent follows from Theorem 3.2.12 and Proposition 2.5.50.

(v) $\implies$ (i): Since $\Sigma$ and $\Sigma_1$ have the same canonical input space $\mathcal{W}_0$ it follows from Theorem 2.2.20 that an i/o representation $(\mathcal{U}, \mathcal{Y})$ is boundedly i/s/o-admissible for $\Sigma$ if and only if it is boundedly i/s/o-admissible for $\Sigma_1$. Let $(\mathcal{U}, \mathcal{Y})$ be such an i/o representation, and denote the corresponding i/s/o systems by $\Sigma_{i/s/o} = \left( [A \ B] ; \mathcal{X}_1, \mathcal{U}, \mathcal{Y} \right)$ and $\Sigma^1_{i/s/o} = \left( [A^1 \ B^1] ; \mathcal{X}_1, \mathcal{U}, \mathcal{Y} \right)$. Then it follows from (3.4.14) and (3.2.23) that $A \mathcal{Z}_1 \subset \mathcal{Z}_1$ and $\mathcal{Z}_1 \subset \ker (C)$, and that $[A^1 \ B^1]$ is given by (3.2.11). By Lemma 3.2.3 $\mathcal{Z}_1$ is unobservably invariant for $\Sigma_{i/s/o}$, and by Proposition 2.5.49 $\mathcal{Z}_1$ is also unobservably invariant for $\Sigma$. The final uniqueness claim follows from Lemmas 1.5.31 and 1.5.40 and Corollary 2.2.28.

**3.4.5. Intertwinements of bounded s/s systems.** At this point the reader may want to recall Definitions 1.5.22 of what we mean by an intertwinement of two s/s systems.

**3.4.16. Lemma.** Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), \ i = 1, 2$, be two bounded s/s systems (with the same signal space $\mathcal{W}$), and let $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ be closed. Let $\Sigma = (V; \text{gph} (P), \mathcal{W})$ be the gph $(P)$-short circuit of $\Sigma_2$ and $\Sigma_1$ (cf. Definition 1.2.29),
(3.4.15) \[
V = \left\{ \begin{bmatrix} z_1 \\ x_1 \\ w \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w \end{bmatrix} \in \text{gph}(P) \; | \; \begin{bmatrix} z_1 \\ x_1 \\ w \end{bmatrix} \in V_1 \quad \text{and} \quad \begin{bmatrix} z_2 \\ x_2 \\ w \end{bmatrix} \in V_2 \right\}.
\]

Then the following conditions are equivalent:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are (forward) intertwined by \( P \);
(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are backward intertwined by \( P \);
(iii) \( \Sigma \) is a bounded \( s/s \) node, and the canonical input spaces of \( \Sigma, \Sigma_1, \) and \( \Sigma_2 \) coincide.
(iv) The following two conditions hold:
   (a) If \( \begin{bmatrix} z_1 \\ w \end{bmatrix} \in \text{gph}(P) \) and \( w \in W \), then \( \begin{bmatrix} z_1 \\ x_1 \\ w \end{bmatrix} \in V_1 \) for some (unique) \( z_1 \in X_1 \) if and only if \( \begin{bmatrix} z_2 \\ w \end{bmatrix} \in V_2 \) for some (unique) \( z_2 \in X_2 \);
   (b) the vectors \( z_1 \) and \( z_2 \) in (a) satisfy \( \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix} \in \text{gph}(P) \).

If these equivalent conditions hold, then \( \Sigma \) and \( \Sigma_1 \) are intertwined by the bounded operator \( P_{X_1|\text{gph}(P)}^X \), \( \Sigma \) and \( \Sigma_2 \) are intertwined by the bounded operator \( P_{X_1|\text{gph}(P)}^{X_2} \), and \( \Sigma, \Sigma_1, \) and \( \Sigma_2 \) are both forward and backward externally equivalent.

**Proof.** (i) \( \Rightarrow \) (iv): Let \( \Sigma_{1/s/o}^1 = \left( \begin{bmatrix} A_1 \\ C_1 \\ D_1 \end{bmatrix} ; X_1, U, Y \right) \) be a bounded i/o representation of \( \Sigma \). If (i) holds, then by Lemma 1.5.27, \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent, and hence by Lemma 3.4.13, \( \Sigma_2 \) has a bounded i/o representation \( \Sigma_{i/s/o}^2 = \left( \begin{bmatrix} A_2 \\ C_2 \\ D_2 \end{bmatrix} ; X_2, U, Y \right) \) with the same input and output spaces. By Proposition 2.5.50, \( \Sigma_{i/s/o}^1 \) and \( \Sigma_{i/s/o}^2 \) are intertwined by \( P \). It then follows from condition (iii) in Lemma 3.2.16 and the representation formula (2.2.33) for \( V_1 \) and \( V_2 \) that (iv) holds.

(iv) \( \Rightarrow \) (i): If condition (iv) holds, then \( \Sigma_1 \) and \( \Sigma_2 \) have the same canonical input space. It therefore follows from Theorem 2.2.27 that \( \Sigma_1 \) and \( \Sigma_2 \) have the same boundedly i/o-admissible i/o representations. In particular, there exists i/o representations \( \Sigma_{i/s/o}^j = \left( \begin{bmatrix} A_j \\ C_j \\ D_j \end{bmatrix} ; X_j, U, Y \right) \) of \( \Sigma_j, j = 1, 2 \), with the same input and output spaces. Condition (iv) combined together with the representation formula (2.2.33) for \( V_1 \) and \( V_2 \) implies that condition (iii) in Lemma 3.2.16 holds, and therefore \( \Sigma_{i/s/o}^1 \) and \( \Sigma_{i/s/o}^2 \) are intertwined by \( P \). By Proposition 2.5.50, also \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \).

(ii) \( \Leftrightarrow \) (iv): The proof of this equivalence is the same as the proof of the equivalence (i) \( \Leftrightarrow \) (iv) with \( \Sigma_1 \) and \( \Sigma_2 \) replaced by their time reflections.

(i) \( \Rightarrow \) (iii): As in the proof of the implication (i) \( \Rightarrow \) (iv) we let \( (U, Y) \) be an i/o representation of \( W \) which is boundedly i/o-admissible for both \( \Sigma_1 \) and \( \Sigma_2 \). Denote the corresponding i/o representations of \( \Sigma_1 \) and \( \Sigma_2 \) by \( \Sigma_{i/s/o} \) and \( \Sigma_{i/s/o}^2 \). Let \( \Sigma_{i/s/o} = \left( \begin{bmatrix} A \\ C \\ D \end{bmatrix} ; gph(P), U, Y \right) \) be the gph(P)-short circuit connection of \( \Sigma_{i/s/o}^1 \) and \( \Sigma_{i/s/o}^2 \), which according to Lemma 3.2.17 is a bounded i/o node. Then \( \Sigma_{i/s/o} \) is a bounded i/o representation of \( \Sigma \), and consequently \( \Sigma \) is bounded.

(iii) \( \Rightarrow \) (i): Since \( \Sigma, \Sigma_1, \) and \( \Sigma_2 \) have the same canonical input space \( W \) it follows from Theorem 2.2.27 that these three bounded i/o have the same boundedly i/o-admissible i/o representations. Let \( (U, Y) \) be such an i/o representation, and denote the corresponding i/o systems by \( \Sigma_{i/s/o}, \Sigma_{i/s/o}^1, \) and \( \Sigma_{i/s/o}^2 \). Then \( \Sigma_{i/s/o} \) is the gph(P)-short circuit of \( \Sigma_{i/s/o}^2 \) and \( \Sigma_{i/s/o}^1 \) since \( \Sigma \) is the gph(P)-short circuit...
of $\Sigma_2$ and $\Sigma_1$. Since $\Sigma_{i/s/o}$ is bounded, it follows from Lemma 3.2.17 that $\Sigma_{i/s/o}^1$ and $\Sigma_{i/s/o}^2$ are intertwined by $P$, and therefore by Proposition 2.5.50 also $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$.

The final claim follows from Lemma 3.2.17 and Proposition 2.5.50.

3.4.17. **Theorem.** Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two bounded s/s systems (with the same signal space). Let $\Sigma = (V; [X_i^1, X_i^2], W)$ be the two-sided short circuit of $\Sigma_2$ and $\Sigma_1$, i.e.,

\[
(3.4.16) \quad V = \left\{ \begin{bmatrix} z_2 \\ x_1 \\ x_2 \end{bmatrix} \in \begin{bmatrix} X_2^1 & X_2^2 \\ X_1^1 & X_1^2 \end{bmatrix} \begin{bmatrix} x_1 \\ w \end{bmatrix} \mid \begin{bmatrix} z_1 \\ x_1 \\ x_2 \\ w \end{bmatrix} \in V_1 \text{ and } \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \in V_2 \right\}
\]

(this s node will not be bounded in general). Then the following claims are true.

(i) $\Sigma_1$ and $\Sigma_2$ are intertwined by some closed $P \in \mathcal{ML}(X_1; X_2)$ if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent.

(ii) Suppose that $\Sigma_1$ and $\Sigma_2$ are externally equivalent. Let $\Sigma_{i/s/o}^j = \left( \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}; X_j, U, Y \right)$ be i/s/o representations of $\Sigma_j$ (with the same input and output spaces; by Theorem 2.2.27 and Lemma 3.4.13 such representations exist), and let $\Sigma_{i/s/o}$ be the difference connection of $\Sigma_{i/s/o}^1$ and $\Sigma_{i/s/o}^2$ (see Example 2.3.39). (Note that $\Sigma_{i/s/o}$ is not an i/s/o representation of $\Sigma$.) Then the following claims are true.

(a) There exists a unique minimal closed $P_{\min} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique closed $P_{\min} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P_{\min}) \subset \text{gph}(P)$ for any other closed $P$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\min}$ can be described in the following two equivalent ways:

1. $\text{gph}(P_{\min})$ coincides with the common forward and backward reachable subspace of $\Sigma$.
2. $\text{gph}(P_{\min})$ coincides with the common forward and backward reachable subspace of $\Sigma_{i/s/o}$.

(b) There exists a unique maximal $P_{\max} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique $P_{\max} \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P) \subset \text{gph}(P_{\max})$ for any other closed $P \in \mathcal{ML}(X_1; X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\max}$ can be described in the following three equivalent ways:

1. $\text{gph}(P_{\max})$ coincides with the set of all possible initial states of all generalized two-sided trajectories of $\Sigma$, and also with the set of all possible initial states of all classical two-sided trajectories of $\Sigma$;
2. $\text{gph}(P_{\max})$ coincides with the common forward and backward unobservable subspace of $\Sigma_{i/s/o}$.

In particular, $P_{\max}$ is closed. Thus, if $P$ is an arbitrary closed multi-valued operator which intertwines $\Sigma_1$ and $\Sigma_2$, then

$$
\text{gph}(P_{\min}) \subset \text{gph}(P) \subset \text{gph}(P_{\max}).
$$
Thus, \( i \) pose that coincide. This proves the alternative description (1) of \( \Sigma \) the backward reachable subspace of \( \Sigma \) is equal to the backward reachable subspace \( \Sigma \) equal to each other because of the assumption that \( \Sigma \) any additional restrictions on \( u \) where

\[
\begin{gathered}
\text{Proof. (i) By Lemma 3.4.16 if } \Sigma_1 \text{ and } \Sigma_2 \text{ are intertwined by some } P \in \mathcal{ML}(X_1; X_2), \text{ then they are externally equivalent. The converse part or } (i) \text{ follows from } (ii), \text{ which we shall prove next. (The proof of } (ii) \text{ does note use } (i).) \\
(ii) \text{ Suppose that } \Sigma_1 \text{ and } \Sigma_2 \text{ are externally equivalent. Then by Proposition 2.5.50 } \Sigma^1_{i/s/o} \text{ and } \Sigma^2_{i/s/o} \text{ are externally equivalent. By Theorem 3.2.18 there exists a unique minimal closed } P_{\min} \in \mathcal{ML}(X_1; X_2) \text{ which intertwines } \Sigma^1_{i/s/o} \text{ and } \Sigma^2_{i/s/o}, \text{ and there also exists a unique maximal } P_{\max} \in \mathcal{ML}(X_1; X_2) \text{ which intertwines } \Sigma^1_{i/s/o} \text{ and } \Sigma^2_{i/s/o}. \text{ The graphs of these operators are equal to the reachable subspace of } \Sigma_{i/s/o} \text{ respectively the unobservable subspace of } \Sigma_{i/s/o}. \text{ By Proposition 2.5.50 the same multi-valued operators also intertwine } \Sigma_1 \text{ and } \Sigma_2, \text{ and they are minimal respectively maximal among all closed } P_{\max} \in \mathcal{ML}(X_1; X_2) \text{ that interchange } \Sigma_1 \text{ and } \Sigma_2.
\end{gathered}
\]

We next prove the alternative characterization (1) in (ii)(a) of \( P_{\min} \). It follows from (3.4.16) that \( \begin{bmatrix} x_1(t) \\ w(t) \end{bmatrix} \) is a generalized trajectory of \( \Sigma \) on some interval \([0, T]\) if and only if \( \begin{bmatrix} x_1 \end{bmatrix} \) is a generalized trajectory of \( \Sigma \) on \([0, T]\), \( i = 1, 2 \). Therefore by Theorem 2.1.14 \( \begin{bmatrix} x_1(t) \\ w(t) \end{bmatrix} \) is a generalized trajectory of \( \Sigma \) on \([0, T]\) with \( \begin{bmatrix} x_2(0) \\ x_1(0) \end{bmatrix} = 0 \) if and only if for all \( t \in [0, T] \) and \( i = 1, 2, \)

\[
\begin{gathered}
x_i(t) = \int_0^t A_i^{t-s} B_i u(s) \, ds,
\end{gathered}
\]

\[
\begin{gathered}
y(t) = \int_0^t C_i A_i^{t-s} B_i u(s) \, ds + D_i u(t),
\end{gathered}
\]

where \( u = P^X_I w \in L^1([0, T]; U) \) and \( y = P^Y_I w. \) The formula for \( y \) does not impose any additional restrictions on \( u \) since the two different expressions for \( y \) are always equal to each other because of the assumption that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent, and hence \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are externally equivalent (see Theorem 3.2.7).

Thus, \( \begin{bmatrix} x_1(t) \\ w(t) \end{bmatrix} \) is a generalized trajectory of \( \Sigma \) on \([0, T]\) if and only if \( \begin{bmatrix} x_1 \end{bmatrix} \) is a generalized trajectory of \( \Sigma_{i/s/o} \) on \([0, T]\). This implies that the (forward) reachable subspace of \( \Sigma \) is equal to the (forward) reachable subspace of \( \Sigma_{i/s/o} \). The same computation can also be carried out in the backward time direction to show that the backward reachable subspace of \( \Sigma \) is equal to the backward reachable subspace of \( \Sigma_{i/s/o} \). By Lemma 3.2.4 the forward and backward reachable subspaces of \( \Sigma_{i/s/o} \) coincide. This proves the alternative description (1) of \( P_{\min} \) in (ii)(a).

It remains to prove the alternative description (1) of \( P_{\max} \) in (ii)(b). Suppose that \( \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \in \mathcal{U}\Sigma_{i/s/o} \), where \( \mathcal{U}\Sigma_{i/s/o} \) is the unobservable subspace of \( \Sigma_{i/s/o} \). By Lemma 3.2.4 this vector is both forward and backward unobservable, and hence \( \Sigma_{i/s/o} \) has a generalized unobservable two-sided trajectory \( \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \) satisfying \( \begin{bmatrix} x_2(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \). It follows from Theorem 2.1.14 that \( x_i(t) = A_i x_i^0, \) \( t \in \mathbb{R}, \) \( i = 1, 2. \) In particular, the trajectory \( \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \) of \( \Sigma_{i/s/o} \) is classical. Define \( y_i(t) = C_i A_i x_i^0, \) \( t \in \mathbb{R}^+, \) \( i = 1, 2. \) Then \( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \) is a classical two-sided trajectory of \( \Sigma_i, \) \( i = 1, 2, \) and
therefore \( \begin{bmatrix} \frac{[x_1]}{u} \\ y_2 - y_1 \end{bmatrix} \) is a classical two-sided trajectory of \( \Sigma_{i/s/o} \). Since such a trajectory is determined uniquely by its initial state and input function, it follows that it follows that \( y_2 - y_1 = 0 \), i.e., \( y_1 = y_2 \). Let us denote \( w = y_1 = y_2 \). Then \( \begin{bmatrix} \frac{[x_1]}{u} \\ w \end{bmatrix} \) is a classical two-sided trajectory of \( \Sigma \). This shows that \( P_{\text{max}} = \mathcal{U}_{\Sigma_{i/s/o}} \) is contained in the set of all possible initial states of all classical two-sided trajectories of \( \Sigma \).

Conversely, let \( \begin{bmatrix} \frac{[x_1]}{u} \\ w \end{bmatrix} \) be an arbitrary generalized two-sided trajectory of \( \Sigma \). Then \( \begin{bmatrix} x_i \\ y \end{bmatrix} \) is a generalized trajectory of \( \Sigma_i \), \( i = 1, 2 \), and \( \begin{bmatrix} x_1 \\ y_2 - y_1 \end{bmatrix} \) is a generalized trajectory of \( \Sigma_{i/s/o} \), \( i = 1, 2 \), where \( u = P_{ij}w \) and \( y = P_{ij}u \). By Theorem 2.1.14 for all \( t \in \mathbb{R} \) and \( j = 1, 2 \),

\[
x_j(t) = \mathcal{A}_j x_j(0) + \int_0^t \mathcal{A}_j^{t-s} B_j \, ds,
\]

(3.4.18)

\[
y(t) = C_j \mathcal{A}_j x_j(0) + \int_0^t C_j \mathcal{A}_j^{t-s} B_j u(s) \, ds + D_j u(t).
\]

For \( j = 1, 2 \) the above trajectory \( \begin{bmatrix} x_j \\ y \end{bmatrix} \) of \( \Sigma_{i/s/o} \) can be written as a sum of the two trajectories \( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \) and \( \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \), where

\[
\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 x_1(0) \\ C_1 \mathcal{A}_1 x_1(0) \end{bmatrix}, \quad \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \int_0^t \mathcal{A}_1^{t-s} B_1 ds \\ \int_0^t C_1 \mathcal{A}_1^{t-s} B_1 u(s) \, ds + D_1 u(t) \end{bmatrix}.
\]

Note that \( x_j(0) = x_0 \) for \( j = 1, 2 \). Since \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent, it follows from Proposition 2.5.50 and Theorem 3.2.7 that \( y_1 = y_2 \), and therefore also \( y_1 = y - y_2 = y - y_1 = y_2 \). This implies that \( \begin{bmatrix} x_1 \\ y_2 - y_1 \end{bmatrix} \) is un unobservable two-sided trajectory of \( \Sigma_{i/s/o} \), and hence \( \begin{bmatrix} x_1(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \in \mathcal{U}_{\Sigma_{i/s/o}} \). This proves the claim that \( P_{\text{max}} = \mathcal{U}_{\Sigma_{i/s/o}} \) coincides with the set of all possible initial states of all generalized two-sided trajectories of \( \Sigma \), and also with the set of all possible initial states of all classical two-sided trajectories of \( \Sigma \).

3.4.18. COROLLARY. Let \( \Sigma_i = (V_i, \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two bounded s/s systems (with the same input and output spaces). Moreover, suppose that both \( \Sigma_1 \) and \( \Sigma_2 \) are controllable and observable. (According to Theorem 3.4.26 below, this is equivalent to the assumption that both \( \Sigma_1 \) and \( \Sigma_2 \) is minimal.) Then \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. Among all the pseudo-similarities between \( \Sigma_1 \) and \( \Sigma_2 \) there is a (unique) minimal one \( P_{\text{min}} \) and a (unique) maximal one \( P_{\text{max}} \), namely those defined in Theorem 3.4.17 (both of which in this case are single-valued densely defined injective operators with dense range).

PROOF. If \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar, then it follows from Theorem 3.4.17(i) that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

Conversely, suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. By Theorem 1.5.27 \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some closed \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \). By Lemma 3.4.17 and the controllability assumption, dom(\( P \)) is dense in \( \mathcal{X}_1 \) and rng(\( P \)) is
dense in $X$. Furthermore, by Lemma 1.5.27 and the observability assumption, both $\ker(P) = \{0\}$ and $\text{mul}(P) = \{0\}$. Thus, $P$ is both injective and single-valued. □

### 3.4.6. Compressions of bounded s/s systems

At this point the reader may want to recall Definition 1.5.28 of what we mean by a compression of a s/s system.

**Lemma.** Let $\Sigma = (V; X, W)$ be a bounded s/s system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. Then the following conditions are equivalent:

(i) $\Sigma$ has a (forward) compression onto $X_1$ along $Z_1$;

(ii) $\Sigma$ has a backward compression onto $X_1$ along $Z_1$.

Suppose that these equivalent conditions hold, and define $V_1$ by

\[(3.4.19)\]

\[
V_1 = \begin{bmatrix}
P_{X_1}^Z & 0 & 0 \\
0 & 1_{X_1} & 0 \\
0 & 0 & 1_{W}
\end{bmatrix}
\begin{bmatrix}
V \\
X
\end{bmatrix}.
\]

Then the bounded s/s system $\Sigma_1 = (V_1; X_1, W)$ is both a forward and a backward compression of $\Sigma$ onto $X_1$ along $Z_1$, and $\Sigma_1$ is the unique bounded forward or backward compression of $\Sigma$ onto $X_1$ along $Z_1$. The two s/s systems $\Sigma_1$ and $\Sigma$ are both forward and backward externally equivalent.

**Proof.** (i) $\iff$ (ii): By Theorem 2.2.27, $\Sigma$ has a bounded i/s/o representation $\Sigma_{i/s/o}$. By Proposition 2.5.50 condition (i) and (ii) are equivalent to conditions (i) and (ii) in Theorem 3.2.20. Since conditions (i) and (ii) in Theorem 3.2.20 are equivalent to each other, it follows that also conditions (i) and (ii) above are also equivalent to each other.

The remaining claims follow from the corresponding claims in Theorem 3.2.20 applied to an arbitrary i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$, together with Proposition 2.5.50 and the representation formula (2.2.33) for $V$. □

**Lemma 3.4.19** differ from Theorems 3.4.14 and 3.4.15 in the sense that unlike Theorems 3.4.14 and 3.4.15 it does not give any necessary and sufficient conditions for the bounded s/s system $\Sigma$ to have a compression which are formulated directly in terms of properties of the generating subspace $V$ or the node and signal bundles $\hat{E}$ and $\hat{F}$ which are can be used to check if $\Sigma$. One such set of conditions is given in our next lemma.

**Lemma.** Let $\Sigma = (V; X, W)$ be a bounded s/s system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, define $V_1$ by (3.4.19), and let $\Sigma_1 = (V_1; X_1, W)$ be the bounded s/s system generated by $V_1$. Let $\Omega'$ be a subset of $\rho_{\text{bnd}}(\Sigma)$ which contains a cluster point in $\rho_{\text{bnd}}(\Sigma)$. Then the following conditions are equivalent:

(i) $\Sigma_1$ is a compression of $\Sigma$ onto $X_1$ along $Z_1$;

(ii) for all $\lambda \in \rho_{\text{bnd}}(\Sigma)$ the node bundles $\hat{E}$ and $\hat{E}_1$ of $\Sigma$ respectively $\Sigma_1$ satisfy

\[(3.4.20)\]

\[
\hat{E}_1(\lambda) = \begin{bmatrix}
1_{X_1} & 0 & 0 \\
0 & P_{X_1}^Z & 0 \\
0 & 0 & 1_{W}
\end{bmatrix}
\begin{bmatrix}
\hat{E}(\lambda) \\
X
\end{bmatrix};
\]

(iii) for all $\lambda \in \Omega'$ the node bundles $\hat{E}$ and $\hat{E}_1$ of $\Sigma$ respectively $\Sigma_1$ satisfy (3.4.20).
If these equivalent conditions hold, then \( \rho_{\infty}^{\text{bnd}}(\Sigma) \subset \rho_{\infty}^{\text{bnd}}(\Sigma_1) \).

Proof. (i) \( \Rightarrow \) (iii): If (i) holds, then by Lemma \[3.4.20\] \( \Sigma \) and \( \Sigma_1 \) are externally equivalent, and by Theorem \[2.2.27\] and Lemma \[3.4.13\] \( \Sigma \) and \( \Sigma_1 \) have bounded i/o representations \( \Sigma_{i/o} = ([A, B]; X, U, Y) \) respectively \( \Sigma_{i/o}^1 = ([A_1, B_1]; X_1, U, Y) \) (with the same input and output spaces). By Proposition \[2.5.50\] \( \Sigma_{i/o}^1 \) is a compression of \( \Sigma_{i/o} \). By using Lemma \[3.2.21\] and the representations \([3.4.1]\) of \( \tilde{E} \) and \( \tilde{E}_1 \) we get the equality \([3.4.20]\) for all \( \lambda \) in the unbounded component of \( \rho(A_1) \cap \rho(A_2) \). In particular, this implies (iii).

(ii) \( \Leftrightarrow \) (iii): This follows from Lemmas \[3.4.3\] and \[A.3.9\].

Proof. (ii) \( \Rightarrow \) (i): It follows from (ii) that the signal bundles of \( \Sigma \) and \( \Sigma_1 \) coincide in \( \rho_{\infty}^{\text{bnd}}(\Sigma) \), and hence by Lemma \[3.4.12\] \( \Sigma \) and \( \Sigma_1 \) are externally equivalent. By Theorem \[2.2.27\] and Lemma \[3.4.13\] \( \Sigma \) and \( \Sigma_1 \) have bounded i/o representations \( \Sigma_{i/o} = ([A, B]; X, U, Y) \) respectively \( \Sigma_{i/o}^1 = ([A_1, B_1]; X_1, U, Y) \) (with the same input and output spaces). This together with \([3.4.20]\) and the representations \([3.4.1]\) of \( \tilde{E} \) and \( \tilde{E}_1 \) implies that condition (v) in Lemma \[3.2.21\] holds with \( \Omega \) equal to the unbounded component of \( \rho(A_1) \cap \rho(A) \). Therefore by that lemma \( \Sigma_1 \) is a compression of \( \Sigma \) onto \( X_1 \) along \( Z_1 \).

It still remains to prove the final claim that \( \rho_{\infty}^{\text{bnd}}(\Sigma) \subset \rho_{\infty}^{\text{bnd}}(\Sigma_1) \). Let \( \lambda \in \rho_{\infty}^{\text{bnd}}(\Sigma) \), and let \( \Sigma_{i/o} = ([A, B]; X, U, Y) \) be an i/o representation of \( \Sigma \) satisfying \( \lambda \in \rho(A) \). Then by Lemma \[3.4.13\] \( \Sigma_1 \) has a bounded i/o representation \( \Sigma_{i/o}^1 = ([A_1, B_1]; X_1, U, Y) \) (with the same input and output spaces). We claim that \( \lambda \in \rho(A_1) \) (and hence \( \lambda \in \rho_{\infty}^{\text{bnd}}(\Sigma_1) \). Since \( \lambda \in \rho(A) \) the node bundle \( \tilde{E} \) of \( \Sigma \) has the representation \([3.4.1]\), and consequently, by \([3.4.20]\) the node bundle \( \tilde{E}_1 \) of \( \Sigma_1 \) has the representation \([3.4.1]\) with \( E = P_{X_1} \). By taking the \( U \)-component of \( w \) to be zero and ignoring the \( Y \)-component of \( w \) we get

\[
\begin{bmatrix}
1_{X_1} & 0 & 0 \\
0 & 1_{X_1} & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
Y
\end{bmatrix}
= \text{rng} \left( \begin{bmatrix}
P_{Z_1} & 1_{X_1} \\
(\lambda - A)^{-1} |_{X_1}
\end{bmatrix} \right).
\]

On the other hand, by instead using the fact that \( \tilde{E}_1(\lambda) = \begin{bmatrix}
-1_{X_1} & \lambda & 0 \\
0 & 1_{X_1} & 0 \\
0 & 0 & 1_{w}
\end{bmatrix} \begin{bmatrix}
V_1
\end{bmatrix} \) and the representation \([2.2.33]\) for \( V_1 \) we get

\[
\begin{bmatrix}
1_{X_1} & 0 & 0 \\
0 & 1_{X_1} & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
Y
\end{bmatrix}
= \text{rng} \left( \begin{bmatrix}
\lambda - A_1 \\
1_{X_1}
\end{bmatrix} \right).
\]

This implies that \( \lambda - A_1 \) has the bounded inverse \( P_{X_1} (\lambda - A)^{-1} |_{X_1} \), and consequently \( \lambda \in \rho(A_1) \).

\[\square\]

3.4.7. The general structure of a bounded s/s compression.

3.4.21. Lemma. Let \( \Sigma = (V; X, W) \) be a bounded s/s system, let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \), let \( W_{\text{ext}} = \begin{bmatrix} W \\ X_1 \\ X_1 \
\end{bmatrix} \), and let \( \Sigma_{\text{ext}} = (V_{\text{ext}}; X, W_{\text{ext}}) \) be the i/o extension of \( \Sigma \) with control operator equal to the embedding operator
3.4. INTERTWINEMENTS AND COMPRESSIONS OF BOUNDED S/S SYSTEMS

\( \mathcal{I}_1 : \mathcal{X}_1 \to \mathcal{X} \), observation operator \( P_{\mathcal{X}_1}^\mathcal{X} \), and feedthrough operator zero, i.e.,

\[
V_{\text{ext}} = \left\{ \begin{bmatrix} \begin{array}{c} z + u \\ x \\ w \\ u \end{array} \end{bmatrix} \in \begin{bmatrix} \mathcal{Z} \\ \mathcal{X} \end{bmatrix} | \left[ \begin{array}{c} z \\ x \\ w \end{array} \right] \in V, u \in \mathcal{X}_1 \right\}.
\]

Then the following claims are true.

(i) There exists a (unique) minimal closed strongly invariant subspace \( \mathcal{X}_{\text{min}} \) for \( \Sigma \) which contains \( \mathcal{X}_1 \) (i.e., \( \mathcal{X}_{\text{min}} \) is closed and strongly invariant for \( \Sigma \), and \( \mathcal{X}_{\text{min}} \) is contained in every other closed strongly invariant subspace of \( \Sigma \) which contains \( \mathcal{X}_1 \)). This subspace has the following alternative descriptions:

- (a) \( \mathcal{X}_{\text{min}} \) is the reachable subspace of \( \Sigma_{\text{ext}} \);
- (b) \( \mathcal{X}_{\text{min}} \) is equal to the subspace \( \mathcal{X}_{\text{min}} \) in Lemma 3.2.23 with \( \Sigma \) replaced by an arbitrary bounded i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \).

(ii) The space \( \mathcal{X}_{\text{min}} \) has the direct sum decomposition \( \mathcal{X}_{\text{min}} = \mathcal{X}_1 \bigoplus \mathcal{Z}_{\text{min}} \),

\[
(3.4.22)
\mathcal{Z}_{\text{min}} = \mathcal{X}_{\text{min}} \cap \mathcal{X}_1 = P_{\mathcal{X}_1}^\mathcal{X} \mathcal{X}_{\text{min}}.
\]

(iii) There exists a (unique) maximal unobservably invariant subspace \( \mathcal{Z}_{\text{max}} \) for \( \Sigma \) which is contained in \( \mathcal{Z}_{\text{ext}} \) (i.e., \( \mathcal{Z}_{\text{max}} \) is observably invariant for \( \Sigma \), and \( \mathcal{Z}_{\text{max}} \) contains every other unobservably invariant subspace for \( \Sigma \) which is contained in \( \mathcal{Z}_{\text{ext}} \)). This subspace has the following alternative descriptions:

- (a) \( \mathcal{Z}_{\text{max}} \) is the unobservable subspace of \( \Sigma_{\text{ext}} \);
- (b) \( \mathcal{Z}_{\text{min}} \) is equal to the subspace \( \mathcal{Z}_{\text{min}} \) in Lemma 3.2.23 with \( \Sigma \) replaced by an arbitrary bounded i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \).

In particular, \( \mathcal{Z}_{\text{max}} \) is closed.

**Proof.** This follows from Proposition 2.5.49 and Lemma 3.2.23 and the fact that \( \Sigma_{i/s/o} = ([A_1 B_1]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is a bounded i/s/o representation of \( \Sigma \) if and only if \( \Sigma_{1i/s/o} = ([A_1^1 B_1^1]; \mathcal{X}, [\mathcal{U}], [\mathcal{X}]) \) is a bounded i/s/o representation of \( \Sigma_1 \) where \( [A_1^1 B_1^1] \) is the operator in (3.2.15). \( \square \)

3.4.22. LEMMA. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a bounded s/s system, and let \( \mathcal{X} = \mathcal{X}_1 \bigoplus \mathcal{Z}_1 \) be a direct sum decomposition of \( \mathcal{X} \). Let \( \Omega' \) be a subspace of \( \rho_{\text{bnd}}^{\text{bnd}}(\Sigma) \) which has a cluster point in \( \rho_{\text{bnd}}^{\text{bnd}}(\Sigma) \), where \( \rho_{\text{bnd}}^{\text{bnd}}(\Sigma) \) is the unbounded component of \( \rho(\Sigma) \). Denote the characteristic node bundle of \( \Sigma \) by \( \mathcal{E} \). Then the following claims are true.

(i) The subspace \( \mathcal{X}_{\text{min}} \) in Lemma 3.4.21 can be computed in the following ways:

\[
(3.4.23)
\mathcal{Z}_{\text{min}} = \bigvee_{\lambda \in \rho_{\text{bnd}}^{\text{bnd}}(\Sigma)} \begin{bmatrix} 0 & 1 \mathcal{X} & 0 \end{bmatrix} \left( \hat{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \right).
\]
(ii) The subspace $Z_{\text{min}}$ in Lemma 3.4.21 can be computed in the following ways:

\[
Z_{\text{min}} = \bigvee_{\lambda \in \rho_{\text{bnd}}(\Sigma)} \begin{bmatrix} 0 & P_{X_1} \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X_1 \\ W \end{bmatrix} \right)
\]

(3.4.24)

\[
= \bigvee_{\lambda \in \Omega'} \begin{bmatrix} 0 & 0 \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X_1 \\ W \end{bmatrix} \right).
\]

(iii) The subspace $Z_{\text{max}}$ in Lemma 3.4.21 can be computed in the following ways:

\[
Z_{\text{max}} = \bigwedge_{\lambda \in \rho_{\text{bnd}}(\Sigma)} \begin{bmatrix} 1 & 0 \\ Z_1 & Z_1 \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X_1 \\ Z_1 \{0\} \end{bmatrix} \right)
\]

(3.4.25)

\[
= \bigwedge_{\lambda \in \Omega'} \begin{bmatrix} 1 & 0 \\ Z_1 & Z_1 \{0\} \end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X_1 \\ Z_1 \{0\} \end{bmatrix} \right).
\]

Proof. Proof of (i): Let $\Sigma_{i/s/o} = ([A, B]; X, U, Y)$ be a bounded $i/s/o$ representation of $\Sigma$. It follows from Lemma 3.2.4 and the representation formula (3.4.1b) that (3.4.23) holds with $\Omega' = \rho_{\text{bnd}}(A)$. By Lemma 3.4.3 the bundle $\hat{E}(\lambda) \cap \begin{bmatrix} X_1 \\ W \end{bmatrix}$ is analytic in $\rho_{\text{bnd}}(\Sigma)$. Since $[0 \ 1 \ 0 \ 0]$ is the projection onto $\begin{bmatrix} \{0\} \\ X \{0\} \end{bmatrix}$ along $\begin{bmatrix} \{0\} \\ W \{0\} \end{bmatrix}$ in $\begin{bmatrix} X \\ W \end{bmatrix}$, this combined with Lemmas 3.4.3 and A.3.10 gives (i).

Proof of (ii): The proof of (ii) is analogous to the proof of claim (v) in Lemma 3.4.11.

Proof of (iii): This proof is analogous to the proof of (ii).

□

3.4.23. Theorem. Let $\Sigma = (V; X, W)$ be a bounded $s/s$ system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. Let $X_{\text{min}}$ be the minimal closed strongly invariant subspace of $\Sigma$ which contains $X_1$, let $Z_{\text{max}}$ be the maximal unobservably invariant subspace of $\Sigma$ which is contained in $Z_1$, and let $Z_{\text{min}} = X_{\text{min}} \cap Z_1$ (cf. Lemma 3.4.21). Then the following conditions are equivalent:

(i) $\Sigma$ has a compression onto $X_1$ along $Z_1$;

(ii) $Z_1$ contains some closed unobservably invariant subspace $Z$ for $\Sigma$ such that $X_1 + Z$ is strongly invariant for $\Sigma$;

(iii) $Z_{\text{min}}$ in an unobservably invariant subspaces for $\Sigma$;

(iv) $X + Z_{\text{max}}$ is a strongly invariant subspace for $\Sigma$;

(v) $Z_{\text{min}} \subset Z_{\text{max}}$.

Two possible choices of the subspace $Z$ in (ii) are $Z = Z_{\text{min}}$ and $Z = Z_{\text{max}}$, and every possible subspace $Z$ in (ii) satisfies $Z_{\text{min}} \subset Z \subset Z_{\text{max}}$.

Suppose that the equivalent conditions (i)–(v) hold, and define $V_1$ by (3.4.19). Then the bounded $s/s$ system $\Sigma_1 = (V_1; X, W)$ is a compression of $\Sigma$ onto $X_1$ along $Z_1$, and $\Sigma_1$ is the unique bounded forward or backward compression of $\Sigma$ onto $X_1$ along $Z_1$.

Proof. This follows from Proposition 2.5.50 and Theorem 3.2.25 applied to an arbitrary $i/s/o$ representation $\Sigma_{i/s/o}$ of $\Sigma$. □
3.4.24. Theorem. Let $\Sigma = (V; X, W)$ be a bounded s/s system, and let $X = X_1 \oplus Z_1$ be a direct sum decomposition of $X$, and suppose that $\Sigma_1 = (V_1; X_1, W)$ is a bounded compression of $\Sigma$ onto $X_1$ along $Z_1$. Let $Z$ satisfy the conditions listed in (ii) in Theorem 3.4.23 and let $Z_c$ be an arbitrary direct complement to $Z$ in $Z_1$.

(i) Let $\Sigma_2$ be the unique bounded restriction of $\Sigma$ to the strongly invariant subspace $X_1 \oplus Z$ for $\Sigma$ given by Theorem 3.4.14. Then $Z$ is unobservably invariant for $\Sigma_2$, and $\Sigma_1$ is the unique bounded projection onto $X_1$ along $Z$ of $\Sigma_2$ given by Theorem 3.4.14.

(ii) Let $\Sigma_3$ be the unique bounded projection of $\Sigma$ onto $X_1 \oplus Z_c$ along $Z$ given by Theorem 3.4.14. Then $X_1$ is strongly invariant for $\Sigma_3$, and $\Sigma_1$ is the unique bounded restriction of $\Sigma_3$ to $X_1$ given by Theorem 3.4.14.

Proof. This follows from Proposition 2.5.50 and Theorem 3.2.27 applied to an arbitrary i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$.

3.4.25. Lemma. Let $\Sigma = (V; X, W)$ and $\Sigma_1 = (V; X, W)$ be two bounded s/s systems with $X = X_1 \oplus Z_1$. Then the following two conditions are equivalent.

(i) $\Sigma_1$ is the compression of $\Sigma$ onto $X_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z$ such that $\Sigma$ and $\Sigma_1$ are intertwined by the operator $P_{X_1}^{|X_1 \oplus Z}$.

Condition (ii) above holds for some particular subspace $Z$ if and only condition (ii) in Theorem 3.4.23 holds for the same subspace $Z$. Thus, in particular, two possible choices of the subspace $Z$ in (ii) are the subspaces $Z = Z_{\min}$ and $Z = Z_{\max}$ defined in Lemma 3.4.27, and every possible subspace $Z$ satisfies $Z_{\min} \subset Z \subset Z_{\max}$.

Proof. This follows from Proposition 2.5.50 and Lemma 3.2.28 applied to an arbitrary i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$.

3.4.8. Compressions into minimal bounded s/s systems.

3.4.26. Theorem. A bounded s/s system $\Sigma = (V; X, W)$ is minimal if and only if $\Sigma$ is both controllable and observable.

Proof. This follows from Proposition 2.5.49 and Theorem 3.2.29 applied to an arbitrary i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$.

As the following theorem shows, every bounded s/s system can be compressed into a minimal bounded s/s system. This result can be used, e.g., to prove the existence of minimal bounded s/s realizations of a given bundle which is analytic at infinity.

3.4.27. Theorem. Every bounded i/s/o system $\Sigma$ has a bounded minimal compression. Two families of such compressions are described below, where we have denoted the reachable and unobservable subspaces of $\Sigma$ by $R_\Sigma$ respectively $U_\Sigma$:

(i) Let $X_1$ be a direct complement to $U_\Sigma$ in $X$, and let $X_o = P_{U_\Sigma}^{|X_1} R_\Sigma$. Define

$$V_o = \left( \begin{array}{c} P_{U_\Sigma}^{|X_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} X_1 \\ X_o \\ W \end{array} \right) \bigcap \left( \begin{array}{c} X_1 \\ X_o \\ W \end{array} \right).$$

Then $\Sigma_o = (V_o; X_o, W)$ is a minimal bounded compression of $\Sigma$. This compression is the bounded s/s system that one gets by first projecting
Σ onto \( X_1 \) along its unobservable subspace \( \mathcal{U}_\Sigma \), and then restricting the resulting system to its reachable subspace \( \mathcal{R}_\Sigma \).

(ii) Let \( \mathcal{X}_* \) be a direct complement to \( \mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma \) in \( \mathcal{R}_\Sigma \), and define

\[
V_* = \begin{bmatrix}
P_{\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma}^* & 0 & 0 \\
0 & P_{\mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma}^* & 0 \\
0 & 0 & 1_W
\end{bmatrix}
\]

Then \( \Sigma_* = (V_*; \mathcal{X}_*, W) \) is a bounded minimal compression of \( \Sigma \). This compression is the system that one gets by first restricting \( \Sigma \) to its reachable subspace \( \mathcal{R}_\Sigma \), and then projecting the resulting system onto \( \mathcal{X}_* \) along its unobservable subspace \( \mathcal{R}_\Sigma \cap \mathcal{U}_\Sigma \).

Proof. This follows from Proposition 2.5.50 and Theorem 3.2.31 applied to an arbitrary i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \). \( \square \)

3.4.28. Lemma. The minimal compression of a bounded s/s system \( \Sigma = ([A B]; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is unique if and only at least one of conditions (i) and (ii) below holds:

(i) \( \Sigma \) is observable, i.e., \( \mathcal{U}_\Sigma = \{0\} \), where \( \mathcal{U}_\Sigma \) is the unobservable subspace of \( \Sigma \).

(ii) the following equivalent conditions hold:

(a) \( \Sigma \) has a compression with state space \( \{0\} \),

(b) the characteristic node signal of \( \Sigma \) is a constant in \( \rho^{\text{bnd}}(\Sigma) \),

(c) \( \mathcal{R}_\Sigma \subset \mathcal{U}_\Sigma \), where \( \mathcal{R}_\Sigma \) is the reachable subspace of \( \Sigma \).

In case (i) the unique minimal compression \( \Sigma_{\text{min}} \) is the restriction of \( \Sigma \) to \( \mathcal{R}_\Sigma \), i.e., \( \Sigma_{\text{min}} = (V_{\text{min}}; \mathcal{R}_\Sigma, W) \) where

\[
V_{\text{min}} = V \cap \begin{bmatrix} \mathcal{R}_\Sigma \\ \mathcal{R}_\Sigma \\ W \end{bmatrix}.
\]

In case (ii) the unique minimal compression is \( \Sigma_{\text{min}} = (V_{\text{min}}; \{0\}, W) \) where

\[
V_{\text{min}} = \begin{bmatrix} \{0\} \\ \{0\} \\ W_0 \end{bmatrix},
\]

where \( W_0 \) is the canonical input space of \( \Sigma \) defined in (1.1.7). If neither (i) nor (ii) holds, then \( \Sigma \) has an infinite number of minimal compressions.

Proof. This follows from Proposition 2.5.50 and Lemma 3.2.32 applied to an arbitrary i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \) (if the characteristic signal bundle \( \hat{\mathcal{F}} \) of \( \Sigma \) is a constant in a neighborhood of infinity, then by analyticity \( \hat{\mathcal{F}} \) is a constant in all of \( \rho^{\text{bnd}}(\Sigma) \)). \( \square \)
3.5. Discrete Time State/Signal Systems (Jan 02, 2016)

Up to now we have interpreted a bounded s/s node \( \Sigma = (V, X, U, Y) \) as the generator of a continuous time s/s system system, but the same node also generates the discrete s/s time system

\[
\begin{bmatrix}
  x(n+1) \\
  x(n) \\
  w(n)
\end{bmatrix} \in V, \quad n \in I,
\]

where \( I \) is some discrete time interval. Here \( u \) and \( y \) are sequences defined on \( I \) with values in \( U \) respectively \( Y \), and \( x \) is a sequence with values in \( X \) which is defined on \( I_{\text{ext}} \) (see Notation 3.3.1). The theory of (bounded) s/s systems is analogous to the theory of (bounded) discrete time i/s/o systems presented in Section 3.3. As in the continuous time setting, discrete time i/s/o systems can be used as i/s/o representations of discrete time s/s systems.

3.5.1. Introduction to discrete time s/s systems.

3.5.1. Definition. Let \( \Sigma = (V, X, U, Y) \) be a bounded s/s node.

(i) By a discrete time trajectory \( [x \ w] \) generated by \( \Sigma \) on the discrete time interval \( I \) we mean a pair of sequences \( x = \{x(n)\}_{n \in I} \) with values in \( X \) respectively \( W \) which satisfy (3.5.1).

(ii) By the (bounded) discrete time i/s/o (input/state/output) system induced by \( \Sigma \) we mean the node \( \Sigma \) itself together with sets of all discrete time trajectories generated by \( \Sigma \). We use the same notation \( \Sigma = (V, X, U, Y) \) for the discrete time s/s system as for bounded s/s node defining this system.

3.5.2. Definition. By a future, past, or two-sided trajectory of a discrete time i/s/o system \( \Sigma = ([A \ B; C \ D]; X, U, Y) \) we mean a trajectory defined on \( \mathbb{Z}^+, \mathbb{Z}^-, \) or \( \mathbb{Z} \), respectively.

Note that in discrete time there is no need to distinguish between “classical” and “generalized” trajectories: all trajectories are automatically “classical” in the sense that they all satisfy (3.5.1).

3.5.3. Lemma. Let \( \Sigma = (V, X, U, Y) \) be a bounded s/s node, and let \( \Sigma_{i/s/o} = (S; X, U, Y) \) be a bounded i/s/o representation of \( \Sigma \) (by Theorem 2.2.27 such a bounded i/s/o representation always exists). Let \( I \) be a discrete time interval.

(i) A pair \( [x \ w] \) is a trajectory of \( \Sigma \) on \( I \) if and only if \( [x \ u] \) is a trajectory of \( \Sigma_{i/s/o} \) on \( I \), where \( u = P_Y^X w \) and \( y = P_Y^U w \).

(ii) If \( I \) has a finite end-point \( m \), then for each \( x^0 \in X \) and each \( U \)-valued sequence \( \{u(n)\}_{n \in I} \) \( \Sigma \) has a unique trajectory \( [x \ w] \) satisfying \( x(m) = x^0 \) and \( P_Y^U w = u \).

Proof. This follows from Lemma 3.3.4 together with the representation (2.2.33) of \( V \) and the respective definitions of trajectories of discrete time s/s and i/s/o systems.

3.5.4. Remark. In this book we shall only discuss discrete time s/s systems which are bounded, i.e., systems induced by bounded s/s nodes. This condition can be interpreted as a natural “well-posedness” condition in discrete time, since boundedness of \( \Sigma \) is a necessary and sufficient condition for \( \Sigma \) to have a bounded
i/s/o representation (see Theorem 2.2.27). It follows from Lemma 3.5.3 that every discrete time system is “uniquely solvable” and has the “continuation property” in the sense of the (forward) discrete time version of Definition 1.3.3. See also Remark 3.3.6.

3.5.2. Properties of discrete time s/s systems.

3.5.5. Definition. Let $\Sigma = (V; X, W)$ be a discrete time system. Then the following discrete times notions are defined analogously to the corresponding continuous time notions (in the forward time direction):

(i) $(P, R, Q)$-similarity, $P$-similarity, and similarity of discrete time s/s systems (see Definition 1.2.11);

(ii) the $(P, Q)$-image of a discrete time i/s/o system (see Definition 1.2.14);

(iii) the part and static projection of a discrete time s/s system (see Definition 1.2.17);

(iv) bounded i/o extensions of a discrete time s/s system (see Definition 1.2.21);

(v) the cross product of two discrete time i/s/o systems (see Definition 1.2.25);

(vi) exactly reachable states, reachable states, and unobservable trajectories and states of a discrete time s/s system (see Definition 1.5.1);

(vii) the exactly reachable subspace, the reachable subspace, and the unobservable subspace of a discrete time s/s system (see Definition 1.5.3);

(viii) strongly and unobservable invariance subspaces of a discrete time s/s system (see Definition 1.5.8);

(ix) controllability and observability of a discrete time s/s system (see Definition 1.5.11);

(x) external equivalence of two discrete time systems (see Definition 1.5.21);

(xi) intertwinement and pseudo-similarity of two discrete time systems (see Definitions 1.5.22 and 1.5.23);

(xii) compressions, dilations, restrictions, and projections of discrete time systems (see Definitions 1.5.28, 1.5.33, and 1.5.37);

(xiii) minimality of discrete time s/s systems (see Definition 1.5.41).

As the following lemma says, most of the results that we have proved for general (bounded or non-bounded) s/s systems in continuous time remain valid for s/s systems in discrete time in the forward time direction. (See the next section for a discussion of how the forward and backward time directions are related to each other in the discrete time case.)

3.5.6. Lemma. The analogues of the following results about s/s systems remain valid for discrete time s/s systems:

(i) Lemmas 1.2.13, 1.2.23, 1.2.24, and 1.2.26 (about trajectories of similarity transformed systems, i/o extended systems, and cross products);

(ii) Lemmas 1.5.4, 1.5.9, 1.5.10, and Corollary 1.5.12 (about the reachable and unobservable subspaces);

(iii) Lemma 1.5.24, 1.5.26, 1.5.27, 1.5.29, 1.5.30, 1.5.32, 1.5.36, and 1.5.40 (about intertwinements, pseudo-similarity, compressions, restrictions, and projections);

(iv) Lemma 1.5.44, 1.5.45, 1.5.46, 1.5.47, and 1.5.48 (about results which are true for s/s systems which have the continuation property).
3.5.3. **Time reflection of discrete time s/s systems.** As in the case of input/state/output systems, the operation of time reflection more complicated in the discrete time setting than in the continuous time setting, due to the fact that we (in this book) require the time reflection of a bounded discrete time system to be bounded. However, the formal description of a time reflected discrete time s/s system is simpler than the corresponding description of a time reflected continuous time system.

By repeating the same argument that we gave for the discrete i/s/o case we find that the appropriate time reflection of a discrete time trajectory \[ x(w) \] of \( \Sigma \) on the discrete time interval \( I \) is given by \[ x(w) = x(-m) \], \( m \in I \) \( \mathbb{R} = \left\{ -(n+1) \mid n \in I \right\} \), and that the discrete time reflection of \( \Sigma \) should be the system \( \Sigma_{R} = (V_{R}, X, W) \), where

\[
V_{R} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ \end{bmatrix} V.
\]

3.5.7. **Definition.** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a discrete time system. 

(i) \( \Sigma \) is time-invertible if the node \( \Sigma_{R} \) in (3.5.2) is bounded. 

(ii) If \( \Sigma \) is time-invertible, then by the time reflection of \( \Sigma \) we mean the discrete time system \( \Sigma_{R} = (V_{R}, \mathcal{X}, \mathcal{W}) \) where \( V_{R} \) is given by (3.5.3).

For time-invertible s/s systems it is possible to formulate an analogue of part (ii) of Lemma 3.5.3 in the backward time direction, but in this book we shall not explicitly study discrete time s/s systems in the backward time direction.

3.5.4. **Connections between continuous and discrete time s/s properties.** Earlier in this chapter we have proved a number or results which are true for bounded s/s systems in continuous time. It turns out that most of these remain valid for s/s systems in discrete time in a somewhat unexpected sense: Most of the properties that we have discussed in this chapter are of the following type:

3.5.8. **Principle.** If \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) is a bounded s/s node, then the continuous time system generated by \( \Sigma \) has a certain property if and only if the discrete time system generated by \( \Sigma \) has the same property (in the forward time direction).

Below we shall list several instances where this principle is valid.

3.5.9. **Lemma.** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) and \( \Sigma_{i} = (V_{i}; \mathcal{X}_{i}, \mathcal{W}_{i}) \) be bounded s/s nodes. Then the following claims are true: (in the list below, when we say “in continuous time” or “in discrete time” we refer to the continuous respectively discrete time system generated by \( \Sigma \), and we only include the forward time direction).

(i) A closed subspace \( \mathcal{Z} \) is strongly or unobservably invariant in continuous time if and only if \( \mathcal{Z} \) is strongly respectively unobservably invariant in discrete time.

(ii) The continuous time reachable and unobservable subspaces of \( \Sigma \) coincide with the discrete time reachable respectively unobservable subspaces of \( \Sigma \).

(iii) \( \Sigma_{1} \) and \( \Sigma_{2} \) are externally equivalent in continuous time if and only if \( \Sigma_{1} \) and \( \Sigma_{2} \) are externally equivalent in discrete time.
(iv) \( \Sigma_1 \) is a continuous time restriction, or projection, or compression of \( \Sigma_2 \) if and only if \( \Sigma_1 \) is a discrete time restriction, or projection, or compression, respectively, of \( \Sigma_2 \).

(v) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \in \mathcal{ML}(X_1;X_2) \) in continuous time if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \) in discrete time.

(vi) The continuous time versions of the subspaces \( X_{\min}, Z_{\min}, \) and \( Z_{\max} \) in Lemma 3.4.21 coincide with the corresponding discrete time versions of these subspaces.

(vii) Discrete time compressions of \( \Sigma_2 \) have the same structure as the continuous time compressions of \( \Sigma_2 \) described in Theorem 3.4.23.

(viii) \( \Sigma \) is minimal in continuous time if and only if \( \Sigma \) is minimal in discrete time.

(ix) \( \Sigma \) is minimal in discrete time if and only if \( \Sigma \) is both controllable and observable in discrete time.

(x) The families of continuous time minimal compressions of \( \Sigma \) described in Theorem 3.4.27 can also be interpreted as discrete time minimal compressions of \( \Sigma \).

(xi) A minimal compression of \( \Sigma \) is unique in continuous time if and only if it is unique in discrete time.

Proof. The proofs of these claims are straightforward. In addition to the representation formulas (2.1.19) and (3.3.2) for trajectories in continuous respectively discrete time they use the following results from this chapter:

(i) See Lemmas 3.4.5 and 3.4.8.

(ii) See Lemma 3.4.11.

(iii) See Lemma 3.4.12.

(iv) See Theorems 3.4.14 and 3.4.15.

(v) See Lemma 3.4.16.

(vi) See Lemma 3.4.22.

(vii)–(xi) These claims follow from (iv). \( \square \)

The results presented in Lemma 3.5.9 make it possible to transfer the dynamical notions that we have defined separately for the continuous time and the discrete time s/s systems generated by a bounded s/s into a notion which applies to the node itself.

3.5.10. Definition. Let \( \Sigma = (V;X,W) \) and \( \Sigma_i = (V_i;X_i,W_i) \) be bounded s/s nodes.

(i) A closed subspace \( Z \) of \( X \) is strongly invariant for \( \Sigma \) if \( Z \) is strongly invariant for the continuous time i/s/o system induced by \( \Sigma \), or equivalently, \( Z \) is strongly invariant for the discrete time i/s/o system induced by \( \Sigma \) (see Lemma 3.4.5(iii)–(ix) for additional equivalent conditions).

(ii) A closed subspace \( Z \) of \( X \) is unobservably invariant for \( \Sigma \) if \( Z \) is unobservably invariant for the continuous time i/s/o system induced by \( \Sigma \), or equivalently, \( Z \) is unobservably invariant for the discrete time i/s/o system induced by \( \Sigma \) (see Lemma 3.4.8(iii)–(ix) for additional equivalent conditions).

(iii) The subspace \( R_\Sigma \) defined in Lemma 3.4.11 is called the reachable subspace of \( \Sigma \) (this is the common reachable subspace of both the continuous time and the discrete time i/s/o system induced by \( \Sigma \)).
(iv) The subspace $U_{\Sigma}$ defined in (3.4.12) is called the unobservable subspace of $\Sigma$ (this is the common unobservable subspace of both the continuous time and the discrete time i/s/o system induced by $\Sigma$).

(v) $\Sigma_1$ and $\Sigma_2$ are externally equivalent if the two continuous time i/s/o systems induced by $\Sigma_i$, $i = 1, 2$, are externally equivalent, or equivalently, if the two continuous time i/s/o systems induced by $\Sigma_i$, $i = 1, 2$, are externally equivalent (see Lemma 3.4.12(iii)–(v) for additional equivalent conditions).

(vi) $\Sigma_1$ and $\Sigma_2$ are intertwined by the closed multi-valued operator $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ if the two continuous time i/s/o systems induced by $\Sigma_i$, $i = 1, 2$, are intertwined by $P$, or equivalently, if the two continuous time i/s/o systems induced by $\Sigma_i$, $i = 1, 2$, are intertwined by $P$ (see Lemma 3.4.16(iii)–(iv) for additional equivalent conditions).

(vii) $\Sigma_1$ is the restriction to $\mathcal{X}_1$, or projection or compression onto $\mathcal{X}_1$ along $Z_1$ of $\Sigma_2$ if the continuous time i/s/o system induced by $\Sigma_1$ is the restriction to $\mathcal{X}_1$, or projection or compression onto $\mathcal{X}_1$, respectively, along $Z_1$ of the continuous time i/s/o system induced by $\Sigma_2$, or equivalently, if the continuous time i/s/o system induced by $\Sigma_1$ is the restriction to $\mathcal{X}_1$, or projection or compression onto $\mathcal{X}_1$, respectively, along $Z_1$ of the continuous time i/s/o system induced by $\Sigma_2$ (see Theorems 3.4.14 and 3.4.15 and Lemma 3.4.20 for additional equivalent conditions).

(viii) $\Sigma$ is minimal if the continuous time i/s/o system induced by $\Sigma$ is minimal, or equivalently, if the discrete time i/s/o system induced by $\Sigma$ is minimal.
All results in this chapter remain valid if we throughout replace $L^1$ by $L^p$ where $1 \leq p < \infty$.

All results in this chapter remain valid if we allow the state spaces of the i/s/o and s/s systems to be $B$-spaces instead of $H$-spaces, except for the results about existence of minimal compressions of a given i/s/o or s/s system, i.e., Theorems 3.2.29, 3.2.31, 3.4.26, and 3.4.27 and Lemmas 3.2.32 and 3.4.28. However, even these excluded results can be extensions to the case where the state space is a $B$-space instead of an $H$-space. Restrictions to strongly invariant subspaces can be carried out as in the $H$-space setting, but the projection onto a direct complement to an unobservably invariant subspace must be replaced by a quotient over the unobservably invariant subspace if such a direct complement does not exits. This procedure is described in [Staffans, 2005, Theorem 9.1.9] in a well-posed setting, and when that result is applied to a bounded i/s/o system the resulting system will be bounded.

All results in this chapter about i/s/o systems remain valid if we allow the input and output spaces of the i/s/o systems to be $B$-spaces instead of $H$-spaces. However, since Theorem 2.2.29 is not valid in a $B$-spaces setting unless $W_0$ is complemented, and since we make heavy use of bounded i/s/o representations in Sections 3.4 and 3.5 many of the proofs of the results about bounded s/s systems in this chapter are not valid in a $B$-space setting without the additional assumption that the canonical input space $W_0$ is complemented.
In this chapter we present the classes of semi-bounded i/s/o and s/s systems which are generalizations of the classes of bounded i/s/o and s/s systems. As in the case of a bounded i/s/o system the generating operator $S$ of a semi-bounded i/s/o system has a block matrix decomposition $S = [A \ B]$, where $B$, $C$, and $D$ are bounded operators, but this time the main operator $A$ is allowed to be the (possibly unbounded) generator of a $C_0$ semigroup. A s/s system is semi-bounded if it has at least one semi-bounded i/s/o representation. Most of the results presented in Chapter 3 for bounded i/s/o and s/s systems remain valid in one form or another for semi-bounded i/s/o and s/s systems, but there is one major exception: In the case of a bounded i/s/o or s/s system the direction of time did not play an important role: most results that were true in the forward time direction were also true in the backward time direction. This is no longer true in the semi-bounded case.
4.1. $C_0$ Semigroups and Their Generators (Feb 02, 2016)

The most striking difference between the theory of bounded i/s/o systems and the theory of semi-bounded i/s/o systems can be seen already in the special case where both the input space $U = \{0\}$ and the output space $Y = \{0\}$ and (2.1.1) can be rewritten in the form

\begin{equation}
\begin{cases}
x(t) \in \text{dom}(A), \\
\dot{x}(t) = Ax(t), 
\end{cases}
t \in I.
\end{equation}

In this section we shall most of the time take $I = \mathbb{R}^+$ and also specify an initial state $x_0$, i.e., we replace (4.1.1) by

\begin{equation}
\begin{cases}
x(t) \in \text{dom}(A), \\
\dot{x}(t) = Ax(t), 
\end{cases}
t \in \mathbb{R}^+, 
\end{equation}

In the literature this equation is known under the name homogeneous Cauchy problem.

4.1.1. The homogeneous Cauchy problem.

4.1.1. Definition. Let $A$ be a closed linear operator in an $H$-space $\mathcal{X}$.

(i) We call $x$ a classical solution of the homogeneous Cauchy problem (4.1.2) with initial state $x^0 \in \text{dom}(A)$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and (4.1.2) holds.

(ii) We call $x$ a generalized solution of the homogeneous Cauchy problem (4.1.2) with initial state $x^0 \in \mathcal{X}$ if $x \in C(\mathbb{R}^+; \mathcal{X})$ is the limit in $C(\mathbb{R}^+; \mathcal{X})$ of a sequence of classical solutions $x_n$ of the homogeneous Cauchy problem (4.1.2) with initial states $x^0_n$ satisfying $x^0_n \to x^0$ in $\mathcal{X}$ as $n \to \infty$.

(Thus, in particular, $x(0) = x^0$.)

(iii) We call $x$ a mild solution of the homogeneous Cauchy problem (4.1.2) with initial state $x^0 \in \mathcal{X}$ if $x \in C(\mathbb{R}^+; \mathcal{X})$ and

\begin{equation}
\begin{cases}
\int_0^t x(s) \, ds \in \text{dom}(A), \\
x(t) = x^0 + A \int_0^t x(s) \, ds, 
\end{cases}
t \in \mathbb{R}^+.
\end{equation}

4.1.2. Lemma. Let $A$ be a closed linear operator in an $H$-space $\mathcal{X}$.

(i) $x$ is a classical, generalized, or mild solution of the Cauchy problem (4.1.2) if and only if $x$ is a classical, generalized, or mild future trajectory, respectively, of the i/s/o system $\Sigma = (A, \mathcal{X}, \{0\}, \{0\})$.

(ii) The following conditions are equivalent:

(a) $x$ is a generalized solution of the homogeneous Cauchy problem (4.1.2).

(b) $x$ is a mild solution of the homogeneous Cauchy problem (4.1.2).

(c) the pair $[f_1, f_2]$, where $x_1(t) := \int_0^t x(s) \, ds$ and $f_1(t) := x^0$, $t \in \mathbb{R}^+$, is a classical future trajectory of the i/s/o system $\Sigma_1 = ([A \ 1_\mathcal{X}], \mathcal{X}, \{0\})$.

(iii) $x$ is a classical solution of the Cauchy problem (4.1.2) if and only if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and $x$ is a generalized (or mild) solution of the Cauchy problem (4.1.2).

Proof. (i) Claim (i) follows from Definitions 2.1.7, 2.4.19 and 4.1.1 and Lemma 2.4.21.
(ii) The equivalence of (a) and (b) follows from (i) and Lemmas 2.4.20 and 2.4.27 and the equivalence of (b) and (c) follows from (i) and Definitions 2.1.7, 2.4.19, and 4.1.1 and Lemma 2.4.21.

(iii) Claim (iii) follows from claim (i) and Theorem 2.4.32. □

4.1.3. Remark. It follows from part (i) of Lemma 4.1.2 that all the results that we have proved in Section 2 about classical, generalized, and mild trajectories of i/s/o systems are also valid for classical, generalized, and mild solutions of the homogeneous Cauchy problem (4.1.2).

At this point the reader may want to recall what we mean by a $C_0$ semigroup (see Definition 2.1.10).

4.1.4. Definition. The generator of the $C_0$ semigroup $\mathfrak{A}$ on a $H$-space $\mathcal{X}$ is the operator $A$ defined by

$$Ax = \lim_{h \downarrow 0} \frac{1}{h} (\mathfrak{A}^h x - x), \quad x \in \text{dom}(A),$$

$$\text{dom}(A) = \{x \in \mathcal{X} \mid \text{the above limit exists}\}.$$

4.1.5. Theorem. Let $A$ be an operator in an $H$-space which satisfies the following conditions:

(i) $A$ is closed and $\text{dom}(A)$ is dense in $\mathcal{X}$;

(ii) For every initial state $x^0 \in \text{dom}(A)$ the homogeneous Cauchy problem (4.1.2) has a unique classical solution;

(iii) There exists constants $M \geq 1$ and $T > 0$ such that the classical solution $x$ of (4.1.2) with initial state $x^0$ satisfies

$$\|x(t)\| \leq M \|x^0\|, \quad t \in [0, T],$$

where $\|\cdot\|_\mathcal{X}$ is some admissible norm in $\mathcal{X}$.

Then the following claims are true:

(iv) For every initial state $x^0 \in \mathcal{X}$ the homogeneous Cauchy problem (4.1.2) has a unique generalized solution (and hence also a unique mild solution);

(v) There exists constants $M \geq 1$ and $\alpha \in \mathbb{R}$ such that the unique generalized solution of the homogeneous Cauchy problem (4.1.2) with initial state $x^0$ satisfies

$$\|x(t)\|_\mathcal{X} \leq M e^{\alpha t} \|x^0\|_\mathcal{X}, \quad t \in \mathbb{R}^+,$$

where $\|\cdot\|_\mathcal{X}$ is some admissible norm in $\mathcal{X}$;

(vi) There exists a unique $C_0$ semigroup such that the generalized solutions of the homogeneous Cauchy problem (4.1.2) with initial state $x^0$ is given by

$$x(t) = \mathfrak{A}^t x^0, \quad t \in \mathbb{R}^+;$$

(vii) The generalized solution of the homogeneous Cauchy problem is classical if and only if its initial state $x^0$ belongs to $\text{dom}(A)$;

(viii) The generator of the $C_0$ semigroup $\mathfrak{A}$ in (vi) is $A$.

By Theorem 4.1.7 below the converse of this theorem is also true in the sense that conditions (i), (ii), and (iii) above hold if (and only if) $A$ is the generator of a $C_0$ semigroup.
Proof of Theorem 4.1.3. Proof of (iv) and (v). We begin by showing that there exists constants $M \geq 1$ and $\alpha \in \mathbb{R}$ such that every classical solution $x$ of (4.1.2) with initial state $x^0$ satisfies (4.1.6). By the time-invariance of (4.1.2), for every $t_0 \in \mathbb{R}$ it is true that every solution $x \in C^1([t_0, \infty); \mathcal{X})$ of (4.1.1) with $I = [t_0, \infty]$ satisfies
\[(4.1.8) \quad \|x(t)\|_x \leq M\|x(t_0)\|_x, \quad t \in [t_0, t_0 + T],\]
with the same constants $M$ and $T$ as in (c). Let $x$ be a classical solution of (4.1.2) with initial state $x^0$. For each $t \in \mathbb{R}^+$ we can choose some $n \in \mathbb{Z}^+$ such that $nT \leq t \leq (n + 1)T$. Then by (4.1.8),
\[
\|x(t)\| \leq M\|x(nT)\|_x \leq M^2\|x((n - 1)T)\|_x \cdots \leq M^{n+1}\|x(0)\|
= Me^{\alpha n T}\|x(0)\| \leq Me^{\alpha t}\|x(0)\|,
\]
where $\alpha = 1/T \log M$. Thus classical trajectories satisfy (4.1.6).

Let $x^0$ be an arbitrary vector in $\mathcal{X}$. Since dom($A$) is dense in $\mathcal{X}$, there exists a sequence $x_n^0 \to x^0$ as $n \to \infty$. Let $x_n$ be the unique classical solution of the Cauchy problem (4.1.2) with initial state $x_n^0$ given by (b). Then it follows from (4.1.6), applied to the difference $x_n - x_m$, that the sequence $x_n$ is a Cauchy sequence in $C(\mathbb{R}^+; \mathcal{X})$. As $C(\mathbb{R}^+; \mathcal{X})$ is complete, this sequence converges to a limit $x \in C(\mathbb{R}^+; \mathcal{X})$ (uniformly on each finite interval $[0,T]$). In particular, $x(0) = \lim_{n \to \infty} x_n(0) = \lim_{n \to \infty} x_n^0 = x^0$. Thus $x$ is a generalized solution of the Cauchy problem (4.1.2). Since each $x_n$ satisfies the estimate (4.1.6), also $x$ must satisfy the same inequality. This inequality implies that $x$ is determined uniquely by $x^0$. Thus, (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

Proof of (vi). For each $x^0 \in \mathcal{X}$ and $t \in \mathbb{R}^+$ we define $\mathfrak{A}^t x^0 = x(t)$, where $x$ is the unique generalized solution of the Cauchy problem given by (iii). It follows from (b) and the linearity of the equation (4.1.2) that each $\mathfrak{A}^t$ is a linear operator defined on all of $\mathcal{X}$, and it follows from (iv) that this operator is bounded.

For each $s \in \mathbb{R}^+$ the left-shifted function $t \mapsto x(t+s)$, $t \in \mathbb{R}^+$, is a generalized solution of the Cauchy problem (4.1.2) with initial state $x(s)$, and therefore
\[
\mathfrak{A}^{t+s} x^0 = x(t+s) = \mathfrak{A}^t \mathfrak{A}^s x^0.
\]
Thus $\mathfrak{A}$ is a semigroup. For each $x^0 \in \mathcal{X}$ the function $t \mapsto x(t) = \mathfrak{A}^t x^0$ is continuous in $\mathcal{X}$ on $\mathbb{R}^+$ since generalized solutions of the Cauchy problem are continuous.

We have now shown that $\mathfrak{A}$ is a $C_0$ semigroup, and that the unique generalized solution of the homogeneous Cauchy problem (4.1.2) are given by (4.1.7). The uniqueness of $\mathfrak{A}$ follows from the uniquenss part of (iv).

Proof of (vii). If $x$ is a classical solution of the Cauchy problem (4.1.2), then $x(t) \in \text{dom}(A)$ for all $t \in \mathbb{R}^+$. In particular, $x^0 = x(0) \in \text{dom}(A)$. Conversely, suppose that $x^0 \in \text{dom}(A)$, and let $x(t) = \mathfrak{A}^t x^0$, $t \in \mathbb{R}^+$ (i.e., $x$ is the unique generalized solution of the Cauchy problem (4.1.2). By (b), the Cauchy problem (4.1.2) has a classical solution $x_1$ with initial state $x^0$. This classical solution is also a generalized solution of (4.1.2), and it has the same initial state as $x$. By (iii), $x_1$ and $x_0$ are classical solutions of the Cauchy problem (4.1.2).

Proofs of (viii). We claim that the generator of the semigroup $\mathfrak{A}$ defined above is $A$. If $x^0 \in \text{dom}(A)$, then by (vii) and the definition of $\mathfrak{A}$ the function $x$ defined by (4.1.7) is a classical solution of the Cauchy problem (4.1.2), i.e., $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $x(t) \in \text{dom}(A)$, and $\dot{x}(t) = Ax(t)$ for all $t \in \mathbb{R}^+$. In particular, $\dot{x}(0) = Ax(0) = x^0$. 


This implies that if we denote the generator of $A$ by $A_1$, then $A$ is the restriction of $A_1$ to $\text{dom}(A)$.

It remains to show that $\text{dom}(A_1) \subset \text{dom}(A)$. Suppose that $x^0 \in \text{dom}(A_1)$, i.e., suppose that the generalized solution of the Cauchy problem (4.1.2) is differentiable at zero. Then by Lemma 2.4.31 applied to the i/s/o system $\Sigma = (A, \mathcal{X}, \{0\}, \{0\})$ $x^0 \in \text{dom}(A)$. Thus $\text{dom}(A_1) = \text{dom}(A)$, and $A_1 = A$. □

4.1.2. Properties of $C_0$ semigroups and their generators. In this subsection we list a number of results about $C_0$ semigroups and their generators that will be needed later. We begin by proving the following theorem, which enables us to prove a converse to Theorem 4.1.5.

4.1.6. Theorem. Let $\mathcal{A}$ be a $C_0$ semigroup on an $H$-space $\mathcal{X}$ with generator $A$, let $\parallel \cdot \parallel_\mathcal{X}$ be an admissible norm in $\mathcal{X}$, and denote the corresponding operator norm in $\mathcal{B}(\mathcal{X})$ by $\parallel \cdot \parallel_\mathcal{B}(\mathcal{X})$.

(i) $\parallel \mathcal{A}^t \parallel_\mathcal{B}(\mathcal{X}) \leq M e^{\alpha t}$ for some $M > 0$, some $\alpha \in \mathbb{R}$ and all $t \in \mathbb{R}^+$. In particular, $\mathcal{A}$ is locally bounded.

(ii) For each $x^0 \in \mathcal{X}$ the function $t \mapsto \mathcal{A}^t x^0$ is continuous on $\mathbb{R}^+$.

(iii) For all $x^0 \in \mathcal{X}$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \mathcal{A}^s x^0 \, ds = \mathcal{A}^t x^0.$$ 

(iv) For all $x^0 \in \mathcal{X}$ and $0 \leq s < t < \infty$, $\int_s^t \mathcal{A}^u x^0 \, dv \in \text{dom}(A)$ and

$$\mathcal{A}^t x^0 - \mathcal{A}^s x^0 = A \int_s^t \mathcal{A}^u x^0 \, dv.$$ 

(v) For all $x^0 \in \text{dom}(A)$ and $t \in \mathbb{R}^+$, $\mathcal{A}^t x^0 \in \text{dom}(A)$, the function $t \mapsto \mathcal{A}^t x^0$ is continuously differentiable in $\mathcal{X}$, and

$$\frac{d}{dt} \mathcal{A}^t x^0 = A \mathcal{A}^t x^0 = \mathcal{A}^t Ax^0, \quad t \in \mathbb{R}^+.$$ 

(vi) For all $x^0 \in \text{dom}(A)$ and $0 \leq s \leq t < \infty$,

$$\mathcal{A}^t x^0 - \mathcal{A}^s x^0 = A \int_s^t \mathcal{A}^u x^0 \, dv = \int_s^t \mathcal{A}^u Ax^0 \, dv.$$ 

(vii) $A$ is a closed linear operator and $\text{dom}(A)$ is dense in $\mathcal{X}$.

(viii) $\mathcal{A}$ is uniquely determined by its generator $A$.

Proof. (i) We begin by showing that there exists constants $M > 1$ and $T > 0$ such that $\| \mathcal{A}^t \|_{\mathcal{B}(\mathcal{X})} \leq M$ for all $t \in [0, T]$. If not, then there exists some sequence $t_n \downarrow 0$ such that $\| \mathcal{A}^t \|_{\mathcal{B}(\mathcal{X})} \to \infty$ as $n \to \infty$. But this contradicts the uniform boundedness principle, since $\mathcal{A}^{t_n} x^0 \to x^0$ as $n \to \infty$ for each $x^0 \in \mathcal{X}$ (see Definition 2.1.10). Thus indeed, there exist constants $M$ and $T$ such that $\| \mathcal{A}^t \|_{\mathcal{B}(\mathcal{X})} \leq M$ for all $t \in [0, T]$.

If $t \in \mathbb{R}^+$ is arbitrary, then we can choose some $n = 0, 1, 2, \ldots$ such that $nT \leq t \leq (n + 1)T$. By the semigroup property

$$\| \mathcal{A}^t \|_{\mathcal{B}(\mathcal{X})} = \| (\mathcal{A}^T)^{nT-n^T} \mathcal{A}^{nT} \|_{\mathcal{B}(\mathcal{X})} \leq \| \mathcal{A}^T \|_{\mathcal{B}(\mathcal{X})}^n \| \mathcal{A}^{nT} \|_{\mathcal{B}(\mathcal{X})} \leq M^{n+1} = M e^{\alpha T} \leq M e^{\alpha t},$$

where $\alpha = 1/T \log M$. 

(ii) Let \( t, h > 0 \). The right continuity of \( \mathfrak{A}^t x \) follows from
\[
\|\mathfrak{A}^{t+h} x - \mathfrak{A}^t x\|_X = \|\mathfrak{A}^t(\mathfrak{A}^h x - x)\|_X \\
\leq \|\mathfrak{A}^t\|_{\mathcal{B}(X)} \|\mathfrak{A}^h x - x\|_X \leq M e^{\alpha t} \|\mathfrak{A}^h x - x\|_X,
\]
and the left continuity from (take \( 0 \leq h \leq t \))
\[
\|\mathfrak{A}^{t-h} x - \mathfrak{A}^t x\|_X = \|\mathfrak{A}^{t-h}(x - \mathfrak{A}^h x)\|_X \\
\leq \|\mathfrak{A}^{t-h}\|_{\mathcal{B}(X)} \|\mathfrak{A}^h x - x\|_X \leq M e^{\alpha t} \|x - \mathfrak{A}^h x\|_X.
\]
(iii) This follows from the continuity of \( s \mapsto \mathfrak{A}^s x \).
(iv) Let \( x \in \mathcal{X} \) and \( h > 0 \). Then
\[
\frac{1}{h}(\mathfrak{A}^h - 1) \int_s^{s+h} \mathfrak{A}^v x \, dv = \frac{1}{h} \int_s^{s+h} \mathfrak{A}^{v+h} x - \mathfrak{A}^v x \, dv \\
= \frac{1}{h} \int_t^{t+h} \mathfrak{A}^v x \, dv - \frac{1}{h} \int_s^{s+h} \mathfrak{A}^v x \, dv.
\]
As \( h \downarrow 0 \) this tends to \( \mathfrak{A}^t x - \mathfrak{A}^s x \).
(v) Let \( x \in \text{dom } (A) \) and \( h > 0 \). Then
\[
\frac{1}{h}(\mathfrak{A}^h - 1)\mathfrak{A}^t x = \mathfrak{A}^t \frac{1}{h}(\mathfrak{A}^h - 1)x \to \mathfrak{A}^t Ax \text{ as } h \downarrow 0.
\]
Thus, \( \mathfrak{A}^t x \in \text{dom } (A) \), and \( A\mathfrak{A}^t x = \mathfrak{A}^t Ax \) is equal to the right-derivative of \( \mathfrak{A}^t x \) at \( t \). To see that it is also a left-derivative we compute
\[
\frac{1}{h}(\mathfrak{A}^t x - \mathfrak{A}^{t-h} x) - \mathfrak{A}^t Ax = \mathfrak{A}^{t-h} \left( \frac{1}{h}(\mathfrak{A}^h x - x) - Ax \right) + (\mathfrak{A}^{t-h} - \mathfrak{A}^t)Ax.
\]
This tends to zero because of the uniform boundedness of \( \mathfrak{A}^{t-h} \) and the strong continuity of \( \mathfrak{A}^t \).
(vi) We get (vi) by integrating (v).
(vii) The linearity of \( A \) is trivial. To prove that \( A \) is closed we let \( x_n \in \text{dom } (A) \), \( x_n \to x \), and \( Ax_n \to y \) in \( \mathcal{X} \), and claim that \( Ax = y \). By part (iv) with \( s = 0 \),
\[
\mathfrak{A}^t x_n - x_n = \int_0^t \mathfrak{A}^s Ax_n \, ds.
\]
Both sides converge as \( n \to \infty \) (the integrand converges uniformly on \( [0,t] \)), hence
\[
\mathfrak{A}^t x - x = \int_0^t \mathfrak{A}^s y \, ds.
\]
Divide by \( t \), let \( t \downarrow 0 \), and use part (iii) to get \( Ax = y \).
That \( \text{dom } (A) \) is dense in \( \mathcal{X} \) follows from (iii) with \( t = 0 \), so \( \int_0^h \mathfrak{A}^s ds \in \text{dom } (A) \).
(viii) Suppose that there is another \( C_0 \) semigroup \( \mathfrak{A}_1 \) with the same generator \( A \). Take \( x \in \text{dom } (A) \), \( t > 0 \), and consider the function \( s \mapsto \mathfrak{A}_1^{t-s} x, s \in [0,t] \).
We can use part (v) and the chain rule to compute its derivative in the form
\[
\frac{d}{ds} \mathfrak{A}_1^{t-s} = \mathfrak{A}_1^{t-s} A \mathfrak{A}_1^{t-s} x - \mathfrak{A}_1^{t-s} x = \mathfrak{A}_1^{t-s} Ax - \mathfrak{A}_1^{t-s} A \mathfrak{A}_1^{t-s} x = 0.
\]
Thus, this function is a constant. Taking \( s = 0 \) and \( s = t \) we get \( \mathfrak{A}_1^t x = \mathfrak{A}_1 x \) for all \( x \in \text{dom } (A) \). By the density of \( \text{dom } (A) \) in \( \mathcal{X} \), the same must be true for all \( x \in \mathcal{X} \).
4.1.7. Theorem. An operator $A$ in an $H$-space is the generator of a $C_0$ semigroup if and only if $A$ satisfies conditions (i), (ii), and (iii) in Theorem \ref{thm:generator_conditions}.

Proof. In one direction this follows from Theorem \ref{thm:generator_conditions} and it remains to prove the opposite direction.

Suppose that $A$ is the generator of a $C_0$ semigroup $\mathfrak{A}$. Then it follows from part (vii) of Theorem \ref{thm:generator_conditions} that condition (i) in Theorem \ref{thm:generator_conditions} holds.

If $x^0 \in \text{dom}(A)$, then by part (v) of Theorem \ref{thm:generator_conditions} the function $t \mapsto \mathfrak{A}^t x^0$ is a solution in $C^1(\mathbb{R}^+; \mathcal{X})$ of the Cauchy problem (4.1.2). That this is the unique solution can be seen as follows. Let $x \in C^1(\mathbb{R}^+; \mathcal{X})$ be a solution of (4.1.2), and consider the function $s \mapsto \mathfrak{A}^s x(t - s), s \in [0, t]$. We can use part (v) of Theorem \ref{thm:generator_conditions} and the chain rule to compute its derivative in the form

$$
\frac{d}{ds}\mathfrak{A}^s x(t - s) = \mathfrak{A}^s x(t - s) - \mathfrak{A}^s x(t - s) = \mathfrak{A}^s Ax(t - s) - \mathfrak{A}^s x(t - s) = 0.
$$

Thus, this function is a constant. Taking $s = 0$ and $s = t$ we get $x(t) = \mathfrak{A}^t x(0)$. This proves that condition (ii) in Theorem \ref{thm:generator_conditions} holds.

Finally, that condition (iii) in Theorem \ref{thm:generator_conditions} holds follows from \ref{thm:generator_conditions}(i) and the fact that the unique classical solution of (4.1.2) is given by $x(t) = \mathfrak{A}^t x_0, t \in \mathbb{R}^+$. \hfill $\square$

4.1.8. Lemma. Let $\mathfrak{A}$ be a $C_0$ semigroup on an $H$-space $\mathcal{X}$, and let $\|\cdot\|_\mathcal{X}$ be an admissible norm in $\mathcal{X}$. Then the limit $\lim_{t \to \infty} \frac{1}{t} \log(\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})})$ exists, and

$$
\omega(\mathfrak{A}) := \lim_{t \to \infty} \frac{\log(\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})})}{t} = \inf_{t > 0} \frac{\log(\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})})}{t} < \infty
$$

(but possibly $\omega(\mathfrak{A}) = -\infty$), where $\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})}$ is the operator norm of $\mathfrak{A}^t$ corresponding to the norm $\|\cdot\|_{\mathcal{X}}$ in $\mathcal{X}$. In particular, for all $t \in \mathbb{R}^+$ it is true that $\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})} \geq e^{\omega(\mathfrak{A}) t}$ and that the spectral radius of $\mathfrak{A}^t$ is equal to $e^{\omega(\mathfrak{A}) t}$. Moreover, for each $\alpha > \omega(\mathfrak{A})$ there is a constant $M \geq 1$ such that

$$
\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})} \leq M e^{\alpha t}, \quad t \in \mathbb{R}^+,
$$

and $e^{-\alpha t}\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})} \to 0$ as $t \to \infty$.

Proof. See, e.g., \cite{Staffans2005}, Theorem 2.5.4(i) for a proof. \hfill $\square$

4.1.9. Definition. The constant $\omega(\mathfrak{A})$ in Lemma \ref{lem:omega} is called the growth bound of the $C_0$ semigroup $\mathfrak{A}$.

4.1.10. Theorem (Hille–Yosida). Let $\mathcal{X}$ be an $H$-space, let $\|\cdot\|_{\mathcal{X}}$ be an admissible norm in $\mathcal{X}$, denote the norm in $\mathcal{B}(\mathcal{X})$ induced by the norm $\|\cdot\|_{\mathcal{X}}$ by $\|\cdot\|_{\mathcal{B}(\mathcal{X})}$. An operator $A$ in $\mathcal{X}$ is the generator of a $C_0$ semigroup $\mathfrak{A}$ satisfying $\|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})} \leq M e^{\alpha t}$ if and only if the following two conditions hold:

(i) $A$ is closed and $\text{dom}(A)$ is dense in $\mathcal{X}$;

(ii) every real $\lambda > \alpha$ belongs to the resolvent set of $A$, and

$$
\|(\lambda - A)^{-n}\|_{\mathcal{B}(\mathcal{X})} \leq \frac{M}{(\lambda - \alpha)^n} \quad \text{for } \lambda > \alpha \text{ and } n \in \mathbb{N}.
$$

Alternatively, condition (ii) can be replaced by the condition

(iii) $\mathbb{C}_\alpha^+ \subset \rho(A)$, and

$$
\|(\lambda - A)^{-n}\|_{\mathcal{B}(\mathcal{X})} \leq \frac{M}{(\Re \lambda - \alpha)^n} \quad \text{for } \lambda \in \mathbb{C}_\alpha^+ \text{ and } n \in \mathbb{N}.
$$
PROOF. That $A$ is the generator of a $C_0$ semigroup satisfying $\|A^t\|_{B(X)} \leq Me^{\alpha t}$ if and only if (i) and (ii) hold is proved in, e.g., [Staffans, 2005, Theorem 3.4.1]. Clearly (ii) $\Rightarrow$ (iii). That (iii) follows from the other conditions can be seen as follows. Let $\lambda = \alpha + j\beta$, where $\alpha > \omega(\mathfrak{A})$ and $\beta \in \mathbb{R}$. Then $(\lambda-A) = (\alpha - (A-j\beta)$, where $A-j\beta$ is the generator of the $C_0$ semigroup $e^{-j\beta t}\mathfrak{A}^t$ which has the same growth bound as $\mathfrak{A}$. Thus, $\alpha \in \rho(A-j\beta)$ for all $\alpha > \omega(\mathfrak{A})$ and the estimates in (ii) hold with $\lambda$ and $A$ replaced by $\alpha$ respectively $A-j\beta$. This is equivalent to the statement that $\lambda \in \rho(A)$ and that (4.1.12) holds.

The following theorem gives an alternative characterization of generators of $C_0$ semigroups.

4.1.11. THEOREM. Let $A$ be a densely defined linear operator in an $H$-space $X$ with a nonempty resolvent set. Then $A$ is a generator of a $C_0$ semigroup if and only if the homogeneous Cauchy problem (4.1.14) has a unique classical solution for every initial state $x_0 \in \text{dom}(A)$.


4.1.12. NOTATION. If $A$ is a generator of a $C_0$ semigroup $\mathfrak{A}$ with growth bound $\omega(\mathfrak{A})$, then we denote the (connected) component of $\rho(A)$ which contains the right half-plane $C^+_{\omega(\mathfrak{A})}$ by $\rho_{+\infty}(A)$.

4.1.13. COROLLARY. If $A$ is the generator of a $C_0$ semigroup in an $H$-space $X$ with growth bound $\omega(\mathfrak{A})$, then $C^+_{\omega(\mathfrak{A})} \subset \rho_{+\infty}(A)$, and for each $\alpha > \omega(\mathfrak{A})$ there exists a constant $M$ such that

$$
(4.1.13) \quad \|(\lambda - A)^{-n}\|_{B(X)} \leq M(\Re\lambda - \alpha + 1)^{-n}, \quad \Re\lambda \geq \alpha,
$$

where $\|(\lambda - A)^{-n}\|_{B(X)}$ is the operator norm of $(\lambda-A)^{-n}$ with respect to some arbitrary admissible norm in $X$.

PROOF. If $\alpha > \omega(\mathfrak{A}) + 2$ (this is, in particular true if $\omega(\mathfrak{A}) = -\infty$), then we define $\alpha_0 = \alpha - 1$, and if $\alpha \leq \omega(\mathfrak{A}) + 2$ (and hence $\omega(\mathfrak{A})$ is finite) then we define $\alpha_0 = \frac{1}{2}(\alpha + \omega(\mathfrak{A}))$. Then in both cases $\omega(\mathfrak{A}) < \alpha_0 < \alpha$ and $\epsilon := \alpha - \alpha_0$ satisfies $0 < \epsilon \leq 1$. Since $\alpha_0 > \omega(\mathfrak{A})$, by Lemma 4.1.8 there exists a constant $M_0$ such that $\|\mathfrak{A}^t\|_{B(X)} \leq M_0 e^{\alpha_0 t}$, $t \in \mathbb{R}^+$. By Theorem 4.1.10 this implies that for $\Re\lambda \geq \alpha_0$ and $n \in \mathbb{N}$,

$$
\|(\lambda - A)^{-n}\|_{B(X)} \leq \frac{M_0}{(\Re\lambda - \alpha)^n} = \frac{M_0}{(\Re\lambda - \alpha + \epsilon)^n}.
$$

Here $\Re\lambda - \alpha + \epsilon = \epsilon((\Re\lambda - \alpha)/\epsilon + 1) \geq \epsilon((\Re\lambda - \alpha) + 1)$, and consequently (4.1.13) holds with $M = M_0/\epsilon$.

The following lemma describes some connections between a semigroup and the resolvent of its generator:

4.1.14. LEMMA. Let $A$ be the generator of a $C_0$ semigroup $\mathfrak{A}$ on $X$ with growth bound $\omega(\mathfrak{A})$.

(i) For all $x^0 \in X$, $\lambda \in C^+_{\omega(\mathfrak{A})}$, and $n \in \mathbb{Z}^+$ the integral $\int_0^\infty t^n e^{-\lambda t} \mathfrak{A}^t x^0 \, dt$ converges (absolutely) and

$$
(4.1.14) \quad (\lambda - A)^{-(n+1)} x = \frac{1}{n!} \int_0^\infty t^n e^{-\lambda t} \mathfrak{A}^t x \, dt.
$$
In particular,  
\begin{equation}
(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}A^t x \, dt.
\end{equation}

Thus, every generalized solution \( x \) of the Cauchy problem (4.1.2) is Laplace transformable, the Laplace transform converges in the half-plane \( \mathbb{C}^+ \), and \( \dot{x}(\lambda) = (\lambda - A)^{-1}x^0 \) for all \( \lambda \in \mathbb{C}_+^{\omega(\mathfrak{A})} \).

(ii) For all \( x \in \mathcal{X} \) and all \( t \in \mathbb{R}^+ \),  
\begin{equation}
\lim_{n \to \infty} \left(1_X - \frac{t}{n}A\right)^{-n} x = \mathfrak{A}^t x,
\end{equation}

and for fixed \( x \in \mathcal{X} \) the convergence is uniform in \( t \) on each finite interval.

(iii) For all \( x \in \mathcal{X} \) and all \( t \in \mathbb{R}^+ \),  
\begin{equation}
\lim_{n \to \infty} e^{A(1_X - \frac{1}{n}A)^{-1}t} x = \mathfrak{A}^t x,
\end{equation}

and for fixed \( x \in \mathcal{X} \) the convergence is uniform in \( t \) on each finite interval.

Proof. See, e.g., [Staffans, 2005, Theorems 3.2.9, 3.7.3, and 3.7.5]. \( \square \)

4.1.15. Lemma. The operator \( A \) is the generator of a \( C_0 \) group in the \( H \)-space \( \mathcal{X} \) if and only if both \( A \) and \( -A \) are generators of \( C_0 \) semigroups. Moreover, the semigroup generated by \( -A \) is equal to \( t \mapsto \mathfrak{A}^{-t} = (\mathfrak{A}^t)^{-1} \), where \( \mathfrak{A} \) is the semigroup generated by \( A \).

Proof. See, e.g., [Pazy, 1983, Theorem 1.6.3, p. 23]. \( \square \)

4.1.16. Lemma. Let \( \mathcal{X} \) be a \( H \)-space, and let \( \| \cdot \|_{\mathcal{B}(\mathcal{X})} \) denote the operator norm in \( \mathcal{B}(\mathcal{X}) \) with respect to some admissible norm in \( \mathcal{X} \). If \( A \) is the generator of a \( C_0 \) semigroup \( \mathfrak{A} \) in \( \mathcal{X} \) which satisfies \( \|\mathfrak{A}^t\|_{\mathcal{B}(\mathcal{X})} \leq Me^{\alpha t} \), \( t \in \mathbb{R}^+ \), for some constants \( M \) and \( \alpha \), and if \( B \in \mathcal{B}(\mathcal{X}) \), then \( A + B \) is the generator of a \( C_0 \) semigroup \( \mathfrak{A}_1 \) in \( \mathcal{X} \) which satisfies \( \|\mathfrak{A}_1^t\|_{\mathcal{B}(\mathcal{X})} \leq Me^{(\alpha + M\|B\|_{\mathcal{B}(\mathcal{X})})t} \), \( t \in \mathbb{R}^+ \). Thus, the growth bound of \( \mathfrak{A}_1 \) is at most \( \omega(\mathfrak{A}) + M\|B\|_{\mathcal{B}(\mathcal{X})} \).

Proof. See, e.g., [Pazy, 1983, Theorem 3.1.1, p. 76]. \( \square \)

4.1.3. The Inhomogeneous Cauchy Problem (Feb 02, 2016). At the beginning of this section we looked at the connection between the notion of a \( C_0 \) semigroup \( \mathfrak{A} \) and the properties of the solutions of the homogeneous Cauchy problem (4.1.2), i.e., the properties of the i/s/o system \( \Sigma = (A; \mathcal{X}, \{0\}, \{0\}) \), where \( A \) is the generator of \( \mathfrak{A} \). Here we add a bounded input to \( \Sigma \) with control operator \( 1_X \) (as described in Definition 2.3.27), and study the i/s/o system \( \Sigma = ([A \ 1_X]; \mathcal{X}, \{0\}) \).

This means that we replace the homogeneous Cauchy problem (4.1.2) by the inhomogeneous Cauchy problem
\begin{equation}
\Sigma: \begin{cases}
x(t) \in \text{dom}(A), & t \in \mathbb{R}^+ ,
\dot{x}(t) = Ax(t) + f(t), & x(0) = x^0,
\end{cases}
\end{equation}

where \( A \) is the generator of a \( C_0 \) semigroup and \( f \) is a function with values in \( \mathcal{X} \).

4.1.17. Definition. Let \( A \) be a closed linear operator in an \( H \)-space \( \mathcal{X} \), let \( x^0 \in \mathcal{X} \), and let \( f \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{X}) \).
We call \( x \) a classical solution of the inhomogeneous Cauchy problem (4.1.18) with initial state \( x^0 \in \text{dom}(A) \) and input function \( f \in C(\mathbb{R}^+; \mathcal{X}) \) if \( x \in C^1(\mathbb{R}^+; \mathcal{X}) \) and (4.1.18) holds. (Thus, in particular, \( x^0 \in \text{dom}(A) \).)

We call \( x \) a generalized solution of the inhomogeneous Cauchy problem (4.1.18) with initial state \( x^0 \in \mathcal{X} \) and input function \( L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \) if \( x \in C(\mathbb{R}^+; \mathcal{X}) \) and there exists a sequence of classical solutions \( x_n \) of the Cauchy problem (4.1.18) with initial states \( x_n^0 \) and input functions \( f_n \) such that \( x_n^0 \to x^0 \) in \( \mathcal{X} \), \( x_n \to x \) in \( C(\mathbb{R}^+; \mathcal{X}) \), and \( f_n \to f \) in \( L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \) as \( n \to \infty \). (Thus, in particular, \( x(0) = x^0 \).)

We call \( x \) a mild solution of the inhomogeneous Cauchy problem (4.1.18) with initial state \( x^0 \in \mathcal{X} \) and input function \( f \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{X}) \) if

\[
(4.1.19) \quad \begin{cases} 
  \int_0^t x(s) \, ds \in \text{dom}(A), \\
  x(t) = x^0 + A \int_0^t x(s) \, ds + \int_0^t f(s) \, ds.
\end{cases} \quad t \in \mathbb{R}^+.
\]

Lemma. Let \( A \) be a closed linear operator in \( \mathcal{X} \).

(i) \( x \) is a classical, generalized, or mild solution of the inhomogeneous Cauchy problem (4.1.18) input function \( f \) if and only if \( [f] \) is a classical, generalized, or mild future trajectory, respectively, of the i/s/o system \( \Sigma = ([A \quad 1 \chi]; \mathcal{X}, \mathcal{X}, \{0\}) \).

(ii) The following conditions are equivalent:

(a) \( x \) is a generalized solution of the inhomogeneous Cauchy problem (4.1.18).

(b) \( x \) is a mild solution of the inhomogeneous Cauchy problem (4.1.18).

(c) the function \( x_1(t) := \int_0^t x(s) \, ds, \, t \in \mathbb{R}^+ \), is a classical solution of the inhomogeneous Cauchy problem (4.1.18) with initial state zero and

input function \( f_1(t) := x^0 + \int_0^t f(s) \, ds, \, t \in \mathbb{R}^+ \).

(d) the pair \( [f_1] \), where \( x_1 \) and \( f_1 \) are defined as in (c), is a classical future trajectory of the i/s/o system \( \Sigma = ([A \quad 1 \chi]; \mathcal{X}, \mathcal{X}, \{0\}) \).

(iii) \( x \) is a classical solution of the inhomogeneous Cauchy problem (4.1.18) if and only if \( x \in C^1(\mathbb{R}^+; \mathcal{X}), \, f \in C(\mathbb{R}^+; \mathcal{X}) \), and \( x \) is a generalized (or mild) solution of the inhomogeneous Cauchy problem (4.1.2).

Proof. The proof is analogous to the proof of Lemma 4.1.2 (replace all references to Definition 4.1.2 by references to Definition 4.1.18).

Remark. It follows from part (i) of Lemma 4.1.18 that all the results that we have proved in Section 2 about classical, generalized, and mild trajectories of i/s/o systems are also valid for classical, generalized, and mild solutions of the inhomogeneous Cauchy problem (4.1.18).

Remark. It follows directly from Definitions 4.1.1 and 4.1.17 that \( x \) is a classical or mild solution of the homogeneous Cauchy problem (4.1.2) with initial state \( x_0 \) if and only if \( x \) is a classical or mild solution of the inhomogeneous Cauchy problem (4.1.18) with initial state \( x_0 \) and input function zero. In the case of generalized solutions the situation is slightly more complex. According to Definition 4.1.1 \( x \) is a generalized solution of the homogeneous Cauchy problem (4.1.2) with
initial state $x_0$ if $x$ can be approximated by a sequence of classical solutions of 
\[ 1.1.2 \] with a initial states $x_n^0$ tending to $x_0$ as $n \to \infty$, whereas $x$ is generalized
solution of the inhomogeneous Cauchy problem \[ 4.1.18 \] with initial state $x_0$ and
input zero if $x$ can be approximated by a sequence of classical solutions of \[ 4.1.18 \]
with initial states $x_n^0$ and input functions $f_n$ satisfying $x_n^0 \to x_0$ and $f_n \to 0$ as
$n \to \infty$. At a first glance the first of these conditions seems to be stronger than the
second. However, it follows from Lemmas \[ 4.1.2 \] and \[ 4.1.18 \] that these two conditions
are actually equalent, since each of these conditions is equivalent to the condition
that $x$ is a mild solution of the same homogeneous Cauchy problem.

4.1.21. Theorem. Let $A$ be the generator of a $C_0$ semigroup in an $H$-space $\mathcal{X}$.

(i) For each $x^0 \in \mathcal{X}$ and $f \in L_{loc}^1(\mathbb{R}^+; \mathcal{X})$ the inhomogeneous Cauchy
problem \[ 4.1.18 \] with initial state $x^0$ and input function $f$ has a unique
generalized solution (and hence also a unique mild solution). This solution is
given by

\[ 4.1.20 \]

\[ x(t) = A^t x^0 + \int_0^t A^{t-s} f(s) \, ds, \quad t \in \mathbb{R}^+, \]

where $x_1(t) := A^t x^0, t \in \mathbb{R}^+$, is the unique generalized solution of the
corresponding homogeneous Cauchy problem \[ 1.1.2 \] with initial state $x^0,
and $x_2(t) := \int_0^t A^{t-s} f(s) \, ds, t \in \mathbb{R}^+$, is the unique generalized solution
of the inhomogeneous Cauchy problem \[ 4.1.18 \] with initial state zero and
input function $f$.

(ii) The unique generalized solution in (i) is classical if and only if both the
generalized solution $x_1(t) = A^t x^0, t \in \mathbb{R}^+$, of the homogeneous Cauchy
problem \[ 1.1.2 \] with initial state $x^0$ and the generalized solution $x_2(t) =
\int_0^t A^{t-s} f(s) \, ds, t \in \mathbb{R}^+$, of the inhomogeneous Cauchy problem \[ 4.1.18 \]
with initial state zero and input function $f$ are classical solutions of the
respective Cauchy problem.

(iii) The function $x_1 := A^t x^0, t \in \mathbb{R}^+$, is a classical solution of the homogene-
ous Cauchy problem \[ 1.1.2 \] if and only if $x^0 \in \text{dom}(A)$.

(iv) The following conditions are equivalent:
(a) The function $x_2(t) := \int_0^t A^{t-s} f(s) \, ds, t \in \mathbb{R}^+,$ is a classical solution
of the inhomogeneous Cauchy problem \[ 4.1.18 \] with initial state zero
and input function $f$.
(b) $f \in C(\mathbb{R}^+; \mathcal{X})$ and $x_2 \in C^1(\mathbb{R}^+; \mathcal{X})$.
(c) $f \in C(\mathbb{R}^+; \mathcal{X}), x_2(t) \in \text{dom}(A)$ for all $t \in \mathbb{R}^+$, and $Ax_2 \in C(\mathbb{R}^+; \mathcal{X})$.

(v) Suppose that $x^0 \in \text{dom}(A)$, and that $f$ satisfies one of the following two conditions:
(a) $f$ is locally absolutely continuous;
(b) $f \in C(\mathbb{R}^+; \mathcal{X}), f(t) \in \text{dom}(A)$ for all $t \in \mathbb{R}^+$, and $Af \in L_{loc}^1(\mathbb{R}^+; \mathcal{X})$.
Then the generalized solution \[ 4.1.20 \] of the inhomogeneous Cauchy
problem \[ 4.1.18 \] is classical.

Proof. Proof of (i). We begin by proving uniqueness. Suppose that both $x_1$ and
$x_2$ are generalized, and hence mild, solutions of the inhomogeneous Cauchy
problem \[ 4.1.18 \] with the same intital state $x^0$ and the same input function $f$.
Then the difference $x = x_1 - x_2$ is a mild solution of the homogeneoues Cauchy
problem with initial state zero, and hence by Theorem 4.1.5 \( x = 0 \) and \( x_1 = x_2 \). This proves the uniqueness claim in (i).

We next prove existence in the special case where \( x^0 = 0 \). We claim that \( x := \int_0^t \mathfrak{A}^t s f(s) \, ds, \ t \in \mathbb{R}^+ \) is a mild (and hence generalized) solution of the inhomogeneous Cauchy problem \( (4.1.18) \) with initial state zero and input function \( f \). To prove this it suffices to show that the function \( x_1(t) = \int_0^t x(s) \, ds \) is a classical solution of the inhomogeneous Cauchy problem \( (4.1.18) \) with initial state zero and input function \( f_1(t) = x^0 + \int_0^t f(s) \, ds, \ t \in \mathbb{R}^+ \). Clearly \( x_1 \in C^1(\mathbb{R}^+; \mathcal{X}) \) and \( f_1 \in C(\mathbb{R}^+; \mathcal{X}) \). Moreover, for all \( t \in \mathbb{R}^+ \) we have by Fubini’s theorem

\[
x_1(t) = \int_0^t \int_0^u \mathfrak{A}^{u-s} f(s) \, ds \, dv = \int_0^t \int_0^t \mathfrak{A}^{v-s} f(s) \, dv \, ds
\]

By Theorem 4.1.6(iv), \( \int_0^t \mathfrak{A}^{u-s} f(s) \, du \in \text{dom}(A) \) and

\[
\left[ \mathfrak{A}^{u-s} f(s) - f(s) \right]_{\int_0^t \mathfrak{A}^{u-s} f(s) \, du} \in \text{gph}(A).
\]

Since \( \text{gph}(A) \) is closed, after integrating the variable \( s \) over \([0, t]\) we get

\[
\left[ \int_0^t \left( \mathfrak{A}^{t-s} f(s) - f(s) \right) \, ds \right]_{\int_0^t \mathfrak{A}^u f(s) \, du} \in \text{gph}(A),
\]

i.e., \( x_1(t) = \int_0^t \int_0^s \mathfrak{A}^u f(s) \, du \, ds \in \text{dom}(A) \) and \( Ax_1(t) = \int_0^t \left( \mathfrak{A}^{t-s} f(s) - f(s) \right) \, ds = x(t) - f_1(t) = x_1(t) - f(t) \). This proves that \( x_1 \) is a classical solution of \( (4.1.18) \) with initial state zero and input function \( f_1 \), and consequently \( x \) is a mild solution of \( (4.1.18) \) with initial state zero and input function \( f \), as we claimed above.

It still remains to prove the existence claim in (i) for nonzero initial states \( x^0 \). By Theorem 4.1.5 the function \( x_1(t) = \mathfrak{A}^t x^0, \ t \in \mathbb{R}^+ \), is a mild solution \( (4.1.18) \) with initial state \( x^0 \) and input function zero, and by the proof that we gave above, the function \( x_2(t) := \int_0^t \mathfrak{A}^{t-s} f(s) \, ds, \ t \in \mathbb{R}^+ \) is a mild solution of \( (4.1.18) \) with initial state zero and input function \( f \). Define \( x = x_1 + x_2 \). Then \( x \) is a mild solution of \( (4.1.18) \) with initial state \( x^0 \) and input function \( f \).

**Proof of (ii).** If both \( x_1 \) and \( x_2 \) are classical solutions of the respective Cauchy problem, then there sum \( x = x_1 + x_2 \) is a classical solution of \( (4.1.18) \) with initial state \( x^0 \) and input function \( f \). Conversely, suppose that \( x \) is a classical solution of \( (4.1.18) \) with initial state \( x^0 \) and input function \( f \). Then \( x(0) \in \text{dom}(A) \), and by Theorem 4.1.5 \( x_1 \) is a classical solution of \( (4.1.2) \) with initial state \( x^0 \), and consequently \( x_2 = x - x_1 \) is a classical solution of \( (4.1.18) \) with initial state zero.

**Proof of (iii).** Claim (iii) follows from Theorem 4.1.5.

**Proof of (iv).** (a) \( \Rightarrow \) (b): That (a) \( \Rightarrow \) (b) follows from Definition 4.1.17 and the converse implication follows from Lemma 4.1.18.

(a) \( \Rightarrow \) (c): This follows from Definition 4.1.17 (since \( Ax_2 = \dot{x}_2 - f \)).
Here the first term in the last expression tends to zero in \( X \) as \( h \downarrow 0 \) because of the local boundedness of \( \mathfrak{A} \) and the fact that \( f(t + h + s) - f(t) \to 0 \) in \( X \) uniformly in \( s \) as \( h \downarrow 0 \), the second term tends to \( f(t) \) in \( X \) because of the strong continuity of \( \mathfrak{A} \), and the last term tends to \( \mathfrak{A} x_2(t) \) since \( x_2(t) \in \text{dom}(A) \). Thus \( x_2 \) is differentiable on \( \mathbb{R}^+ \) and \( \dot{x}_2(t) = \mathfrak{A} x_2(t) + f(t) \). In particular, \( x_2 \in C^1(\mathbb{R}^+; X) \), i.e., (b) holds.

**Proof of (v).** It follows from (ii) and (iii) that we may, without loss of generality, assume that \( x^0 = 0 \), and hence \( x(t) = \int_0^t \mathfrak{A}^{t-s} f(s) \, ds \), \( t \in \mathbb{R}^+ \).

Suppose first that (v)(a) holds. Then \( f(t) = f(0) + \int_0^t \dot{f}(s) \, ds \) where \( \dot{f} \in L^1_{\text{loc}}(\mathbb{R}^+; X) \), and \( \frac{1}{h}(f(t + h) - f(t)) \) tends to \( \dot{f} \) in \( L^1_{\text{loc}}(\mathbb{R}^+; X) \) as \( h \downarrow 0 \). For each \( t \in \mathbb{R}^+ \) and \( h > 0 \) we have

\[
\frac{1}{h} (x(t+h) - x(t)) = \frac{1}{h} \left( \int_0^{t+h} \mathfrak{A}^{t+h-s} f(s) \, ds - \int_0^{t} \mathfrak{A}^{t-s} f(s) \, ds \right) = \frac{1}{h} \int_0^{h} \mathfrak{A}^{t+h-s} f(s) \, ds + \int_0^{t} \mathfrak{A}^{t-s} \frac{1}{h} (f(s + h) - f(s)) \, ds = \frac{1}{h} \int_0^{h} \mathfrak{A}^{t+h} f(h-s) \, ds + \int_0^{t} \mathfrak{A}^{t-s} \frac{1}{h} (f(s + h) - f(s)) \, ds = \frac{1}{h} \int_0^{h} \mathfrak{A}^{t+h} f(h-s) \, ds + \int_0^{t} \mathfrak{A}^{t-s} \frac{1}{h} (f(s + h) - f(s)) \, ds.
\]

Here the first term in the last expression tends to zero in \( X \) because of the local boundedness of \( \mathfrak{A} \) and the fact that \( f(h-s) - f(0) \to 0 \) in \( X \) uniformly in \( s \) as \( h \downarrow 0 \), the second term tends to \( \mathfrak{A} f(0) \) in \( X \) because of the strong continuity of \( \mathfrak{A} \), and the last term tends to \( \int_0^{t} \mathfrak{A}^{t-s} \dot{f}(s) \, ds \) since \( \mathfrak{A} \) is locally bounded and \( \frac{1}{h}(f(t + h) - f(t)) \) tends to \( \dot{f} \) in \( L^1_{\text{loc}}(\mathbb{R}^+; X) \) as \( h \downarrow 0 \). Thus \( x_2 \) is differentiable on \( \mathbb{R}^+ \) and \( \dot{x}_2(t) = \mathfrak{A} f(0) + \int_0^{h} \mathfrak{A}^{t-h} \dot{f}(s) \, ds \). The right-hand side is continuous, and thus \( x \in C^2(\mathbb{R}^+; X) \). By (iv) \( x \) is a classical solution of the inhomogenous Cauchy problem.
Suppose, finally, that (v)(b) holds. We can then repeat the computation in (4.1.21) with \( x_2 = x \), but replace the last line in this computation by
\[
\frac{1}{h}(x(t + h) - x(t)) = \frac{1}{h} \int_0^h A^s(f(t + h - s) - f(t)) \, ds + \frac{1}{h} \int_0^h A^s f(t) \, ds
\]
+ \( \int_0^t A^{t-s} \frac{1}{h}(A^h - 1_x) f(s) \, ds \).

As before, the sum of the first two terms tend to \( f(t) \) as \( h \downarrow 0 \). Since \( f(s) \in \text{dom}(A) \) for all \( s \in \mathbb{R}^+ \), the integrand in the last term tends pointwise to \( A^{t-s} f(s) \) as \( h \downarrow 0 \), and by the Lebesgue dominated convergence this term tends to \( \int_0^t A^{t-s} f(s) \, ds \) as \( h \downarrow 0 \). Thus, \( x \) is differentiable on \( \mathbb{R}^+ \), and \( \dot{x}(t) = f(t) + \int_0^t A^{t-s} f(s) \, ds \). This implies that \( x \in C^1(\mathbb{R}^+; \mathcal{X}) \). By (iv) \( x \) is a classical solution of the inhomogenous Cauchy problem. \( \Box \)

4.1.22. REMARK. Everything that we have proved above about solutions of the inhomogeneous Cauchy problem remains true if we throughout replace the time interval \( \mathbb{R}^+ \) by a finite half-open interval \( [0, T) \) or a finite closed interval \( [0, T] \). In the case of the interval \( [0, T] \) all the proofs remain the same. In the case of the closed interval \( [0, T] \) the proofs also remain the same, except that all references to Lemma 2.4.27 should be replaced by references to Lemma 2.4.28. That this is possible follows from Lemma 4.1.23 below.

4.1.23. LEMMA. Every mild or classical solution of the inhomogeneous Cauchy problem (4.1.18) can be extended to a mild respectively classical solution of (4.1.18) on \( \mathbb{R}^+ \).

PROOF. Suppose that \( x \) is a mild solution of (4.1.18) on the interval \( [0, T] \) with initial state \( x^0 \) and input function \( f \in L^1_{\text{loc}}([0, T]; \mathcal{X}) \). Extend \( f \) to a function defined on \( \mathbb{R}^+ \) by taking \( f(t) = 0 \) for \( t > T \). Then by Theorem 4.1.21 the Cauchy problem (4.1.18) has a unique mild solution on \( \mathbb{R}^+ \) with initial state \( x^0 \) and the extended input function \( f \). In the case where \( f \) is a classical solution of (4.1.18) on the interval \( [0, T] \) we may instead extend \( f \) by taking \( f(t) = f(T) \) for \( t > T \), and in this case the corresponding solution of (4.1.18) on \( \mathbb{R}^+ \) is classical. \( \Box \)

4.1.4. Invariant subspaces and restrictions of \( C_0 \) semigroups. At this point the reader may want to recall what we mean by an invariant subspace of a \( C_0 \) semigroup \( \mathfrak{A} \) and by the notation \( \rho_{t\to\infty}(A) \) where \( A \) is the generator of \( \mathfrak{A} \) (see Definition 3.1.5 and Notation 4.1.12). In addition we shall need the following definition.

4.1.24. DEFINITION (cf. Definitions 3.1.5 and 3.1.10). Let \( A: \mathcal{X} \supset \text{dom}(A) \to \mathcal{X} \) be a linear operator in the \( H \)-space \( \mathcal{X} \), and let \( \mathcal{Z} \) be a subspace of \( \mathcal{X} \).

(i) By the notation \( A|_{\mathcal{Z}} \) we mean the operator which is the restriction of \( A \) to \( \mathcal{Z} \) in the sense that \( \text{dom}(A|_{\mathcal{Z}}) = \text{dom}(A) \cap \mathcal{Z} \) and \( A|_{\mathcal{Z}} x = Ax \) for \( x \in \text{dom}(A|_{\mathcal{Z}}) \). We call \( A|_{\mathcal{Z}} \) the restriction of \( A \) to \( \mathcal{Z} \).

(ii) \( \mathcal{Z} \) is called an invariant subspace for \( A \) if \( \text{rng}(A|_{\mathcal{Z}}) \subset \mathcal{Z} \), i.e., if \( Ax \in \mathcal{Z} \) for every \( x \in \text{dom}(A|_{\mathcal{Z}}) = \text{dom}(A) \cap \mathcal{Z} \). In this case we also say that \( \mathcal{Z} \) is \( A \)-invariant.

(iii) We say that \( A|_{\mathcal{Z}} \) is a restriction of \( A \) in \( \mathcal{L}(\mathcal{Z}) \) if \( \mathcal{Z} \) is \( A \)-invariant (and \( A|_{\mathcal{Z}} \) is the restriction of \( A \) to \( \mathcal{Z} \)).
We warn the reader that a “restriction of $A$ in $\mathcal{L}(Z)$” is not the same thing as the “restriction of $A$ to $Z$”. The restriction $A|_Z$ of $A \in \mathcal{L}(X)$ always exists as an operator in $\mathcal{L}(Z;X)$. Only in the case where $Z$ is $A$-invariant it is true that $A|_Z$ is a restriction of $A$ in $\mathcal{L}(Z)$.

4.1.25. Lemma (cf. Lemma 3.1.8). Let $X$ be an $H$-space, let $A$ be a closed operator in $X$, and $Z$ be a closed $A$-invariant subspace of $X$. Then, for each $\lambda \in \rho(A)$ the following claims are equivalent:

(i) $(\lambda - A)^{-1}Z \subset Z$,
(ii) $\lambda \in \rho(A|_Z)$,

If these equivalent conditions hold, then

(iii) $(\lambda - A)^{-1}Z = Z \cap \text{dom}(A)$,
(iv) $(\lambda - A|_Z)^{-1} = (\lambda - A)^{-1}|_Z$.

Proof. Let us first observe that $A|_Z \in \mathcal{L}(Z)$ since $Z$ is $A$-invariant. Throughout the rest of this proof we assume that $\lambda \in \rho(A)$.

We next observe that for all $z \in \text{dom}(A) \cap Z$ we have $(\lambda - A|_Z)z \subset Z$ and

$$(\lambda - A)^{-1}|_Z(\lambda - A|_Z)z = (\lambda - A)^{-1}(\lambda - A)z = z.$$ 

Thus, $(\lambda - A)^{-1}|_Z$ is always a left-inverse of $(\lambda - A|_Z)$ (whenever $Z$ is $A$-invariant and $\lambda \in \rho(A)$).

(i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv): We claim that if (i) holds, then $(\lambda - A)^{-1}|_Z$ is a right-inverse of $(\lambda - A|_Z)$ as well. This is true because for all $z \in Z$ we have

$$(\lambda - A|_Z)(\lambda - A)^{-1}|_Zz = (\lambda - A)(\lambda - A)^{-1}z = z.$$ 

Thus $(\lambda - A)^{-1}|_Z$ is an inverse of $(\lambda - A|_Z)$, and hence $\lambda \in \rho(A|_Z)$. This shows that (i) implies both (ii) and (iv).

(ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii): If (ii) holds, then $A|_Z$ maps $\text{dom}(A|_Z) = \text{dom}(A) \cap Z$ one-to-one onto $Z$, and hence $\lambda - A$ maps $\text{dom}(A|_Z) = \text{dom}(A) \cap Z$ one-to-one onto $Z$. Consequently $(\lambda - A)^{-1}$ maps $Z$ one-to-one onto $\text{dom}(A) \cap Z$. This implies (i) and (iv). \qed

4.1.26. Theorem (cf. Theorem 3.1.9). Let $X$ be an $H$-space, let $Z$ be a closed subspace of $X$, and let $\mathfrak{A}$ be a $C_0$ semigroup in $X$ with generator $A$. Then the following conditions are equivalent:

(i) $Z$ is an invariant subspace for $\mathfrak{A}$;
(ii) The family $\mathfrak{A}|_Z : t \mapsto \mathfrak{A}|_Z^t$ is a $C_0$ semigroup in $Z$;
(iii) $Z$ is an invariant subspace for $A$ and $\rho(A|_Z) \cap \rho_{+\infty}(A) \neq \emptyset$;
(iv) $(\lambda - A)^{-1}Z = Z \cap \text{dom}(A)$ for all $\lambda \in \rho_{+\infty}(A)$;
(v) $(\lambda - A)^{-1}Z \subset Z$ for some $\lambda \in \rho_{+\infty}(A)$.

If these equivalent conditions hold, then

(vi) The generator of $\mathfrak{A}|_Z$ is $A|_Z$;
(vii) $(\lambda - A|_Z)^{-1} = (\lambda - A)^{-1}|_Z$ for all $\lambda \in \rho(A|_Z) \cap \rho(A)$,
(viii) $\rho_{+\infty}(A) \subset \rho(A|_Z)$, and hence $(\lambda - A|_Z)^{-1} = (\lambda - A)^{-1}|_Z$ for all $\lambda \in \rho_{+\infty}(A)$,
(ix) $\omega(\mathfrak{A}|_Z) \leq \omega(\mathfrak{A})$, where $\omega(\mathfrak{A}|_Z)$ and $\omega(\mathfrak{A})$ are the growth bounds of $\mathfrak{A}|_Z$ respectively $\mathfrak{A}$. 

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Proof. (i) \(\Leftrightarrow\) (ii): If (i) holds, then for all \( s, t \in \mathbb{R}^+ \) and all \( x^0 \in Z \),

\[ A^s|_Z A^t|_Z x^0 = A^{s+t}|_Z x^0 = A^{s+t} x^0. \]

In addition \( A^0 = 1_Z \). Thus \( A|_Z \) is a semigroup in \( Z \). That \( A|_Z \) is strongly continuous follows from the strong continuity of \( A \). Conversely, if (ii) holds, then \( A^t|_Z \in B(Z) \), i.e., \( A^t Z \subset Z \), \( t \in \mathbb{R}^+ \).

(i) \(\Rightarrow\) (v): Let \( \lambda > \omega(A) \), where \( \omega(A) \) is the growth bound of \( A \), and let \( x^0 \in Z \). Then by Lemma 4.1.14(\(\lambda - A\))\(^{-1}\) is strongly continuous for all \( t \in \mathbb{R}^+ \) also \( (\lambda - A)^{-1} x^0 \in Z \).

(v) \(\Rightarrow\) (iv): See the proof of the implication (vi) \(\Rightarrow\) (v) in Lemma 3.1.7.

(iv) \(\Rightarrow\) (i): If (iv) holds, then for each \( t > 0 \), \( x^0 \in Z \), and all \( n \) sufficiently large we have \((1 - \frac{t}{n} A)^{-n} x^0 \in Z \). Therefore by Lemma 4.1.14(ii), for all \( t \in \mathbb{R}^+ \) and \( x^0 \in Z \),

\[ A^t x^0 = \lim_{n \to \infty} \left(1 - \frac{t}{n} A\right)^{-n} x^0 \in Z. \]

(i) \(\Leftrightarrow\) (iii) and (i) \(\Rightarrow\) (vi): If (i) holds, then for all \( x^0 \in \text{dom}(A) \cap Z \) we have \( Ax = \lim_{h \to 0} \frac{1}{h}(A^h x - x) \in Z \), i.e., \( Z \) is an invariant subspace for \( A \).

As we noticed above, (i) \(\Rightarrow\) (ii). Denote the generator of the semigroup \( A|_Z \) by \( A_1 \). We have just seen that \( \text{dom}(A) \cap Z \subset \text{dom}(A_1) \) and that \( A_1 x^0 = Ax^0 \) for all \( x^0 \in \text{dom}(A) \cap Z \), i.e., \( A_1 \) is an extension of \( A|_Z \). Conversely, if \( x^0 \in \text{dom}(A_1) \), then \( x^0 \in Z \) and \( \frac{1}{h}(A^h x^0 - x^0) \) has a limit in \( Z \) as \( h \downarrow 0 \), and hence \( x^0 \in \text{dom}(A) \cap Z \). Thus \( A_1 = A|_Z \). This proves that (ii) \(\Rightarrow\) (vi).

That \( \rho(A|_Z) \cap \rho_{+\infty}(A) \neq \emptyset \) follows from Theorem 4.1.10 since both \( A \) and \( A|_Z \) are generators of \( C_0 \) semigroups. Thus (i) \(\Rightarrow\) (iii).

(iii) \(\Rightarrow\) (v): Let \( \lambda \in \rho(A|_Z) \cap \rho_{+\infty}(A) \). Then \( (\lambda - A|_Z) \) maps \( \text{dom}(A|_Z) \) to \( \text{dom}(A|_Z) \) one-to-one onto \( Z \), and consequently also \( (\lambda - A) \) maps \( \text{dom}(A) \cap Z \) one-to-one onto \( Z \). Thus, for each \( x^0 \in Z \) there exists a unique \( z^0 \in Z \) such that \( (\lambda - A|_Z) z^0 = (\lambda - A) z^0 = x^0 \), and thus \( (\lambda - A|_Z)^{-1} x^0 = (\lambda - A)^{-1} x^0 = z^0 \in Z \).

Proof of (vii): That (vii) holds follows from Lemma 4.1.25.

Proof of (viii): That (vii) holds follows from (iv) and Lemma 4.1.25.

Proof of (ix): That (ix) holds follows from (4.1.9) applied to both semigroups \( A \) and \( A|_Z \) and the fact that \( ||A^t\|_B(Z) \leq ||A^t\|_B(X) \). \(\Box\)

4.1.27. Definition (cf. Definition 3.1.10). Let \( X \) be an \( H \)-space, let \( Z \) be a closed subspace of \( X \), and let \( A \) be a \( C_0 \) semigroup in \( X \). A \( C_0 \) semigroup \( A_t \) in \( Z \) is called a restriction of \( A \) to \( Z \) if \( A_t|_Z = A^t|_Z \) for all \( t \in \mathbb{R}^+ \).

Thus by Theorem 4.1.26, a necessary and sufficient condition for a \( C_0 \) semigroup \( A \) in \( X \) to have a restriction to a closed subspace \( Z \) of \( X \) that \( Z \) is \( A \)-invariant.

4.1.5. Projections of \( C_0 \) semigroups. At this point the reader may want to recall the definition of what we mean by a projection \( A_{\text{proj}} \in B(X_1) \) of an operator \( A \in B(X) \) along \( Z_1 \) where \( X = X_1 + Z_1 \) is a direct sum decomposition of \( X \) (see Definition 3.1.7). Here we shall also need a generalization of this definition to the case where \( A \in L(X) \) is allowed to be unbounded.

4.1.28. Definition (cf. Definition 3.1.11). Let \( X \) be an \( H \)-space, let \( A \in L(X) \), and let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \).
(i) An operator operator \( A_{\text{proj}} \in \mathcal{L}(\mathcal{X}_1) \) is called a projection of \( A \) in \( \mathcal{L}(\mathcal{X}_1) \) along \( Z_1 \) if
\[
\text{gph} (A_{\text{proj}}) = \begin{bmatrix} \frac{x}{x_1} & 0 \\ 0 & \frac{x_1}{x_1} \end{bmatrix} \text{gph} (A),
\]
\[\text{i.e.,} \]
\[
dom (A_{\text{proj}}) = \frac{x}{x_1} \text{dom} (A),
\]
\[
A_{\text{proj}} P_{\mathcal{X}_1} x = P_{\mathcal{X}_1} A x, \quad x \in \text{dom} (A).
\]

(ii) We say that \( A \) has a projection in \( \mathcal{L}(\mathcal{X}_1) \) along \( Z_1 \) if the right-hand side of (4.1.23) is the graph of a operator in \( \mathcal{L}(\mathcal{X}_1) \).

A multi-valued version of this definition will be given in Definition 6.4.23.

(There both \( A \) and \( A_{\text{proj}} \) are allowed to be multi-valued.)

4.1.29. Lemma (cf. Lemma 3.1.12). Let \( \mathcal{X} \) be an \( H \)-space, let \( A \) be a closed operator in \( \mathcal{X} \), and let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \). Then the following conditions are equivalent:
(i) \( Z_1 \) is an invariant subspace for \( A \);
(ii) \( P_{\mathcal{X}_1} A x = 0 \) whenever \( x \in \text{dom} (A) \) and \( P_{\mathcal{X}_1} x = 0 \).
(iii) \( A \) has a projection in \( \mathcal{L}(\mathcal{X}_1) \) along \( Z_1 \)

where \( (\lambda - A)^{-1} \) is the projection of \( (\lambda - A)^{-1} \) in \( \mathcal{B}(\mathcal{X}_1) \) along \( Z_1 \).

Proof. The easy proof is left to the reader. □

4.1.30. Lemma (cf. Lemma 3.1.13). Let \( \mathcal{X} \) be an \( H \)-space, let \( A \) be a closed operator in \( \mathcal{X} \), and let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \). In addition, suppose that \( Z_1 \) is invariant for \( A \). Then, for each \( \lambda \in \rho(A) \) the following claims are equivalent:
(i) \( (\lambda - A)^{-1} Z_1 \subset Z_1 \),
(ii) \( \lambda \in \rho(A_{\text{proj}}) \), where \( A_{\text{proj}} \) is the projection of \( A \) in \( \mathcal{L}(\mathcal{X}_1) \) along \( Z_1 \).

If these equivalent conditions hold, then
(iii) \( (\lambda - A)^{-1} A_{\text{proj}} = (\lambda - A)^{-1} \) proj,

where \( (\lambda - A)^{-1} \) proj is the projection of \( (\lambda - A)^{-1} \) in \( \mathcal{B}(\mathcal{X}_1) \) along \( Z_1 \).

Proof. Let us first observe that by Lemma 4.1.29 \( A \) has a projection \( A_{\text{proj}} \in \mathcal{L}(\mathcal{X}_1) \) along \( Z_1 \). Throughout the rest of this proof we assume that \( \lambda \in \rho(A) \).

We next observe that for all \( x \in \mathcal{X}_1 \) we have \( (\lambda - A)^{-1} A_{\text{proj}} x = P_{\mathcal{X}_1} (\lambda - A)^{-1} x \in \text{dom} (A_{\text{proj}}) \) and
\[\begin{align*}
(\lambda - A_{\text{proj}})(\lambda - A)^{-1} A_{\text{proj}} x &= (\lambda - A_{\text{proj}})P_{\mathcal{X}_1} (\lambda - A)^{-1} x \\
&= P_{\mathcal{X}_1} (\lambda - A)(\lambda - A)^{-1} x = x.
\end{align*}\]
Thus \( (\lambda - A)^{-1} A_{\text{proj}} \) is always a right-inverse of \( (\lambda - A_{\text{proj}}) \) (whenever \( Z_1 \) is A-invariant and \( \lambda \in \rho(A) \)).

(i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii): We claim that if (i) holds, then \( (\lambda - A)^{-1} A_{\text{proj}} \) is a left-inverse of \( (\lambda - A_{\text{proj}}) \) as well. This is true because every \( x_1 \in \text{dom} (A_{\text{proj}}) \) is of
the form \( x_1 = P_{X_1}^{\lambda} x \in \text{dom}(A_{\text{proj}}) \) for some \( x \in \text{dom}(A) \) and

\[
(\lambda - A)^{-1}_{\text{proj}} P_{X_1}^{\lambda} x = P_{X_1}^{\lambda} (\lambda - A)^{-1} P_{X_1}^{\lambda} (\lambda - A)x = P_{X_1}^{\lambda} (\lambda - A)^{-1} (P_{X_1}^{\lambda} + P_{Z_1}^{\lambda}) (\lambda - A)x = P_{X_1}^{\lambda} (\lambda - A)^{-1} (\lambda - A)x = P_{X_1}^{\lambda} x.
\]

Thus \( (\lambda - A)^{-1}_{\text{proj}} \) is an inverse of \( (\lambda - A_{\text{proj}}) \), and hence \( \lambda \in \rho(A_{\text{proj}}) \). This shows that (i) implies both (ii) and (iii).

(ii) \( \Rightarrow \) (i): For all \( z \in Z_1 \) we have \( P_{X_1}^{\lambda} (\lambda - A)^{-1} z \in \text{dom}(A_{\text{proj}}) \) and

\[
(\lambda - A_{\text{proj}}) P_{X_1}^{\lambda} (\lambda - A)^{-1} z = P_{X_1}^{\lambda} (\lambda - A) (\lambda - A)^{-1} z = P_{X_1}^{\lambda} z = 0.
\]

If (ii) holds, then \( \lambda - A_{\text{proj}} \) is injective, and consequently \( P_{X_1}^{\lambda} (\lambda - A)^{-1} z = 0 \), i.e., \( (\lambda - A)^{-1} z \in Z \). Thus (ii) \( \Rightarrow \) (i).

By combining Lemmas 4.1.25 and 4.1.30 we get the following result:

4.1.31. Lemma (cf. Lemma 3.1.14). Let \( X \) be an \( H \)-space, let \( A \) be a closed operator in \( X \), and let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \). In addition, suppose that \( Z_1 \) is invariant for \( A \). Then, for each \( \lambda \in \rho(A) \) the following claims are equivalent:

(i) \( (\lambda - A)^{-1} Z_1 \subset Z_1 \),
(ii) \( \lambda \in \rho(A|_{Z_1}) \),
(iii) \( \lambda \in \rho(A_{\text{proj}}) \), where \( A_{\text{proj}} \) is the projection of \( A \) in \( L(X_1) \) along \( Z_1 \).

If these equivalent conditions hold, then

(iv) \( (\lambda - A)^{-1} Z_1 = Z_1 \cap \text{dom}(A) \),
(v) \( (\lambda - A|_{Z_1})^{-1} = (\lambda - A)^{-1}|_{Z_1} \),
(vi) \( (\lambda - A_{\text{proj}})^{-1} = (\lambda - A)^{-1}_{\text{proj}} \).

Proof. This follows from Lemmas 4.1.25 and 4.1.30.

4.1.32. Theorem (cf. Theorem 3.1.15). Let \( X \) be an \( H \)-space, let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \), and let \( A \) be a \( C_0 \) semigroup in \( X \) with generator \( A \). Then the following conditions are equivalent:

(i) \( Z_1 \) is an invariant subspace for \( A \);
(ii) \( Z_1 \) is an invariant subspace for \( A \) and \( \rho(A|_{Z_1}) \cap \rho_{+\infty}(A) \neq \emptyset \);
(iii) \( A^t \) has a projection in \( B(X_1) \) along \( Z_1 \) for all \( t \in \mathbb{R}^+ \);
(iv) \( A \) has a projection \( A_{\text{proj}} \in L(X_1) \) along \( Z_1 \), and \( \rho(A_{\text{proj}}) \cap \rho_{+\infty}(A) \neq \emptyset \);
(v) \( (\lambda - A)^{-1} \) has a projection in \( B(X_1) \) along \( Z_1 \) for all \( \lambda \in \rho_{+\infty}(A) \);
(vi) \( (\lambda - A)^{-1} \) has a projection in \( B(X_1) \) along \( Z_1 \) for some \( \lambda \in \rho_{+\infty}(A) \).

Suppose that these equivalent conditions hold, and denote the projections of \( A, A^t, \) and \( (\lambda - A)^{-1} \) onto \( X_1 \) along \( Z_1 \) by \( A_{\text{proj}}, A^t_{\text{proj}}, \) and \( (\lambda - A)^{-1}_{\text{proj}} \), respectively. Then

(vii) \( (\lambda - A_{\text{proj}})^{-1} = (\lambda - A)^{-1}_{\text{proj}} \) for all \( \lambda \in \rho(A_{\text{proj}}) \cap \rho(A) \);
(viii) \( \rho_{+\infty}(A) \subset \rho(A_{\text{proj}}) \), and hence \( (\lambda - A_{\text{proj}})^{-1} = (\lambda - A)^{-1}_{\text{proj}} \) for all \( \lambda \in \rho_{+\infty}(A) \);
(ix) The family \( A_{\text{proj}} : t \mapsto A^t_{\text{proj}} \) is \( C_0 \) semigroup in \( X_1 \) with generator \( A_{\text{proj}} \).
(x) \( \omega(A_{\text{proj}}) \leq \omega(A) \), where \( \omega(A_{\text{proj}}) \) and \( \omega(A) \) are the growth bounds of \( A_{\text{proj}} \) respectively.
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Proof. (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) ⇔ (vi): This follows from Theorem 4.1.26 and Lemma 3.1.12.

Proof of (vii): This follows from Lemma 4.1.30.

Proof of (viii): That (vii) holds follows from (v) and Lemmas 3.1.12 and 4.1.30.

Proof of (ix): By Lemma 4.1.29, $Z_1$ is an invariant subspace for $A^t$ for all $t \in \mathbb{R}^+$, i.e., $P_{X_1} Z_1 A^t P_{X_1} = 0$. From this follows that for all $s, t \in \mathbb{R}^+$ and all $x^0 \in X_1$ we have

$$A^s_{\text{proj}} A^t_{\text{proj}} x^0 = P_{X_1} Z_1 A^s_{\text{proj}} A^t_{\text{proj}} x^0 = P_{X_1} Z_1 A^s_{\text{proj}} A^t_{\text{proj}} x^0 = P_{X_1} Z_1 A^{s+t}_{\text{proj}} x^0 = A^{s+t}_{\text{proj}} x^0.$$

Thus $A^s_{\text{proj}}$ is a semigroup. That this semigroup is strongly continuous follows from the strong continuity of $A$.

Denote the generator of $A_{\text{proj}}$ by $A_1$. Then by Lemma 4.1.14 for all $\lambda \in \mathbb{C}^+$ and $x^0 \in X_1$ we have

$$(\lambda - A_1)^{-1} = \int_0^\infty e^{-\lambda t} A^t_{\text{proj}} x^0 dt = \int_0^\infty e^{-\lambda t} P_{X_1} Z_1 A^t_{\text{proj}} x^0 dt$$

$$= P_{X_1} \int_0^\infty e^{-\lambda t} A^t_{\text{proj}} x^0 dt = P_{X_1} Z_1 (\lambda - A)^{-1} x^0$$

$$= (\lambda - A)^{-1}_{\text{proj}} x^0 = (\lambda - A_{\text{proj}})^{-1} x^0.$$

Since an operator is determined uniquely by its resolvent, this together with (vii) implies that $A_1 = A_{\text{proj}}$.

Proof of (x): That (x) holds follows from (4.1.9) and the fact that $\|A^t_{\text{proj}}\|_{B(X)} \leq \|P_{X_1} Z_1 A^t\|_{B(X)}$. \qed

4.1.33. Definition (cf. Definition 3.1.16). Let $X$ be an $H$-space, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, and let $A$ be a $C_0$ semigroups in $X$. A $C_0$ semigroup $A_1$ in $X_1$ is called a projection of $A$ onto $X_1$ along $Z_1$ if $A^t_{\text{proj}} = P_{X_1} A^t_{\text{proj}}$ for all $t \in \mathbb{R}^+$.

Thus by Theorem 4.1.26 a necessary and sufficient condition for a $C_0$ semigroup $A$ in $X$ to have a projection onto a closed subspace $X_1$ along the direct complement $Z_1$ is that $Z_1$ is $A$-invariant.

4.1.6. Intertwinements of $C_0$ semigroups. In Chapter 3 we discussed the intertwining of bounded linear operators and also intertwining of the uniformly continuous groups generated by these operators (see, in particular, Lemma 3.1.24). Analogous results, restricted to the time interval $\mathbb{R}^+$, remain true for the intertwining of $C_0$ semigroups and their generators.

4.1.34. Definition (cf. Definition 3.1.17). Let $A_1$ and $A_2$ be linear operators in the $H$-spaces $X_1$ respectively $X_2$, and let $P \in \mathcal{ML}(X_1; X_2)$.

(i) We say that $A_1$ and $A_2$ are intertwined by $P$ if

$$\begin{bmatrix} A_2 x_2 \\ A_1 x_1 \end{bmatrix} \in \text{gph} (P) \text{ whenever } \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \in \text{gph} (P) \cap \left[ \text{dom} (A_1) \right] \left[ \text{dom} (A_2) \right].$$

(ii) $A_1$ and $A_2$ are pseudo-similar if $A_1$ and $A_2$ are intertwined by an closed single-valued injective linear operator $P : X \to X_1$ with dense domain and dense range, called the pseudo-similarity operator.
A and $A_2$ are similar if $A_1$ and $A_2$ are intertwined by a bounded linear operator $P \in \mathcal{B}(\mathcal{X}_1;\mathcal{X}_2)$ with a bounded inverse $P^{-1} \in \mathcal{B}(\mathcal{X}_2;\mathcal{X}_1)$, called the similarity operator.

4.1.35. **Lemma** (cf. Lemma 3.1.19). Let $A_1$ and $A_2$ be linear operators in the $H$-spaces $\mathcal{X}_1$ respectively $\mathcal{X}_2$, and let $P \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2)$. Then $A_1$ and $A_2$ are intertwined by $P$ if and only if $\text{gph}(P)$ is an invariant subspace for $A_2 \times A_1$.

**Proof.** This follows immediately from Definitions 3.1.5 and 1.1.34.

4.1.36. **Lemma** (cf. Lemma 3.1.20). Let $A_1$ and $A_2$ be linear operators in the $H$-spaces $\mathcal{X}_1$ respectively $\mathcal{X}_2$. Then the following claims are true:

(i) $\rho (A_1 \times A_2) = \rho (A_1) \cap \rho (A_2)$ and (3.1.11) holds.

(ii) Suppose, in addition, that $A_1$ and $A_2$ are the generators of $C_0$ semigroups $\mathfrak{A}_1$ respectively $\mathfrak{A}_2$. For each $t \in \mathbb{R}^+$, define $(\mathfrak{A}_1 \times \mathfrak{A}_2)^{\downarrow} := \mathfrak{A}_1^t \times \mathfrak{A}_2^t$. Then $\mathfrak{A}_1 \times \mathfrak{A}_2$ is a $C_0$ semigroup in $\mathcal{X}_1 \times \mathcal{X}_2$ with generator $A_1 \times A_2$.

(iii) Under the same assumption as in (ii) $\rho_{+\infty} (A_1 \times A_2)$ is the component of $\rho_{+\infty} (A_1) \cap \rho_{+\infty} (A_2)$ which is unbounded to the right. In particular, if $\rho_{+\infty} (A_1) \cap \rho_{+\infty} (A_2)$ is connected, then $\rho_{+\infty} (A_1 \times A_2) = \rho_{+\infty} (A_1) \cap \rho_{+\infty} (A_2)$.

**Proof.** The easy proof is left to the reader.

4.1.37. **Lemma** (cf. Lemma 3.1.21). Let $A_1$ and $A_2$ be linear operators in the $H$-spaces $\mathcal{X}_1$ respectively $\mathcal{X}_2$, and let $P \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2)$.

(i) $A_1$ and $A_2$ are intertwined by $P$ if and only if $A_2$ and $A_1$ are intertwined by $P^{-1}$.

(ii) If $A_1$ and $A_2$ are intertwined by $P$, then both $\text{dom} (P)$ and $\ker (P)$ are invariant subspaces for $A_1$, and both $\text{rng} (P)$ and $\text{mul} (P)$ are invariant subspaces for $A_2$.

(iii) If $P$ is single-valued, then $A_1$ and $A_2$ are intertwined by $P$ if and only if $A_2PX_1 = PA_1x_1$ for all those $x_1 \in \text{dom}(A_1) \cap \text{dom}(P)$ for which $Px_1 \in \text{dom}(A_2)$.

**Proof.** The proof is analogous to the proof of Lemma 3.1.21.

4.1.38. **Lemma** (cf. Lemma 3.1.22). Let $A_i$ be linear operators in the $H$-spaces $\mathcal{X}_i$, $i = 1, 2, 3$. If $A_1$ and $A_2$ are intertwined by $P_1 \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2)$ and $A_2$ and $A_3$ are intertwined by $P_2 \in \mathcal{ML}(\mathcal{X}_2;\mathcal{X}_3)$, then $A_1$ and $A_3$ are intertwined by $P_3 := P_2P_1$.

**Proof.** This follows from Definition 4.1.34 and the definition of the composition of two multi-valued operators.

4.1.39. **Lemma** (cf. Lemma 3.1.23). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}_1$, and let $A$ and $A_1$ be linear operators in $\mathcal{X}$ respectively $\mathcal{X}_1$. Then the following claims are true:

(i) $A_1$ is the restriction of $A$ in $\mathcal{L}(\mathcal{X}_1)$ if and only if $A_1$ and $A$ are intertwined by the embedding operator $\mathcal{X}_1 \hookrightarrow \mathcal{X}$.

(ii) $A_1$ is the projection of $A$ in $\mathcal{L}(\mathcal{X}_1)$ along $\mathcal{Z}_1$ if and only if $A$ and $A_1$ are intertwined by the projection operator $P_{\mathcal{Z}_1}^{\mathcal{X}_1}$, interpreted as an operator in $\mathcal{B}(\mathcal{X};\mathcal{X}_1)$. 
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4.1.40. **Theorem (cf. Theorem 3.1.24).** Let $\mathfrak{A}_i$ be $C_0$ semigroups in the $H$-spaces $\mathcal{X}_i$ with generators $A_i$, $i = 1, 2$, and let $P \in \mathcal{M}\mathcal{L}(\mathcal{X}_1; \mathcal{X}_2)$ be closed. Then the following conditions are equivalent:

(i) $A_1$ and $A_2$ are intertwining by $P$ and

\[ \rho((A_2 \times A_1)|_{\text{gph}(R)}) \cap \rho_{+\infty}(A_2 \times A_1) \neq \emptyset. \]

(ii) $\mathfrak{A}_1^t$ and $\mathfrak{A}_2^t$ are intertwining by $P$ for all $t \in \mathbb{R}$;

(iii) $(\lambda - A_1)^{-1}$ and $(\lambda - A_2)^{-1}$ are intertwining by $P$ for all $\lambda$ in $\rho_{+\infty}(A_2 \times A_1)$;

(iv) $(\lambda - A_1)^{-1}$ and $(\lambda - A_2)^{-1}$ are intertwining by $P$ for some $\lambda$ in $\rho_{+\infty}(A_2 \times A_1)$.

**Proof.** This follows from Theorem 4.1.26 and Lemmas 4.1.36 and 4.1.35. \(\square\)

Motivated by Theorem 4.1.40 we make the following definition.

4.1.41. **Definition (cf. Definition 3.1.25).** Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be $H$-spaces, let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be $C_0$ semigroups in $\mathcal{X}_1$ respectively $\mathcal{X}_2$, and let $P \in \mathcal{M}\mathcal{L}(\mathcal{X}_1; \mathcal{X}_2)$. We say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are **intertwined by $P$** if $\mathfrak{A}_1^t$ and $\mathfrak{A}_2^t$ are intertwined by $P$ for all $t \in \mathbb{R}^+$. 

4.1.42. **Lemma (cf. Lemma 3.1.30).** Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$, and let $\mathfrak{A}$ be a $C_0$ semigroup group in $\mathcal{X}$. Then the following conditions are equivalent:

(i) The family $t \mapsto P_{\mathcal{X}_1}^Z\mathfrak{A}^t|_{\mathcal{X}_1}$ is a $C_0$ semigroup in $\mathcal{X}_1$;

(ii) for each $t \in \mathbb{R}^+$ the operator $\mathfrak{A}^t$ has a compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;

(iii) $\mathfrak{A}$ satisfies the condition

\[ P_{\mathcal{X}_1}^Z\mathfrak{A}^{s+t}|_{\mathcal{X}_1} = P_{\mathcal{X}_1}^Z\mathfrak{A}^s P_{\mathcal{X}_1}^Z\mathfrak{A}^t|_{\mathcal{X}_1}, \quad s, t \in \mathbb{R}^+. \]

(iv) $\mathfrak{A}$ satisfies the condition

\[ P_{\mathcal{X}_1}^Z\mathfrak{A}^s P_{\mathcal{X}_1}^Z\mathfrak{A}^t|_{\mathcal{X}_1} = 0, \quad s, t \in \mathbb{R}^+. \]

**Proof.** The proof is essentially the same as the proof of the equivalences (ii) \(\Leftrightarrow\) (iii) \(\Leftrightarrow\) (iv) in Lemma 3.1.30 (simply replace “uniformly continuous group” by “$C_0$ semigroup” and $\mathbb{R}$ by $\mathbb{R}^+$ throughout that proof). \(\square\)

Motivated by Lemma 4.1.42 we make the following definition.

4.1.43. **Definition (cf. Definition 3.1.31).** Let $\mathcal{X}$ be an $H$-space, let $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\mathfrak{A}$ be a $C_0$ semigroup in $\mathcal{X}$. A $C_0$ semigroup $\mathfrak{A}_1$ in $\mathcal{X}_1$ is called the **compression of $\mathfrak{A}$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$** if

\[ \mathfrak{A}_1^t = P_{\mathcal{X}_1}^Z\mathfrak{A}^t|_{\mathcal{X}_1}, \quad t \in \mathbb{R}^+. \]
Thus by Lemma 4.1.42, a necessary and sufficient condition for a $C_0$ semigroup $\mathfrak{A}$ in $\mathcal{X}$ to have a compression onto a closed subspace $X_1$ along the direct complement $Z_1$ is that the family $t \mapsto P_{X_1}^{Z_1}\mathfrak{A}t|_{X_1}$ is a $C_0$ semigroup in $X_1$.

4.1.44. **Lemma** (cf. Lemma 3.1.32). Let $\mathcal{X}$ be an $H$-space, let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\mathfrak{A}$ and $\mathfrak{A}_1$ be $C_0$ semigroups in $\mathcal{X}$ respectively $X_1$.

(i) The following conditions are equivalent:
   (a) $\mathfrak{A}_1$ is a restriction of $\mathfrak{A}$ to $X_1$;
   (b) $X_1$ is an invariant subspace for $\mathfrak{A}$ and $\mathfrak{A}_1$ is a compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$.

(ii) The following conditions are equivalent:
   (a) $\mathfrak{A}_1$ is a projection of $\mathfrak{A}$ onto $X_1$ along $Z_1$;
   (b) $Z_1$ is an invariant subspace for $\mathfrak{A}$ and $\mathfrak{A}_1$ is a compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$.

**Proof.** This follows from Theorems 4.1.26, 4.1.32, and 4.1.42 and Definitions 4.1.27, 4.1.33, and 4.1.43.

4.1.45. **Lemma** (cf. Lemma 3.1.33). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = X_1 + Z_1$, and $\mathfrak{A}$ be a $C_0$ semigroup in $\mathcal{X}$ with generator $A$. Then the following conditions are equivalent:

(i) $\mathfrak{A}$ has a $C_0$ compression onto $X_1$ along $Z_1$;

(ii) for all $\lambda \in \rho_{+\infty}(A)$ the operator $(\lambda - A)^{-1}$ has a compression onto $X_1$ along $Z$;

(iii) for some $\lambda \in \rho_{+\infty}(A)$ the operator $(\lambda - A)^{-1}$ has a compression onto $X_1$ along $Z$.

(iv) The resolvent of $A$ satisfies the condition

\[
P_{X_1}^{Z_1}(\lambda - A)^{-1}(\mu - A)^{-1}|_{X_1} = P_{X_1}^{Z_1}(\lambda - A)^{-1}P_{X_1}^{Z_1}(\mu - A)^{-1}|_{X_1},
\]

for all $\lambda, \mu \in \rho_{+\infty}(A)$;

(v) The resolvent of $A$ satisfies the condition

\[
P_{X_1}^{Z_1}(\lambda - A)^{-1}P_{Z_1}^{X_1}(\mu - A)^{-1}|_{X_1} = 0
\]

for all $\lambda, \mu \in \rho_{+\infty}(A)$;

(vi) the function

\[
\lambda \mapsto P_{X_1}^Z(\lambda - A)^{-1}|_{X_1}, \quad \lambda \in \rho(A),
\]

satisfies the resolvent identity (3.1.5) for all $\lambda, \mu \in \rho_{+\infty}(A)$;

(vii) at least one of condition (iv), (v), and (vi) holds with $\rho_{+\infty}(A)$ replaced by an arbitrary subset $\Omega'$ of $\rho_{+\infty}(A)$ which has a cluster point in $\rho_{+\infty}(A)$.

Suppose that the above equivalent conditions hold, denote the generator of the compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$ by $A_{\text{cmp}}$, and denote the compression of $(\lambda - A)^{-1}$ onto $X_1$ along $Z_1$ by $(\lambda - A)^{-1}_{\text{cmp}}$, $\lambda \in \rho_{+\infty}(A)$. Then

(viii) $\rho_{+\infty}(A) \subset \rho(A_{\text{cmp}})$ and

\[
(\lambda - A_{\text{cmp}})^{-1} = (\lambda - A)^{-1}_{\text{cmp}}, \quad \lambda \in \rho_{+\infty}(A).
\]
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Proof. (i) ⇒ (iii): Suppose that (i) holds, and denote the $C_0$ semigroup in (i) by $\mathfrak{A}_{\text{cmp}}$ and its generator by $A_{\text{cmp}}$. Then by (4.1.14), for all $x^0 \in \mathcal{X}_1$ and all $\lambda$ with $\Re \lambda > \max\{\omega(\mathfrak{A}), \omega(\mathfrak{A}_1)\}$ we have

$$P^Z_{\mathcal{X}_1}(\lambda - A)^{-(n+1)}x^0 = \frac{1}{n!} \int_0^\infty t^n e^{-\lambda t} P^Z_{\mathcal{X}_1} \mathfrak{A}^t x^0 dt$$

$$= \frac{1}{n!} \int_0^\infty t^n e^{-\lambda t} A_{\text{cmp}}^t x^0 dt$$

$$= (\lambda - A_{\text{cmp}})^{-(n+1)}x^0.$$  

Thus $(\lambda - A_{\text{cmp}})^{-1}$ is a compression onto $\mathcal{X}_1$ along $Z_1$ of $(\lambda - A)^{-1}$.

(iii) ⇒ (v): Repeat the proof of the same implication in Theorem 3.1.33 (see also Theorem 5.2.3).  

(iv) ⇔ (v): This is obvious.  

(iv) ⇔ (vi): This is true since the resolvent of $A$ satisfies the resolvent identity (3.1.5) in $\rho(A)$ (see Theorem 5.2.3).  

(iv) ⇒ (i): Suppose that (iv) holds. By differentiating (4.1.30) with respect to $\lambda$ and $\mu$ we find that

$$(4.1.33) \quad P^Z_{\mathcal{X}_1}(\lambda - A)^{-(m+1)} P^Z_{\mathcal{X}_1}(\mu - A)^{-(n+1)}|_{\mathcal{X}_1} = 0$$

for all $\lambda, \mu \in \rho_{\infty}(A)$ and all $m, n \in \mathbb{Z}_+$. Let $s, t \in \mathbb{R}_+$. Then by Lemma 4.1.14 for each $x^0 \in \mathcal{X}_1$,

$$P^Z_{\mathcal{X}_1} \mathfrak{A}^s P^Z_{\mathcal{X}_1} \mathfrak{A}^t x^0 = \lim_{m \to \infty} P^Z_{\mathcal{X}_1} \left(1 + \frac{t}{m} A\right)^{-m} P^Z_{\mathcal{X}_1} \lim_{n \to \infty} \left(1 + \frac{s}{n} A\right)^{-n} x^0$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} P^Z_{\mathcal{X}_1} \left(1 + \frac{t}{m} A\right)^{-m} P^Z_{\mathcal{X}_1} \left(1 + \frac{s}{n} A\right)^{-n} x^0 = 0.$$  

By Lemma 4.1.42 $\mathfrak{A}$ has a compression onto $\mathcal{X}_1$ along $Z_1$.

(vi) ⇒ (vii) ⇒ (i): Repeat the proof of the same implications in Theorem 3.1.33.

Proof of (viii): Repeat the proof of claim (viii) in Theorem 3.1.33 with $\rho_{\infty}(A)$ replaced by $\rho_{\infty}(A)$.

By combining Lemmas 3.1.30 and 3.1.33 we get the following theorem.

4.1.46. THEOREM (cf. Theorem 3.1.34). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$, and let $\mathfrak{A}$ be a $C_0$ semigroup in $\mathcal{X}$. Then the following conditions are equivalent:

(i) $\mathfrak{A}$ has a compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(ii) for all $t \in \mathbb{R}_+$ the operator $\mathfrak{A}^t$ has a compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(iii) for all $\lambda \in \rho_{\infty}(A)$ the operator $(\lambda - A)^{-1}$ has a compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(iv) for some $\lambda \in \rho_{\infty}(A)$ the operator $(\lambda - A)^{-1}$ has a compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(v) The function $\lambda \mapsto P^Z_{\mathcal{X}_1}(\lambda - A)^{-1}|_{\mathcal{X}_1}$ satisfies the resolvent identity (3.1.5) in $\rho_{\infty}(A)$.

Suppose that these equivalent conditions hold, denote the generator of the compression of $\mathfrak{A}$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ by $A_{\text{cmp}}$, and denote the compression of $(\lambda - A)^{-1}$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ by $(\lambda - A)^{-1}_{\text{cmp}}$. Then
\( (vi) \) \( \rho_{+\infty}(A) \subset \rho(A_{\text{cmp}}) \) and \( (\lambda - A_{\text{cmp}})^{-1} = (\lambda - A)^{-1}_{\text{cmp}} \) for all \( \lambda \in \rho_{+\infty}(A) \).

**Proof.** This follows from Lemmas 4.1.42 and 4.1.45. \( \square \)

### 4.1.8. The general structure of a compression of a \( C_0 \) semigroup.

**4.1.47. Lemma** (cf. Lemma 3.1.36). Let \( \mathcal{X} \) be an \( H \)-space with the direct sum decomposition \( \mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1 \), and let \( \mathcal{A} \) be a \( C_0 \) semigroup in \( \mathcal{X} \).

(i) **Define**

\[
\mathcal{X}_{\text{min}} := \bigvee_{t \in \mathbb{R}^+} \text{rng} (\mathcal{A}^t|_{\mathcal{X}_1}) .
\]

Then \( \mathcal{X}_{\text{min}} \) is the minimal closed \( \mathcal{A} \)-invariant subspace which contains \( \mathcal{X}_1 \) (i.e., \( \mathcal{X}_{\text{min}} \) is closed and \( \mathcal{A} \)-invariant, and \( \mathcal{X}_{\text{min}} \) is contained in every other closed \( \mathcal{A} \)-invariant subspace which contains \( \mathcal{X}_1 \)).

(ii) **The space** \( \mathcal{X}_{\text{min}} \) **has the direct sum decomposition** \( \mathcal{X}_{\text{min}} = \mathcal{X}_1 + \mathcal{Z}_{\text{min}} \), **where**

\[
\mathcal{Z}_{\text{min}} = \mathcal{X}_{\text{min}} \cap \mathcal{Z}_1 = P_{\mathcal{Z}_1} \mathcal{X}_{\text{min}} .
\]

(iii) **Define**

\[
\mathcal{Z}_{\text{max}} := \bigcap_{t \in \mathbb{R}^+} \{ x \in \mathcal{X} \mid \mathcal{A}^t x \in \mathcal{Z}_1 \} .
\]

Then \( \mathcal{Z}_{\text{max}} \) is the maximal \( \mathcal{A} \)-invariant subspace which is contained in \( \mathcal{Z}_1 \) (i.e., \( \mathcal{Z}_{\text{max}} \) is \( \mathcal{A} \)-invariant, and \( \mathcal{Z}_{\text{max}} \) contains every other \( \mathcal{A} \)-invariant subspace which is contained in \( \mathcal{Z}_1 \)).

**Proof.** The proof is a copy of the proof of Lemma 3.1.36 with the following changes: replace "\( A \)-invariant" by "\( \mathcal{A} \)-invariant", \( A^n \) by \( \mathcal{A}^t \), and "\( n \in \mathbb{Z}^+ \)" by "\( t \in \mathbb{R}^+ \)". \( \square \)

**4.1.48. Lemma** (cf. Lemma 3.1.37). Let \( \mathcal{X} \) be an \( H \)-space with the direct sum decomposition \( \mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1 \), and let \( \mathcal{A} \) be a \( C_0 \) semigroup in \( \mathcal{X}_1 \) with generator \( A \). Let \( \Omega' \) be an arbitrary subset of \( \rho_{+\infty}(A) \) which has a cluster point in \( \rho_{+\infty}(A) \), and let \( \lambda_0 \in \rho_{+\infty}(A) \). Then the following claims are true:

(i) **The subspace** \( \mathcal{X}_{\text{min}} \) **defined in** (4.1.34) **can be computed in the following alternative ways:**

\[
\mathcal{X}_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng} (\mathcal{A}^t|_{\mathcal{X}_1}) = \bigvee_{\lambda \in \rho_{+\infty}(A)} \text{rng} ((\lambda - A)^{-1}|_{\mathcal{X}_1})
\]

\[
= \bigvee_{\lambda \in \Omega'} \text{rng} ((\lambda - A)^{-1}|_{\mathcal{X}_1}) = \bigvee_{n \in \mathbb{Z}^+} \text{rng} ((\lambda_0 - A)^{-(n+1)}|_{\mathcal{X}_1}) .
\]

(ii) **The subspace** \( \mathcal{Z}_{\text{max}} \) **defined in** (4.1.36) **can be computed in the following alternative ways:**

\[
\mathcal{Z}_{\text{max}} = \bigcap_{t \in \mathbb{R}^+} \{ x \in \mathcal{X} \mid \mathcal{A}^t x \in \mathcal{Z}_1 \} = \bigcap_{\lambda \in \rho_{+\infty}(A)} \{ x \in \mathcal{X} \mid (\lambda - A)^{-1}x \in \mathcal{Z}_1 \}
\]

\[
= \bigcap_{\lambda \in \Omega'} \{ x \in \mathcal{X} \mid (\lambda - A)^{-1}x \in \mathcal{Z}_1 \} = \bigcap_{n \in \mathbb{Z}^+} \{ x \in \mathcal{X} \mid (\lambda_0 - A)^{-(n+1)}x \in \mathcal{Z}_1 \} .
\]
Proof. Proof of (i): The first equality in (4.1.37) is true by the definition of $X_{\min}$, and the third and fourth equalities follow from Lemma A.3.6.

To prove the last equality in (4.1.37) we take the set $\Omega'$ in (4.1.37) to be the open interval $\Omega' = (\omega(\mathfrak{A}), \infty)$. For each $x^0 \in X$ and $t \in \mathbb{R}^+$ we have by Lemmas 4.1.14 and A.3.6

$$\mathfrak{A}^t x^0 = \lim_{n \to \infty} \left( 1 + \frac{t}{n} A \right)^{-n} x^0 = \bigvee_{\lambda \in \Omega'} \text{rng} \left( (\lambda - A)^{-1} | x^0 \right).$$

Taking the closed linear span over all $x^0 \in X_1$ and $t \in \mathbb{R}^+$ we get

$$\bigvee_{t \in \mathbb{R}^+} \text{rng} \left( \mathfrak{A}^t | x^0 \right) \subset \bigvee_{\lambda \in \Omega'} \text{rng} \left( (\lambda - A)^{-1} | x^0 \right).$$

Conversely, for each $x^0 \in X_1$ and $\lambda \in \Omega'$ we have by Lemma 4.1.14

$$(\lambda - A)^{-1} x^0 \in \int_0^\infty e^{-\lambda t} \mathfrak{A}^t x^0 \ dt \in \bigvee_{t \in \mathbb{R}^+} \text{rng} \left( \mathfrak{A}^t | x^0 \right).$$

Taking the closed linear span over all $x^0 \in X_1$ and $t \in \Omega'$ we get

$$\bigvee_{\lambda \in \Omega'} \text{rng} \left( (\lambda - A)^{-1} | x^0 \right) \subset \bigvee_{t \in \mathbb{R}^+} \text{rng} \left( \mathfrak{A}^t | x^0 \right).$$

Proof of (ii): The first equality in (4.1.38) is true by the definition of $Z_{\max}$, and the third and fourth equalities follow from Lemma A.3.6. To prove the last equality in (4.1.38) we again take the set $\Omega'$ in (4.1.37) to be the open interval $\Omega' = (\omega(\mathfrak{A}), \infty)$, and use an argument analogous to the one given in the proof of (i) above. We leave the details to the reader. \(\square\)

4.1.49. Theorem (cf. Theorem 3.1.35). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = X_1 + Z_1$, and let $\mathfrak{A}$ be a $C_0$ semigroup in $\mathcal{X}$. Then the following conditions are equivalent:

(i) $\mathfrak{A}$ has a compression onto $X_1$ along $Z_1$.

(ii) There exists a closed subspace $Z$ of $Z_1$ such that both $Z$ and $X_1 + Z$ are $\mathfrak{A}$-invariant subspaces.

(iii) $Z_{\min}$ in an $\mathfrak{A}$-invariant subspace where $Z_{\min}$ is given by (4.1.35).

(iv) $X_1 + Z_{\max}$ is an $\mathfrak{A}$-invariant subspace, where $Z_{\max}$ is given by (4.1.36).

(v) $Z_{\min} \subset Z_{\max}$.

Two possible choices of the subspace $Z$ in (ii) are $Z = Z_{\min}$ and $Z = Z_{\max}$, and every possible subspace $Z$ in (ii) satisfies $Z_{\min} \subset Z \subset Z_{\max}$.

Proof. (i) $\Rightarrow$ (iii): Suppose that (i) holds, and let $\mathfrak{A}_1 : t \to P_{X_1}^t \mathfrak{A}^t | x_1$ be the compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$. It follows from Lemma 4.1.42 that for all $s \in \mathbb{R}^+$, $t \in \mathbb{R}^+$ and all $x \in X_1$ we have $P_{X_1}^s \mathfrak{A}_1^s P_{X_1}^t \mathfrak{A}_1^t x = 0$. Taking the closed linear span over all $x \in X_1$ and all $t \in \mathbb{R}^+$ we find that $P_{X_1}^s \mathfrak{A}_1^s P_{X_1}^t \mathfrak{X}_1 = 0$, where $\mathfrak{X}_1$ is defined by (4.1.34). Here $P_{X_1}^s \mathfrak{X}_1 = Z_{\min}$, so $P_{X_1}^s \mathfrak{A}^s Z_{\min} = 0$, i.e., $\mathfrak{A}^s Z_{\min} \subset Z_1$. On the other hand, $\mathfrak{A}^s Z_{\min} \subset \mathfrak{A}^s X_{\min} \subset X_{\min}$ since $Z_{\min} \subset X_{\min}$ and $X_{\min}$ is $\mathfrak{A}$-invariant. Thus, $\mathfrak{A}^s Z_{\min} \subset X_{\min} \cap Z_1 = Z_{\min}$ for all $s \in \mathbb{R}^+$. This proves (iii).

(iii) $\Rightarrow$ (ii): This follows from Lemma 4.1.47 (take $Z = X_{\min}$).
(ii) ⇒ (i): Suppose that (ii) holds, and define $\mathfrak{A}_t^t := P_{X_t}^s \mathfrak{A}^t|_{X_t}$, $t \in \mathbb{R}^+$. It follows from (ii) that for all $t \in \mathbb{R}^+$,

$$\mathfrak{A}^t(X_t + Z) \subset X_t + Z, \quad \mathfrak{A}^t Z \subset Z.$$ 

Thus, $P_{X_t}^s \mathfrak{A}^t X_t \subset Z$, and for all $s \in \mathbb{R}^+$ we have $\mathfrak{A}^s P_{X_t}^s \mathfrak{A}^t X_t \subset Z$. Therefore $P_{X_t}^s \mathfrak{A}^t P_{X_t}^s \mathfrak{A}^t X_t = \{0\}$. By Lemma 4.1.42, $t \mapsto \mathfrak{A}_t^t$ is the compression of $\mathfrak{A}$ onto $X_t$ along $Z_t$.

(i) ⇒ (iv): Suppose that (i) holds. By Lemma 4.1.42 for all $x \in X_t$ and $s$, $t \in \mathbb{R}^+$ we have $P_{X_t}^s \mathfrak{A}^t P_{X_t}^s \mathfrak{A}^t x = 0$, i.e., $\mathfrak{A}^s P_{X_t}^s \mathfrak{A}^t x \in Z_t$. By (4.1.36), this means that $P_{X_t}^s \mathfrak{A}^t x \in Z_{\text{max}}$, and consequently $\mathfrak{A}^t x \in X_t + Z_{\text{max}}$. This shows that $\mathfrak{A}^t X_t \subset X_t + Z_{\text{max}}$. By Lemma 4.1.47, $\mathfrak{A}^t Z_{\text{max}} \subset Z_{\text{max}}$. Thus $\mathfrak{A}^t (X_t + Z_{\text{max}}) \subset X_t + Z_{\text{max}}$, $t \in \mathbb{R}^+$.

(iv) ⇒ (ii): If (iv) holds, then it follows from Lemma 4.1.47 that (ii) holds with $Z = Z_{\text{max}}$.

(iii) ⇒ (v): This follows from Lemma 4.1.47.

(v) ⇒ (iii): By Lemma 4.1.47 both $X_t + Z_{\text{min}}$ and $Z_{\text{max}}$ are $\mathfrak{A}$-invariant. The $\mathfrak{A}$-invariance of $X_t + Z_{\text{min}}$ implies that $\mathfrak{A}^t Z_{\text{min}} \subset X_t + Z_{\text{min}}$ for all $t \in \mathbb{R}^+$, and the $\mathfrak{A}$-invariance of $Z_{\text{max}}$ together with the condition $Z_{\text{min}} \subset Z_{\text{max}}$ implies that $\mathfrak{A}^t Z_{\text{min}} \subset \mathfrak{A}^t Z_{\text{max}} \subset Z_{\text{max}} \subset Z_t$ for all $t \in \mathbb{R}^+$. Thus $\mathfrak{A}^t Z_{\text{min}} \subset (X_t + Z_{\text{min}}) \cap Z_t = Z_{\text{min}}$ for all $t \in \mathbb{R}^+$, and (iii) holds.

4.1.50. COROLLARY (cf. Corollary 3.1.39). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = X_t + Z_t$, let $\mathfrak{A}$ be a $C_0$ semigroup in $X_t$, and define $\mathfrak{A}_t^t := P_{X_t}^s \mathfrak{A}^t|_{X_t}$, $t \in \mathbb{R}^+$. Then $\mathfrak{A}_t^t$ is the compression of $\mathfrak{A}$ onto $X_t$ along $Z_t$ if and only if $Z_t$ has a direct sum decomposition $Z_t = Z + Z_c$ such that $\mathfrak{A}_t^t$ has the following structure with respect to the decomposition $\mathcal{X} = X_t + Z_t + Z_c$ of $\mathcal{X}$ for all $t \in \mathbb{R}^+$ (where we use “*” to mark irrelevant entries):

\[
\mathfrak{A}_t^t = \begin{bmatrix}
\mathfrak{A}_t^t_{X} & * & * \\
0 & \mathfrak{A}_t^t_{Z} & * \\
0 & 0 & \mathfrak{A}_t^t_{Z_c}
\end{bmatrix}.
\]

(4.1.39)

From this structure follows that both $Z$ and $Z + X_t$ are $\mathfrak{A}$-invariant, that $t \mapsto \mathfrak{A}_t^t_{X}$ is the restriction of $\mathfrak{A}$ to $Z$, and that $t \mapsto \mathfrak{A}_t^t_{Z_c}$ is the projection of $\mathfrak{A}$ onto $Z_c$ along $X_t + Z$. The subspace $Z$ in this decomposition can be chosen to be the same as the subspace $Z$ in condition (ii) in Theorem 4.1.49 and the subspace $Z_c$ can be chosen to be an arbitrary direct complementary to $Z$ in $Z_t$. In particular, two possible choices of $Z$ are $Z = Z_{\text{min}}$ and $Z = Z_{\text{max}}$, where $Z_{\text{min}}$ and $Z_{\text{max}}$ are defined by (4.1.35) and (4.1.36).

PROOF. This follows from the equivalence of (i) and (ii) in Theorem 4.1.49 (take $Z_c$ to be an arbitrary direct complement to $Z$ in $Z_t$).

4.1.51. THEOREM (cf. Theorem 3.1.40). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = X_t + Z_t$, let $\mathfrak{A}$ be a $C_0$ semigroup in $\mathcal{X}$ with generator $A$, define $\mathfrak{A}_t^t := P_{X_t}^s \mathfrak{A}^t|_{X_t}$, $t \in \mathbb{R}^+$, and suppose that $\mathfrak{A}_t^t$ is the compression of $\mathfrak{A}$ onto $X_t$ along $Z_t$ with generator $A_t$. Let $Z$ satisfy the conditions listed in (ii) in Theorem 4.1.49 and let $Z_c$ be an arbitrary direct complement to $Z$ in $Z_t$. Then the following claims are true:

(i) $\mathfrak{A}_t^t$ is the projection onto $X_t$ along $Z$ of the restriction of $\mathfrak{A}$ to $X_t + Z$;

(ii) $\mathfrak{A}_t^t$ is the restriction to $X_t$ of the projection of $\mathfrak{A}$ onto $X_t + Z_c$ along $Z$. 


(iii) $A_1$ is the projection in $\mathcal{L}(X_1)$ along $Z$ of the restriction of $A$ in $\mathcal{L}(X_1 + Z)$;
(iv) $A_1$ is also the restriction in $\mathcal{L}(X_1)$ of the projection of $A$ in $\mathcal{L}(X_1 + Z_c)$ along $Z$.

**Proof.** This follows from Corollary 4.1.50.

4.1.52. **Lemma** (cf. Lemma 3.1.41). Let $\mathcal{X}$ be an $H$-space with a direct sum decomposition $\mathcal{X} = X_1 + Z_1$, let $\mathfrak{A}$ be a $C_0$ semigroup in $\mathcal{X}$, and suppose that $\mathfrak{A}$ has a compression $\mathfrak{A}_1$ onto $X_1$ along $Z_1$. Let $Z$ satisfy the conditions listed in (ii) in Theorem 4.1.49, and let $Z_c$ be an arbitrary direct complement to $Z$ in $Z_1$. Then $\mathfrak{A}$ and $\mathfrak{A}_1$ are intertwined by the operator $P_{X_1}^Z|_{X_1 + Z}$, interpreted as a continuous operator from $\mathcal{X}$ to $X_1$ with closed domain $X_1 + Z$.

**Proof.** This follows from Lemmas 3.1.22 and 3.1.23 and Theorem 4.1.51. □
4.2. Semi-Bounded I/S/O Systems (Feb 02, 2016)

In this section we shall study a class of i/s/o systems that we call semi-bounded. This class consists of all bounded i/o extensions (in the sense of Definition 2.3.27) of i/s/o systems of the type \( \Sigma = (A; X, \{0\}, \{0\}) \) where \( A \) is the generator of a \( C_0 \) semigroup in \( X \).

4.2.1. Introduction to semi-bounded i/s/o systems.

4.2.1. Definition. An i/s/o system \( \Sigma = (S, X, U, Y) \) is called a semi-bounded i/s/o system if the system operator \( S \) of \( \Sigma \) has the following properties:

(i) \( S \) is single-valued and closed;
(ii) The main operator \( A \) of \( S \) is the generator of a \( C_0 \) semigroup;
(iii) The observation operator of \( S \) can be extended to an operator in \( B(X; Y) \).
(iv) \( \text{dom } (S) = \left[ \text{dom} (A) \cup U \right] \).

4.2.2. Lemma. Every bounded i/s/o system is semi-bounded.

Proof. This is true since every bounded operator \( A \) generates a uniformly continuous group (whose restriction to \( \mathbb{R}^+ \) is a uniformly continuous semigroup).

4.2.3. Lemma. Let \( \Sigma = (S, X, U, Y) \) be a semi-bounded i/s/o system. Then the following claims are true:

(i) \( \Sigma \) is regular;
(ii) The classical control operator \( B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and the classical feedthrough operator \( D := \begin{bmatrix} 0 & 1 \end{bmatrix} S \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) of \( \Sigma \) are bounded.

Proof. (i) This follows from condition (i), (ii) and (iv) in Definition 4.2.1 and the fact that the generator of a \( C_0 \) always has dense domain.
(ii) The operator \( S|_{\{0\} \cup U} \) is closed with domain \( \left[ \{0\} \cup U \right] \), and hence by the closed graph theorem it is bounded. Since \( \begin{bmatrix} B \\ D \end{bmatrix} = S|_{\{0\} \cup U} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) both \( B \) and \( D \) are bounded.

4.2.4. Notation. Let \( \Sigma = (S; X, U, Y) \) be a semi-bounded i/s/o system.

(i) The semigroup \( \mathfrak{A} \) generated by the main operator \( A \) of \( \Sigma \) is called the evolution semigroup of \( \Sigma \).
(ii) The operators \( B \) and \( D \) defined in part (ii) of Lemma 4.2.3 are called the control respectively feedthrough operators of \( \Sigma \).
(iii) The extension \( C \) of the observation operator of \( \Sigma \) to an operator in \( B(X; Y) \) is called the extended observation operator of \( \Sigma \).
(iv) We denote the system operator of \( \Sigma \) by \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A \) is the main operator of \( \Sigma \), \( B \) and \( D \) are the classical control and feedthrough operators of \( \Sigma \), respectively, and \( C \) is the extended observation operator of \( \Sigma \). (Here we use the convention introduced in Notation 2.1.2 according to which the domain of \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is \( \left[ \text{dom}(A) \cup U \right] \).)

4.2.5. Lemma. Let \( \Sigma = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}; X, U, Y) \) be a semi-bounded i/s/o system. Then a triple \( \begin{bmatrix} x \\ u \end{bmatrix} \) is a classical solution of \( \Sigma \) on the interval \( I \) if and only if \( x \in C^1(I; X) \),
$u \in C(I;U)$, and
\begin{align}
\begin{cases}
x(t) \in \text{dom}(A), \\
x(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t) + Du(t),
\end{cases} & \quad t \in I.
\end{align}
(4.2.1)

**Proof.** This follows from Lemma 4.2.3, Definition 2.1.1, Notation 4.2.4, and the fact that $y$ is continuous whenever both $x$ and $u$ are continuous. \(\square\)

4.2.6. **Theorem (cf. Theorem 2.1.14).** Let $\Sigma = ([A\ B];\ X, U, Y)$ be a semi-bounded i/s/o system with evolution semigroup $\mathfrak{A}$.

(i) For every $x^0 \in X$, every interval $I$ with finite left-endpoint $t_0 \in I$, and every $u \in L^1_{\text{loc}}(I;U)$ the system $\Sigma$ has a unique generalized trajectory $[x\ y]$ on $I$ (with input function $u$) satisfying $x(t_0) = x^0$. The state $x$ and the output $y$ of this trajectory is given by
\begin{align}
x(t) = \mathfrak{A}^{t-t_0}x^0 + \int_{t_0}^t \mathfrak{A}^{t-s}Bu(s)\,ds, & \quad t \in I. \\
y(t) = C\mathfrak{A}^{t-t_0}x^0 + \int_{t_0}^t C\mathfrak{A}^{t-s}Bu(s)\,ds + Du(t), & \quad t \in I.
\end{align}
(4.2.2)

(ii) The trajectory in (i) is classical if and only if $x^0 \in \text{dom}(A)$, $u \in C(I;U)$, and (one of) the following two equivalent conditions hold:

(a) the function $x_1(t) := \int_{t_0}^t \mathfrak{A}^{t-s}Bu(s)\,ds$, $t \in I$, belongs to $C(I;X)$;

(b) the function $x_1$ in (a) satisfies $x_1(t) \in \text{dom}(A)$ for all $t \in I$ and $Ax_1 \in C(I;X)$.

(iii) The trajectory in (i) is classical if $u$ is locally absolutely continuous on $I$ or $Bu(t) \in \text{dom}(A)$ for all $t \in I$ and $ABu \in L^1_{\text{loc}}(I;X)$.

**Proof.** This follows from Lemma 2.4.1, Theorem 1.1.21 and Lemma 4.2.5 if $I$ is infinite, i.e., $I = [t_0, \infty)$. If $I$ is finite, then this follows from the same results together with Remark 4.1.22. \(\square\)

4.2.7. **Lemma.** Every semi-bounded i/s/o system is uniquely solvable and has the continuation property.

**Proof.** This follows from the representation formula (4.2.2) which gives existence and uniqueness of classical and generalized trajectories on any time interval. \(\square\)

4.2.2. **Strongly invariant and unobservably invariant subspaces.** At this point the reader may want to recall the notions of strongly invariant and unobservably invariant subspaces of an i/s/o system introduced in Definition 2.5.8.

4.2.8. **Lemma (cf. Lemma 3.2.1).** Let $\Sigma = ([A\ B];\ X, U, Y)$ be a semi-bounded i/s/o system.

(i) If $Z$ is a strongly invariant or unobservable invariant subspace for $\Sigma$, then the closure of $Z$ is also strongly invariant respectively unobservably invariant for $\Sigma$.

(ii) If both $Z_1$ and $Z_2$ are strongly invariant for $\Sigma$, then $Z_1 + Z_2$ and $Z_1 \vee Z_2$ are strongly invariant for $\Sigma$. 


(iii) If both \( Z_1 \) and \( Z_2 \) are unobservably invariant for \( \Sigma \), then \( Z_1 \cap Z_2 \) is unobservably invariant for \( \Sigma \).

**Proof.** This follows from Definition 2.5.8 and the representation formula \[ 4.2.2 \] for trajectories of semi-bounded i/s/o systems.

**Lemma 4.2.9.** Let \( \Sigma = ([A \ B] \ ; \ C \ D) \), \( \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a semi-bounded i/s/o system with evolution group \( \mathfrak{A} \), and let \( Z \) be a closed subspace of \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( Z \) is a strongly invariant subspace for \( \Sigma \);
(ii) \( \text{rng}(B) \subseteq Z \) and \( Z \) is an invariant subspace for \( \mathfrak{A} \);
(iii) \( \text{rng}(B) \subseteq Z \), \( Z \) is an invariant subspace for \( A \), and \( \rho(A|z) \cap \rho_{+\infty}(A) \neq \emptyset \);
(iv) \( \text{rng}(B) \subseteq Z \) and \( (\lambda - A)^{-1}Z \subseteq Z \) for all \( \lambda \in \rho_{+\infty}(A) \);
(v) \( \text{rng}(B) \subseteq Z \) and \((\lambda - A)^{-1}Z \subseteq Z \) for some \( \lambda \in \rho_{+\infty}(A) \);
(vi) \( \text{rng}((\lambda - A)^{-1}B) \subseteq Z \) and \((\lambda - A)^{-1}Z \subseteq Z \) for all \( \lambda \in \rho_{+\infty}(A) \);
(vii) \( \text{rng}((\lambda - A)^{-1}B) \subseteq Z \) and \((\lambda - A)^{-1}Z \subseteq Z \) for some \( \lambda \in \rho_{+\infty}(A) \).

**Proof.** (i) \( \iff \) (ii): See Theorem \[ 4.1.26 \] and Theorem \[ 4.2.6 \].
(iii) \( \iff \) (iv): This follows from Definition 2.5.8 and Theorem \[ 4.2.6 \].
(iv) \( \iff \) (vi): See the proof of the equivalence (v) \( \iff \) (vii) in Lemma 3.2.2.
(vi) \( \iff \) (vii): See the proof of the equivalence (v) \( \iff \) (vii) in Theorem \[ 3.2.2 \].

**Lemma 4.2.10.** Let \( \Sigma = ([A \ B] \ ; \ C \ D) \), \( \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a semi-bounded i/s/o system with evolution group \( \mathfrak{A} \), and let \( Z \) be a closed subspace of \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( Z \) is an unobservably invariant subspace for \( \Sigma \);
(ii) \( Z \subseteq \ker(C) \), \( \ker(C) \subseteq Z \) is an invariant subspace for \( \mathfrak{A} \);
(iii) \( Z \subseteq \ker(C) \), \( Z \) is an invariant subspace for \( A \), and \( \rho(A|z) \cap \rho_{+\infty}(A) \neq \emptyset \);
(iv) \( Z \subseteq \ker(C) \) and \((\lambda - A)^{-1}Z \subseteq Z \) for all \( \lambda \in \rho_{+\infty}(A) \);
(v) \( Z \subseteq \ker(C) \) and \((\lambda - A)^{-1}Z \subseteq Z \) for some \( \lambda \in \rho_{+\infty}(A) \);
(vi) \( Z \subseteq \ker((\lambda - A)^{-1}) \) and \((\lambda - A)^{-1}Z \subseteq Z \) for all \( \lambda \in \rho_{+\infty}(A) \);
(vii) \( Z \subseteq \ker((\lambda - A)^{-1}) \) and \((\lambda - A)^{-1}Z \subseteq Z \) for some \( \lambda \in \rho_{+\infty}(A) \).

**Proof.** The proof is analogous to the proof of Lemma 3.2.3.

**Lemma 4.2.11.** Let \( \Sigma = ([A \ B] \ ; \ C \ D) \), \( \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a semi-bounded i/s/o system with evolution semigroup \( \mathfrak{A} \). Let \( \Omega \) be an arbitrary subset of \( \rho_{+\infty}(A) \) which has a cluster point in \( \rho_{+\infty}(A) \), and let \( \lambda_0 \in \rho_{+\infty}(A) \). Then the following claims are true:

(i) The reachable subspace \( \mathcal{R}_\Sigma \) of \( \Sigma \) can be computed in the following alternative ways:

\[
\mathcal{R}_\Sigma = \bigvee_{t \in \mathbb{R}_+} \text{rng} (\mathfrak{A}^t B) = \bigvee_{\lambda \in \rho_{+\infty}(A)} \text{rng} ((\lambda - A)^{-1} B) = \bigvee_{\lambda \in \Omega} \text{rng} ((\lambda - A)^{-1} B) = \bigvee_{n \in \mathbb{N}_+} \text{rng} ((\lambda_0 - A)^{-n-1} B).
\]

(4.2.3)
(ii) The unobservable subspace $\mathcal{U}_\Sigma$ of $\Sigma$ can be computed in the following alternative ways:

$$\mathcal{U}_\Sigma = \bigcap_{t \in \mathbb{R}^+} \ker(C\mathbb{A}^t) = \bigcap_{\lambda \in \rho_+(A)} \ker(C(\lambda - A)^{-1})$$

(4.2.4)

$$= \bigcap_{\lambda \in \Omega'} \ker(C(\lambda - A)^{-1}) = \bigcap_{n \in \mathbb{Z}^+} \ker(C(\lambda_0 - A)^{-(n+1)}) .$$

(iii) $\mathcal{R}_\Sigma$ is the minimal closed $\mathcal{A}$-invariant subspace which contains $\operatorname{rng}(B)$, and it is also the minimal closed strongly invariant subspace for $\Sigma$.

(iv) $\mathcal{U}_\Sigma$ is the maximal $\mathcal{A}$-invariant subspace which is contained in $\ker(C)$, and it is also the maximal unobservable invariant subspace of $\Sigma$.

PROOF. The proof is analogous to the proof of Lemma 3.2.4 \hfill \Box

4.2.12. LEMMA. Let $\mathcal{X}$ be an $H$-space with the direct sum decomposition $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$, let $\mathcal{A}$ be a $C_0$ semigroup with generator $A$, and let $\Sigma$ be the semi-bounded i/s/o system $\Sigma = (\left[\begin{array}{c} \mathcal{X}_1 \\ \mathcal{Z}_1 \end{array}\right]; \mathcal{X}, \mathcal{X}_1, \mathcal{Z}_1)$. Then the subspace $\mathcal{X}_{\text{min}}$ in (4.1.34) is the reachable subspace of $\Sigma$ and the subspace $\mathcal{Z}_{\text{max}}$ in (4.1.35) is the unobservable subspace of $\Sigma$.

PROOF. This follows from Lemmas 4.1.48 and 4.2.11 \hfill \Box

4.2.3. External equivalence of semi-bounded i/s/o systems. At this point the reader may want to recall what we mean by the future behavior of an i/s/o system (see Definition 2.5.43).

4.2.13. LEMMA (cf. Lemma 3.2.6). Let $\Sigma = (\left[\begin{array}{c} A \\ C \end{array}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-bounded i/s/o system with evolution semigroup $\mathcal{A}$. Then the future behavior of $\Sigma$ consists of all $\left[\begin{array}{c} y \\ \omega \end{array}\right] \in L^1_{\text{loc}}(\left[\begin{array}{c} \mathcal{Y} \\ \mathcal{Z} \end{array}\right])$ which satisfy

$$y(t) = \int_0^t C\mathbb{A}^{t-s} Bu(s) \, ds + Du(t), \quad t \in \mathbb{R}^+$$

(4.2.5)

(this identity should be interpreted in the $L^1$ sense, i.e., it only holds almost everywhere).

PROOF. This follows from Theorem 4.2.6 and Definition 2.5.43 \hfill \Box

At this point the reader may want to recall what we mean by external equivalence of two i/s/o systems (see Definition 2.5.21).

4.2.14. THEOREM (cf. Theorem 3.2.7). Let $\Sigma_i = (\left[\begin{array}{c} A_i \\ C_i \end{array}\right]; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}), \ i = 1, 2$, be two semi-bounded i/s/o systems with evolution semi-groups $\mathcal{A}_i$. Let $\Omega'$ be an arbitrary subset of $\rho_+(A_1 \times A_2)$ which has a cluster point in $\rho_+(A_1 \times A_2)$, and let $\lambda_0 \in \rho_+(A_1 \times A_2)$. Then the following conditions are equivalent:

(i) $\Sigma_1$ and $\Sigma_2$ are externally equivalent;

(ii) $D_1 = D_2$ and $C_1\mathcal{A}_1 B_1 = C_2\mathcal{A}_2 B_2$ for all $t \in \mathbb{R}$;

(iii) $D_1 = D_2$ and $C_1(\lambda - A_1)^{-1} B_1 = C_2(\lambda - A_2)^{-1} B_2$ for all $\lambda \in \rho_+(A_1 \times A_2)$.

(iv) $D_1 = D_2$ and $C_1(\lambda - A_1)^{-1} B_1 = C_2(\lambda - A_2)^{-1} B_2$ for all $\lambda \in \Omega'$.

(v) $D_1 = D_2$ and $C_1(\lambda_0 - A_1)^{-n} B_1 = C_2(\lambda_0 - A_2)^{-n} B_2$ for all $n \in \mathbb{Z}^+$.

PROOF. The proof is analogous to the proof of Theorem 3.2.7 \hfill \Box
4.2.15. **Definition.** Let \( \Sigma = ([A \ B] ; X, U, Y) \) be a semi-bounded i/s/o system with evolution group \( \mathfrak{A} \).

(i) The mapping from \( u \in L^1_{\text{loc}}(\mathbb{R}^+ ; U) \) to \( y \in L^1_{\text{loc}}(\mathbb{R}^+ ; Y) \) defined by \( \Sigma \) is called the future i/o map (input/output map) of \( \Sigma \).

(ii) The \( B(U;Y) \)-valued function
\[
\hat{D}(\lambda) := C(\lambda - A)^{-1}B = D, \quad \lambda \in \rho(A),
\]

is called the i/o (input/output) resolvent of \( \Sigma \).

4.2.16. **Remark.** Using the terminology of **Definition 4.2.15** conditions (iii) and (v) in **Theorem 4.2.14** can be reformulated as follows:

(iii') \( \Sigma_1 \) and \( \Sigma_2 \) have the same i/o map;

(v') The i/o resolvents of \( \Sigma_1 \) and \( \Sigma_2 \) coincide in \( \rho_{+\infty}(\mathfrak{A}_1 \times \mathfrak{A}_2) \).

4.2.17. **Theorem** (cf. **Theorems 3.2.11** and **3.2.12**). Let \( \Sigma = ([A \ B] ; X, U, Y) \) be a semi-bounded i/s/o system with evolution semigroup \( \mathfrak{A} \), and let \( X = X_1 \oplus Z_1 \) be a direct sum decomposition of \( X \).

(i) \( \Sigma \) has a restriction to \( X_1 \) if and only if \( X_1 \) is a strongly invariant subspace for \( \Sigma \).

(ii) Suppose that the equivalent conditions in (i) hold. Then \( X_1 \) is an invariant subspace for \( A \), and \( A \) has a restriction \( A|_{X_1} \) in \( \mathcal{L}(X_1) \) (in the sense of **Definition 4.1.24**). Define
\[
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
A|_{X_1} & B \\
C|_{X_1} & D
\end{bmatrix}.
\]

Then \( \Sigma_1 = ([A_1 \ B_1] ; X_1, U, Y) \) is a semi-bounded i/s/o system with is a restriction of \( \Sigma \) to \( X_1 \), and \( \Sigma_1 \) is the unique semi-bounded restriction of \( \Sigma \). The evolution semigroup \( \mathfrak{A}_1 \) of \( \Sigma_1 \) is the restriction to \( X_1 \) of the evolution semigroup \( \mathfrak{A} \) of \( \Sigma \), and \( \Sigma_1 \) and \( \Sigma \) are externally equivalent.

(iii) \( \Sigma \) has a projection onto \( X_1 \) along \( Z_1 \) if and only if \( Z_1 \) is an unobservably invariant subspace for \( \Sigma \).

(iv) Suppose that the equivalent conditions in (iii) hold. Then \( Z_1 \) is an invariant subspace for \( A \), and \( A \) has a projection \( A_{\text{proj}} \) in \( \mathcal{L}(X_1) \) (in the sense of **Definition 4.1.28**). Define
\[
\begin{bmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{bmatrix} = \begin{bmatrix}
A_{\text{proj}} & P^Z_{X_1}B \\
C|_{X_1} & D
\end{bmatrix}.
\]

Then \( \Sigma_2 = ([A_2 \ B_2] ; X_1, Y) \) is a semi-bounded i/s/o system which is a projection of \( \Sigma \) onto \( X_2 \) along \( Z_2 \), and \( \Sigma_2 \) is the unique semi-bounded projection of \( \Sigma \) onto \( X_2 \) along \( Z_2 \). The evolution semigroup \( \mathfrak{A}_2 \) of \( \Sigma_2 \) is the projection onto \( X_2 \) along \( Z_2 \) of the evolution semigroup \( \mathfrak{A} \) of \( \Sigma \), and \( \Sigma_2 \) and \( \Sigma \) are externally equivalent.

**Proof.** The proof is analogous to the proofs of **Theorems 3.2.11** and **3.2.12**. \( \square \)
4.2.5. Interconnections of semi-bounded i/s/o systems. In Section 2.3, we introduced the notions of the cross product and the parallel, the difference, and the cascade connection of two i/s/o systems $\Sigma_1$ and $\Sigma_2$.

4.2.18. Lemma (cf. Lemma 3.2.14). Let $\Sigma_i = ([A_i \ B_i] ; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i)$, $i = 1, 2$ be two semi-bounded i/s/o systems with i/o maps $\mathfrak{D}_i$ and i/o resolvents $\hat{\mathfrak{D}}_i$, $i = 1, 2$.

(i) The cross product $\Sigma_\times := \Sigma_1 \times \Sigma_2$ of $\Sigma_1$ and $\Sigma_2$ has the following properties:

(a) $\Sigma_\times$ is a semi-bounded i/s/o system $\begin{bmatrix} A_\times & B_\times \\ C_\times & D_\times \end{bmatrix}$ is given by (2.3.20).

(b) the i/o map $\Sigma_\times$ is $\mathfrak{D}_1 \times \mathfrak{D}_2 = \begin{bmatrix} \mathfrak{D}_1 & 0 \\ 0 & \mathfrak{D}_2 \end{bmatrix}$.

(c) the i/o resolvent $\hat{\mathfrak{D}}_\times$ of $\Sigma_\times$ satisfies $\hat{\mathfrak{D}}_\times(\lambda) = \hat{\mathfrak{D}}_1(\lambda) \times \hat{\mathfrak{D}}_2(\lambda)$.

(ii) If $\mathcal{U}_1 = \mathcal{U}_2$ and $\mathcal{Y}_1 = \mathcal{Y}_2$, so that the parallel and difference connections $\Sigma_\parallel := \Sigma_1 \parallel \Sigma_2$ and $\Sigma_\ominus := \Sigma_1 \ominus \Sigma_2$ of $\Sigma_1$ and $\Sigma_2$ are defined, then

(a) $\Sigma_\parallel$ is a semi-bounded i/s/o system $\begin{bmatrix} A_\parallel & B_\parallel \\ C_\parallel & D_\parallel \end{bmatrix}$ is given by (2.3.24) and $\Sigma_\ominus$ is a semi-bounded i/s/o system $\begin{bmatrix} A_\ominus & B_\ominus \\ C_\ominus & D_\ominus \end{bmatrix}$ is given by (2.3.25).

(b) the i/o map $\Sigma_\parallel$ is $\mathfrak{D}_1 + \mathfrak{D}_2$, and the i/o map of $\Sigma_\ominus$ is $\mathfrak{D}_1 - \mathfrak{D}_2$.

(c) the i/o resolvent $\hat{\mathfrak{D}}_\parallel$ of $\Sigma_\parallel$ satisfies $\hat{\mathfrak{D}}_\parallel(\lambda) = \hat{\mathfrak{D}}_1(\lambda) + \hat{\mathfrak{D}}_2(\lambda)$ and the i/o resolvent $\hat{\mathfrak{D}}_\ominus$ of $\Sigma_\ominus$ satisfies $\hat{\mathfrak{D}}_\ominus(\lambda) = \hat{\mathfrak{D}}_1(\lambda) - \hat{\mathfrak{D}}_2(\lambda)$, $\lambda \in \rho(A_1) \cap \rho(A_2)$.

(iii) If $\mathcal{Y}_1 = \mathcal{Y}_2$, so that the cascade connection $\Sigma_\circ := \Sigma_2 \circ \Sigma_1$ and $\Sigma_2$ is defined, then

(a) $\Sigma_\circ$ is a semi-bounded i/s/o system $\begin{bmatrix} A_\circ & B_\circ \\ C_\circ & D_\circ \end{bmatrix}$ is given by (2.3.30).

(b) the i/o map $\Sigma_\circ$ is $\mathfrak{D}_2 \mathfrak{D}_1$.

(c) the i/o resolvent $\hat{\mathfrak{D}}_\circ$ of $\Sigma_\circ$ satisfies $\hat{\mathfrak{D}}_\circ(\lambda) = \hat{\mathfrak{D}}_2(\lambda) \hat{\mathfrak{D}}_1(\lambda)$, $\rho(A_1) \cap \rho(A_2)$.

Proof. Claims (i) and (ii) follow from from Definitions 2.3.32, 2.3.38, 2.3.43 and Lemma 4.1.36.

Proof of (iii): As in the bounded case discussed in Example 2.3.33, it is straightforward to check that the generator operator $S_\circ$ of $\Sigma_\circ$ is given by (2.3.30) with $C_1$ replaced by $C_1|_{\text{dom}(A_1)}$ and $C_2$ replaced by $C_2|_{\text{dom}(A_2)}$. By part Lemmas 4.1.36 and 4.1.16, the operator $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} 0 & B_1 C_1 \\ 0 & B_2 C_2 \end{bmatrix}$ is the generator of a $C_0$ semigroup, and the other operators in (2.3.30) are bounded. Thus $\Sigma_\circ$ is semi-bounded. The remaining claims follow from (2.3.30) and Definition 4.2.15.

4.2.6. Intertwinements of semi-bounded i/s/o systems. At this point the reader might want to recall what we mean by the intertwinement of two i/s/o systems (see Definition 2.5.22).

4.2.19. Lemma (cf. Lemma 3.2.15). If two semi-bounded i/s/o systems $\Sigma_i = ([A_i \ B_i] ; \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i)$, $i = 1, 2$, are intertwined by $P \in \mathcal{ML}(\mathcal{X}_1 ; \mathcal{X}_2)$, then $\Sigma_1$ and $\Sigma_2$ are also intertwined by the closure of $P$. 

□
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PROOF. The proof is analogous to the proof of Lemma 3.2.15. □

4.2.20. LEMMA (cf. Lemma 3.2.16). Let \( \Sigma_i = ([A_i, B_i]; X_i, U, Y) \), \( i = 1, 2 \), be two semi-bounded i/s/o systems, let \( P \in \mathcal{ML}(X_1; X_2) \), and let \( \Sigma = ([A, B]; X, U, Y) = \Sigma_2 \upharpoonright \Sigma_1 = \text{the difference of } \Sigma_2 \text{ and } \Sigma_1 \) (see Lemma 4.2.18). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \);
(ii) \( \text{gph}(P) \) is an invariant subspace for \( A_2 \times A_1 \), \( \text{rng}([B_2]) \subset \text{gph}(P) \subset \ker\left( [C_2 - C_1] \right) \), and \( D_1 = D_2 \);
(iii) \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent and \( \text{gph}(P) \) is both a strongly invariant and an unobservably invariant subspace for \( \Sigma \);
(iv) \( \text{gph}(P) \) is both a strongly invariant and an unobservably invariant subspace for \( \Sigma_\text{p} \), and the i/o resolvent of \( \Sigma_\text{p} \) vanishes on \( \rho_\infty(\Sigma) \).

PROOF. The proof is analogous to the proof of Lemma 3.2.16. □

4.2.21. LEMMA (cf. Lemma 3.2.17). Let \( \Sigma_i = ([A_i, B_i]; X_i, U, Y) \), \( i = 1, 2 \), be two semi-bounded i/s/o systems, let \( P \in \mathcal{ML}(X_1; X_2) \) be closed, and let \( \Sigma = (S; \text{gph}(P), U, Y) \) be the gph(P)-short circuit connection of \( \Sigma_2 \) and \( \Sigma_1 \) (see Definition 2.3.37).

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \) if an only if \( \Sigma \) is a semi-bounded i/s/o node.
(ii) Suppose that the equivalent conditions in (i) hold. Then \( \Sigma = ([A, B]; \Sigma_1, U, \Sigma_2) \) where \([ \Sigma_1 \Sigma_2 ] \) is given by (3.2.14). Moreover, \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by the bounded operator \( P_{\Sigma_1}^{\text{gph}(P)} \), \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

PROOF. The proof is analogous to the proof of Lemma 3.2.17. □

4.2.22. THEOREM (cf. Theorem 3.2.18). Let \( \Sigma_i = ([A_i, B_i]; X_i, U, Y) \), \( i = 1, 2 \), be two semi-bounded i/s/o systems (with the same input and output spaces). Then the following claims are true.

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some \( P \in \mathcal{ML}(X_1; X_2) \) if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.
(ii) If \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent, then the following claims are true.
   (a) There exists a unique minimal closed operator \( P_{\text{min}} \in \mathcal{ML}(X_1; X_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \), i.e., there exists a unique closed \( P_{\text{min}} \in \mathcal{ML}(X_1; X_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \) such that \( \text{gph}(P_{\text{min}}) \subset \text{gph}(P) \) for any other closed \( P \in \mathcal{ML}(X_1; X_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \).
   (b) There exists a unique maximal operator \( P_{\text{max}} \in \mathcal{ML}(X_1; X_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \), i.e., there exists a unique \( P_{\text{max}} \in \mathcal{ML}(X_1; X_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \) such that \( \text{gph}(P_{\text{max}}) \subset \text{gph}(P) \) for any other \( P \in \mathcal{ML}(X_1; X_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \).
   (c) Let \( \Sigma_\text{p} = \Sigma_2 \upharpoonright \Sigma_1 \) be the difference of \( \Sigma_1 \) and \( \Sigma_2 \) (see Lemma 4.2.18). Then the graph of \( P_{\text{min}} \) is equal to the reachable subspace of \( \Sigma_\text{p} \), and the graph of \( P_{\text{max}} \) is equal to the unobservable subspace of \( \Sigma_\text{p} \). In particular, \( P_{\text{max}} \) is closed.
Thus, if $P$ is an arbitrary closed multi-valued operator in $\mathcal{ML}(X_1;X_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, then
\[ \text{gph}(P_{\min}) \subseteq \text{gph}(P) \subseteq \text{gph}(P_{\max}). \]

**Proof.** The proof is analogous to the proof of Theorem 3.2.18. \(\square\)

### 4.2.23. Corollary (cf. Corollary 3.2.19).
Let $\Sigma_i = ([A_i, B_i]; X_i, U, \mathcal{Y})$, $i = 1, 2$, be two semi-bounded i/s/o systems (with the same input and output spaces). Moreover, suppose that both $\Sigma_1$ and $\Sigma_2$ are controllable and observable. Then $\Sigma_1$ and $\Sigma_2$ are pseudo-similar if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent. Among all the pseudo-similarities between $\Sigma_1$ and $\Sigma_2$ there is a (unique) minimal one $P_{\min}$ and a (unique) maximal one $P_{\max}$, namely those defined in Theorem 4.2.22 (both of which in this case are single-valued densely defined injective operators with dense range).

**Proof.** The proof is analogous to the proof of Corollary 3.2.19. \(\square\)

### 4.2.7. Compressions of semi-bounded i/s/o systems.
At this point the reader may want to recall Definition 2.5.28 of what we mean by a compression of an i/s/o system.

#### 4.2.24. Theorem (cf. Theorem 3.2.20).
Let $\Sigma = ([A, B]; \mathcal{X}, U, \mathcal{Y})$ be a semi-bounded i/s/o system with evolution group $\mathfrak{A}$, and let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Then the following conditions are equivalent:

(i) $\Sigma$ has a compression onto $X_1$ along $Z_1$.

(ii) The following four conditions hold:

(a) $\mathfrak{A}^t$ has a compression onto $X_1$ along $Z_1$ for all $t \in \mathbb{R}$,

(b) $P_{Z_1}^{\mathfrak{A}^t}X_1B = P_{Z_1}^{\mathfrak{A}^t}X_1B$ for all $t \in \mathbb{R}$,

(c) $C\mathfrak{A}^t|_{X_1} = CP_{Z_1}^{\mathfrak{A}^t}|_{X_1}$ for all $t \in \mathbb{R}$,

(d) $C\mathfrak{A}^tB = CP_{Z_1}^{\mathfrak{A}^t}X_1B$ for all $t \in \mathbb{R}$.

(iii) The following four conditions hold:

(a) $(\lambda - A)^{-1}$ has a compression onto $X_1$ along $Z$ for all $\lambda \in \rho_+(A)$,

(b) $P_{Z_1}^{(\lambda - A)^{-1}}X_1B = P_{Z_1}^{(\lambda - A)^{-1}}X_1B$ for all $\lambda \in \rho_+(A)$,

(c) $C(\lambda - A)^{-1}|_{X_1} = CP_{Z_1}^{(\lambda - A)^{-1}}|_{X_1}$ for all $\lambda \in \rho_+(A)$,

(d) $C(\lambda - A)^{-1}B = CP_{Z_1}^{(\lambda - A)^{-1}}X_1B$ for all $\lambda \in \rho_+(A)$.

(iv) Conditions (a)-(d) in (iii) hold with $\rho_+(A)$ replaced by an arbitrary subset $\Omega'$ of $\rho_{+\infty}(A)$ which has a cluster point in $\rho_{+\infty}(A)$.

Suppose that these equivalent conditions hold, and let $A_{\text{cmp}}$ be the generator of the evolution semigroup of the compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$. Define

\begin{equation}
(4.2.9)
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} = \begin{bmatrix}
A_{\text{cmp}} & P_{Z_1}X_1B \\
C|_{X_1} & D
\end{bmatrix}.
\end{equation}

Then $\Sigma_1 = ([A_1, B_1]; X_1, \mathcal{Y})$ is a semi-bounded i/s/o system which is a compression of $\Sigma$ onto $X_1$ along $Z_1$, and $\Sigma_1$ is the unique semi-bounded compression of $\Sigma$ onto $X_1$ along $Z_1$. The evolution semigroup $\mathfrak{A}_1$ of $\Sigma_1$ is the compression onto $X_1$ along $Z_1$ of the evolution semigroup $\mathfrak{A}$ of $\Sigma$, and $\Sigma_1$ and $\Sigma$ are externally equivalent.

**Proof.** (i) ⇒ (ii): See the proof of the implication (i) ⇒ (iii) in Theorem 3.2.20.

(ii) ⇒ (i): See the proof of the implication (iii) ⇒ (i) in Theorem 3.2.20.
(ii) $\iff$ (iii): This follows from Lemma 4.1.45.

(iii) $\iff$ (iv): This follows from the analyticity of all the functions appearing in (iv) and the fact that $\rho_{+\infty}(A)$ is connected.

Theorem 4.2.24 can be reformulated in many different ways. One such reformulation which puts equal emphasis on the compression $\Sigma_1$ and the dilation $\Sigma$ is the following:

4.2.25. Lemma (cf. Lemma 3.2.21). Let $\Sigma = ([A \ B] ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-bounded i/s/o system with evolution group $\mathfrak{A}$, let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\Sigma_1 = ([A_1 \ B_1] ; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ be another semi-bounded i/s/o system with evolution group $\mathfrak{A}_1$. Then the following conditions are equivalent:

(i) $\Sigma_1$ is the (unique) semi-bounded compression of $\Sigma$;

(ii) The following four conditions hold:
   (a) $\mathfrak{A}_1^t = P_{X_1}^t \mathfrak{A}^t|_{X_1}$ for all $t \in \mathbb{R}$ (i.e., $\mathfrak{A}_1$ is the compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$);
   (b) $\mathfrak{A}_1^t B_1 = P_{X_1}^t \mathfrak{A}^t B$ for all $t \in \mathbb{R}$,
   (c) $C_1 \mathfrak{A}_1^t = C \mathfrak{A}^t|_{X_1}$ for all $t \in \mathbb{R}$,
   (d) $D_1 = D$ and $C_1 \mathfrak{A}_1^t B_1 = C \mathfrak{A}^t B$ for all $t \in \mathbb{R}$.

(iii) The following four conditions hold:
   (a) $\rho_{+\infty}(A) \cap \rho(A_1)$ and $(\lambda - A_1)^{-1} = P_{X_1}^t (\lambda - A)^{-1}|_{X_1}$ for all $\lambda \in \rho_{+\infty}(A)$ (or equivalently, $(\lambda - A_1)^{-1}$ is a compression in $X_1$ along $Z_1$ of $(\lambda - A)^{-1}$ for all $\lambda \in \rho_{+\infty}(A)$);
   (b) $(\lambda - A_1)^{-1} B_1 = P_{X_1}^t (\lambda - A)^{-1} B$ for all $\lambda \in \rho_{+\infty}(A)$,
   (c) $C_1 (\lambda - A_1)^{-1} = C (\lambda - A)^{-1}|_{X_1}$ for all $\lambda \in \rho_{+\infty}(A)$,
   (d) $D_1 = D$ and $C_1 (\lambda - A_1)^{-1} B_1 = C (\lambda - A)^{-1} B$ for all $\lambda \in \rho_{+\infty}(A)$.

(iv) Conditions (a)–(d) in (iv) hold with $\rho_{+\infty}(A)$ replaced by an arbitrary subset $\Omega'$ of $\rho_{+\infty}(A)$ which has a cluster point in $\rho_{+\infty}(A)$.

Proof. This follows immediately from Theorem 4.2.24.

Another reformulation of Theorem 4.2.24 is the following.

4.2.26. Lemma (cf. Lemma 3.2.22). Let $\Sigma = ([A \ B] ; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-bounded i/s/o system with evolution group $\mathfrak{A}$, and let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Then the following conditions are equivalent:

(i) Conditions (a)–(d) in part (ii) of Theorem 4.2.24 are equivalent to the following four conditions:
   (a) $P_{X_1}^s \mathfrak{A}^s P_{X_1}^{s'} \mathfrak{A}^t|_{X_1} = 0$ for all $s, t \in \mathbb{R}$,
   (b) $P_{X_1}^s \mathfrak{A}^s P_{X_1}^{s'} \mathfrak{A}^t B = 0$ for all $s, t \in \mathbb{R}$,
   (c) $C \mathfrak{A}^s P_{X_1}^{s'} \mathfrak{A}^t|_{X_1} = 0$ for all $s, t \in \mathbb{R}$,
   (d) $C \mathfrak{A}^s P_{X_1}^{s'} \mathfrak{A}^t B = 0$ for all $s, t \in \mathbb{R}$.

(ii) Conditions (a)–(d) in part (iii) of Theorem 4.2.24 are equivalent to the following four conditions:
   (a) $P_{X_1}^s (\lambda - A)^{-1} P_{X_1}^{s'} (\mu - A)^{-1}|_{X_1} = 0$ for all $\lambda, \mu \in \rho_{+\infty}(A)$,
   (b) $P_{X_1}^s (\lambda - A)^{-1} P_{X_1}^{s'} (\mu - A)^{-1} B = 0$ for all $\lambda, \mu \in \rho_{+\infty}(A)$,
   (c) $C (\lambda - A)^{-1} P_{X_1}^{s'} (\mu - A)^{-1}|_{X_1} = 0$ for all $\lambda, \mu \in \rho_{+\infty}(A)$,
   (d) $C (\lambda - A)^{-1} P_{X_1}^{s'} (\mu - A)^{-1} B = 0$ for all $\lambda, \mu \in \rho_{+\infty}(A)$.
(iii) Conditions (a)–(d) in part (iv) of Theorem 4.2.24 are equivalent to conditions (a)–(d) in (ii) above with $\rho_{+\infty}(A)$ replaced by $\Omega'$.

**Proof.** The proof is analogous to the proof of Lemma 3.2.22. □

4.2.8. The general structure of a semi-bounded i/o compression.

4.2.27. Lemma (cf. Lemma 3.2.23). Let $\Sigma = ([A\ B] : \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-bounded i/o system, let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\Sigma_{\text{ext}} = ([A_{\text{ext}}\ B_{\text{ext}}] : \mathcal{X}, [\mathcal{U}], [\mathcal{Y}])$ be the semi-bounded i/o system with system operator

$$\begin{bmatrix} A_{\text{ext}} & B_{\text{ext}} \\ C_{\text{ext}} & D_{\text{ext}} \end{bmatrix} = \begin{bmatrix} A & B & I_{X_1} \\ C & D & 0 \\ P_{X_1} & 0 & 0 \end{bmatrix}.$$

(i) There exists a (unique) minimal closed strongly invariant subspace $X_{\text{min}}$ for $\Sigma$ which contains $X_1$ (i.e., $X_{\text{min}}$ is closed and strongly invariant for $\Sigma$, and $X_{\text{min}}$ is contained in every other closed strongly invariant subspace of $\Sigma$ which contains $X_1$). This subspace has the following alternative descriptions:

(a) $X_{\text{min}}$ is the reachable subspace of $\Sigma_{\text{ext}}$;

(b) $X_{\text{min}}$ is the closed linear span of the reachable subspace of $\Sigma$ and the subspace $X_{\text{min}}$ defined in Lemma 4.1.47.

(ii) The space $X_{\text{min}}$ has the direct sum decomposition $X_{\text{min}} = X_1 + Z_{\text{min}}$, where

$$Z_{\text{min}} = X_{\text{min}} \cap Z_1 = P_{X_1} X_{\text{min}}.$$

(iii) There exists a (unique) maximal unobservably invariant subspace $Z_{\text{max}}$ for $\Sigma$ which is contained in $Z_1$ (i.e., $Z_{\text{max}}$ is observably invariant for $\Sigma$, and $Z_{\text{max}}$ contains every other unobservably invariant subspace for $\Sigma$ which is contained in $Z_1$). This subspace has the following alternative descriptions:

(a) $Z_{\text{max}}$ is the unobservable subspace of $\Sigma_{\text{ext}}$;

(b) $Z_{\text{max}}$ is the intersection of the unobservable subspace of $\Sigma$ and the subspace $Z_{\text{max}}$ defined in Lemma 4.1.47.

In particular, $Z_{\text{max}}$ is closed.

**Proof.** The proof is analogous to the proof of Lemma 3.2.23. □

4.2.28. Lemma (cf. Lemma 3.2.24). Let $\Sigma = ([A\ B] : \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-bounded i/o system with evolution semigroup $\mathfrak{A}$, and let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\rho_{+\infty}(A)$, and let $\lambda_0 \in \rho_{+\infty}(A)$. Denote the injection operator $X_1 \hookrightarrow \mathcal{X}$ by $I_{X_1}$.

(i) The subspace $X_{\text{min}}$ in Lemma 4.2.27 can be computed in the following alternative ways:

$$X_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng} (\mathfrak{A}^t [B I_{X_1}]) = \bigvee_{\lambda \in \rho_{+\infty}(A)} \text{rng} ((\lambda - A)^{-1} [B I_{X_1}])$$

$$= \bigvee_{\lambda \in \Omega'} \text{rng} ((\lambda - A)^{-1} [B I_{X_1}]) = \bigvee_{n \in \mathbb{Z}^+} \text{rng} ((\lambda_0 - A)^{-(n+1)} [B I_{X_1}]).$$
\(Z\) are equivalent:

Let \(X\) be an unobservably invariant subspace for \(\Sigma\), and let \(X\) satisfies \(A \cap Z_{\min} \neq \emptyset\), and \(X\) contains some closed unobservably invariant subspace \(\Sigma\) which contains \(X\). Then the following conditions are equivalent:

(i) \(\Sigma\) has a compression onto \(X\) along \(Z_{\min}\);
(ii) \(Z_{\min}\) contains some closed unobservably invariant subspace \(Z\) for \(\Sigma\) such that \(X + Z\) is strongly invariant for \(\Sigma\);
(iii) \(Z_{\min}\) in an unobservably invariant subspaces for \(\Sigma\);
(iv) \(X + Z_{\max}\) is a strongly invariant subspace for \(\Sigma\);
(v) \(Z_{\min} \subset Z_{\max}\).

Two possible choices of the subspace \(Z\) in (ii) are \(Z = Z_{\min}\) and \(Z = Z_{\max}\), and every possible subspace \(Z\) in (ii) satisfies \(Z_{\min} \subset Z \subset Z_{\max}\).

Suppose that the equivalent conditions (i)--(v) hold, denote the compression generator of the compression of \(A\) onto \(X\) along \(Z_1\) by \(A_{\text{cmp}}\), and define \(A_{\text{cmp}}^t = B_{\Sigma_1}^t A_{\Sigma_1}^t\) by (4.2.9). Then the semi-bounded i/s/o system \(\Sigma_1 = ([A, B_1], [C, D_1], \{X_1, U, Y\})\) is a compression of \(\Sigma\) onto \(X_1\) along \(Z_1\), and \(\Sigma_1\) is the unique semi-bounded compression of \(\Sigma\).

\[Z_{\min} = \bigcup_{t \in \mathbb{R}^+} \text{rng} \left( P_{Z_1} X^t [B \ I_{X_1}] \right) = \bigcup_{\lambda \in \rho_{\infty}(A)} \text{rng} \left( (P_{Z_1} \lambda - A)^{-1} [B \ I_{X_1}] \right) = \bigcup_{\lambda \in \Omega'} \text{rng} \left( P_{Z_1} (\lambda - A)^{-1} [B \ I_{X_1}] \right) = \bigcup_{n \in \mathbb{Z}^+} \text{rng} \left( P_{Z_1} (\lambda_0 - A)^{-1} [(n+1) [B \ I_{X_1}] \right).

\[Z_{\max} = \bigcap_{t \in \mathbb{R}^+} \{ x \in X \mid X^t x = 0 \} \]

\[= \bigcap_{\lambda \in \rho_{\infty}(A)} \{ x \in X \mid (\lambda - A)^{-1} x \in Z_{\min} \} \cap \{ x \in X \mid (\lambda - A)^{-1} x \in Z_{\min} \} \cap \{ x \in X \mid (\lambda_0 - A)^{-1} x \in Z_{\min} \} = \bigcap_{n \in \mathbb{Z}^+} \{ x \in X \mid (\lambda_0 - A)^{- (n+1)} x \in Z_{\min} \} \cap \{ x \in X \mid (\lambda_0 - A)^{- (n+1)} x \in Z_{\min} \}.

Proof. This follows from Lemmas 4.2.11 and 4.2.27. \(\square\)

4.2.29. Theorem (cf. Theorem 3.2.25). Let \(\Sigma = ([A, B] : X, U, Y)\) be a semi-bounded i/s/o system, and let \(X = X_1 + Z_1\) be a direct sum decomposition of \(X\). Let \(X_{\min}\) be the minimal closed strongly invariant subspace of \(\Sigma\) which contains \(X_1\), let \(Z_{\max}\) be the maximal unobservably invariant subspace of \(\Sigma\) which is contained in \(Z_1\), and let \(Z_{\min} = X_{\min} \cap Z_1\) (cf. Lemma 4.2.27). Then the following conditions are equivalent:

(i) \(\Sigma\) has a compression onto \(X_1\) along \(Z_1\);
(ii) \(Z_1\) contains some closed unobservably invariant subspace \(Z\) for \(\Sigma\) such that \(X_1 + Z\) is strongly invariant for \(\Sigma\);
(iii) \(X_{\min}\) in an unobservably invariant subspaces for \(\Sigma\);
(iv) \(X + Z_{\max}\) is a strongly invariant subspace for \(\Sigma\);
(v) \(Z_{\min} \subset Z_{\max}\).

Two possible choices of the subspace \(Z\) in (ii) are \(Z = Z_{\min}\) and \(Z = Z_{\max}\), and every possible subspace \(Z\) in (ii) satisfies \(Z_{\min} \subset Z \subset Z_{\max}\).

Suppose that the equivalent conditions (i)--(v) hold, denote the compression generator of the compression of \(A\) onto \(X_1\) along \(Z_1\) by \(A_{\text{cmp}}\), and define \(A_{\text{cmp}}^t = B_{\Sigma_1}^t A_{\Sigma_1}^t\) by (4.2.9). Then the semi-bounded i/s/o system \(\Sigma_1 = ([A, B_1], [C, D_1], \{X_1, U, Y\})\) is a compression of \(\Sigma\) onto \(X_1\) along \(Z_1\), and \(\Sigma_1\) is the unique semi-bounded compression of \(\Sigma\).

Proof. Repeat the proof of Theorem 3.2.25 with \(m, n \in \mathbb{Z}^+\) replaced by \(s, t \in \mathbb{R}^+\), and \(A^m\) and \(A^n\) replaced by \(A^s\) respectively. \(\square\)
4.2.30. Corollary (cf. Corollary 3.2.26). Let $\Sigma = ([A, B] ; X, U, Y)$ be a semi-bounded i/s/o system with evolution semigroup $A$, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, and let $\Sigma_1 = ([A_1, B_1] ; X_1, U, Y)$ be a semi-bounded i/s/o system with evolution semigroup $A_1$. Then $\Sigma_1$ is a compression of $\Sigma$ onto $X_1$ along $Z_1$ if and only if $Z_1$ has a direct sum decomposition $Z_1 = Z + Z_c$ such that $A, B, C,$ and $D$ have the following structure with respect to the decomposition $X = X_1 + Z_1 + Z_c$ of $X$ (where we use "*" to mark irrelevant entries):

\begin{equation}
\begin{bmatrix}
A_t & B
\end{bmatrix} = \begin{bmatrix}
A_{X_1}^t & * & * & 0
0 & A_{X_1}^t & * & B_1
0 & 0 & A_{Z_c}^t & 0
0 & C_1 & * & D_1
\end{bmatrix}, \quad t \in \mathbb{R}^+.
\end{equation}

From this structure follows that $Z + X_1$ is strongly invariant for $\Sigma$, that $Z$ is unobservably invariant of $\Sigma$, that $A_{Z_c}$ is the restriction of $A$ to $Z$, and that $A_{X_1}$ is the projection onto $X_1$ of $A$ along $X_1 + Z$. The subspace $Z$ in this decomposition can be chosen to be the same as the subspace $Z$ in condition (ii) in Theorem 4.2.29, and the subspace $Z_c$ can be chosen to be an arbitrary direct complement to $Z$ in $Z_1$. In particular, two possible choices of $Z$ are $Z = Z_{\min}$ and $Z = Z_{\max}$, where $Z_{\min}$ and $Z_{\max}$ are the subspaces defined in Lemma 4.2.30.

Proof. This follows from the equivalence of (i) and (ii) in Theorem 4.2.29 (take $Z_c$ to be an arbitrary direct complement to $Z$ in $Z_1$).

4.2.31. Theorem (cf. Theorem 3.2.27). Let $\Sigma = ([A, B] ; X, U, Y)$ be a semi-bounded i/s/o system, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, and suppose that $\Sigma_1 = ([A_1, B_1] ; X_1, U, Y)$ is a semi-bounded compression of $\Sigma$ onto $X_1$ along $Z_1$. Let $Z$ satisfy the conditions listed in (ii) in Theorem 4.2.29 and let $Z_c$ be an arbitrary direct complement to $Z$ in $Z_1$.

(i) Let $\Sigma_2$ be the unique semi-bounded restriction of $\Sigma$ to the strongly invariant subspace $X_1 + Z$ for $\Sigma$ given by Theorem 4.2.17. Then $Z$ is unobservably invariant for $\Sigma_2$, and $\Sigma_1$ is the unique semi-bounded projection onto $X_1$ along $Z$ of $\Sigma_2$ given by Theorem 4.2.17.

(ii) Let $\Sigma_3$ be the unique semi-bounded projection of $\Sigma$ onto $X_1 + Z_c$ along $Z$ given by Theorem 4.2.17. Then $X_1$ is strongly invariant for $\Sigma_3$, and $\Sigma_1$ is the unique semi-bounded restriction of $\Sigma_3$ to $X_1$ given by Theorem 4.2.17.

(iii) The main operator $A_1$ of $\Sigma_1$ is the projection in $\mathcal{L}(X_1)$ along $Z$ of the restriction in $\mathcal{L}(X_1 + Z)$ of $A$, and it is also the restriction in $\mathcal{L}(X_1)$ of the projection of $A$ in $\mathcal{L}(X_1 + Z_c)$ along $Z$.

Proof. The proof is analogous to the proof of Theorem 3.2.27.

4.2.32. Lemma (cf. Lemma 3.2.28). Let $\Sigma = ([A, B] ; X, U, Y)$ and $\Sigma_1 = ([A_1, B_1] ; X_1, U, Y)$ be two semi-bounded i/s/o systems with $X = X_1 + Z_1$. Then the following two conditions are equivalent.

(i) $\Sigma_1$ is the compression of $\Sigma$ onto $X_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z$ such that $\Sigma$ and $\Sigma_1$ are intertwined by the operator $P_{X_1}^{Z_1} |_{X_1 + Z}$.

Condition (ii) above holds for some particular subspace $Z$ if and only condition (ii) in Theorem 4.2.29 holds for the same subspace $Z$. Thus, in particular, two possible
operators with dense range). (both of which in this case are single-valued densely defined injective
of \( \Sigma \) described below, where we have denoted the reachable and unobservable subspaces
has a semi-bounded minimal compression. Two families of such compressions are
minimal one \( P \) equivalent. Among all the pseudo-similarities between
\( \Sigma \) and \( \Sigma_1 \) has a unique semi-bounded restriction. This follows from Corollary 4.2.23 and Theorem 4.2.33.

Proof. The proof is analogous to the proof of Theorem 3.2.29. □

4.2.9. Compressions into minimal semi-bounded i/s/o systems. The
results that we have proved above permit us to prove the following theorem.

4.2.33. Theorem (cf. Theorem 3.2.29). A semi-bounded i/s/o system
\( \Sigma = ([A \ B] \ \ X, U, Y) \) is minimal if and only if \( \Sigma \) is both controllable and observable.

Proof. The proof is analogous to the proof of Theorem 3.2.29. □

4.2.34. Corollary (cf. Corollary 3.2.30). Let \( \Sigma_i = ([A_i \ B_i] \ \ X_i, U, Y), \ i = 1, 2 \), be two minimal semi-bounded i/s/o systems (with the same input and output spaces). Then \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. Among all the pseudo-similarities between \( \Sigma_1 \) and \( \Sigma_2 \) there is a (unique) minimal one \( P_{\min} \) and a (unique) maximal one \( P_{\max} \), namely those defined in Theorem 4.2.22 (both of which in this case are single-valued densely defined injective operators with dense range).

Proof. This follows from Corollary 4.2.29 and Theorem 4.2.33. □

4.2.35. Theorem (cf. Theorem 3.2.31). Every semi-bounded i/s/o system \( \Sigma \) has a minimal semi-bounded minimal compression. Two families of such compressions are described below, where we have denoted the reachable and unobservable subspaces of \( \Sigma \) by \( R_\Sigma \) respectively \( U_\Sigma \):

(i) Let \( X_1 \) be a direct complement to \( U_\Sigma \) in \( X \), and let \( \Sigma_1 = ([A_1 \ B_1] \ \ X_1, U, Y) \) be the the unique semi-bounded projection of \( \Sigma \) onto \( X_1 \) along its unobservably invariant subspace \( U_\Sigma \). Then \( X_\circ = P_{\Sigma_1 \Sigma} R_\Sigma \) is strongly invariant for \( \Sigma_1 \), and hence \( \Sigma_1 \) has a unique semi-bounded restriction \( \Sigma_\circ = ([A_\circ \ B_\circ] \ ; X_\circ, U, Y) \) to \( X_\circ \). The system \( \Sigma_\circ \) is a minimal semi-bounded compression of \( \Sigma \) onto \( X_\circ \) along \( U_\Sigma + Z_\circ \) where \( Z_\circ \) is an arbitrary direct complement to \( X_\circ \) in \( X_1 \). The main operator \( A_\circ \) of \( \Sigma_\circ \) is restriction in \( L(X_\circ) \) of the projection of \( A \) in \( L(X_1) \) along \( U_\Sigma \).

(ii) By Lemma 4.2.11 \( A(\mathcal{R}_\Sigma \cap \text{dom}(A)) \subset \mathcal{R}_\Sigma \) and \( \text{rng}(B) \subset \mathcal{R}_\Sigma \), so that we may interpret \( A|_{\mathcal{R}_\Sigma} \) and \( B \) as operators with range space \( \mathcal{R}_\Sigma \). Let \( X_\bullet \) be a direct complement to \( R_{\mathcal{R}_\Sigma} \cap U_\Sigma \) in \( R_\Sigma \), and define (with the above interpretation of \( A|_{X_\bullet} = (A|_{\mathcal{R}_\Sigma})|_{X_\bullet} \) and \( B \))

\[
\begin{bmatrix}
A_\bullet & B_\bullet \\
C_\bullet & D_\bullet
\end{bmatrix} = 
\begin{bmatrix}
P^{R_{\mathcal{R}_\Sigma} \cap U_\Sigma} A|_{X_\bullet} & P^{R_{\mathcal{R}_\Sigma} \cap U_\Sigma} B
\end{bmatrix}.
\]

Then \( \Sigma_\bullet = ([A_\bullet \ B_\bullet] ; X_\bullet, U, Y) \) is a semi-bounded minimal compression of \( \Sigma \). This compression is the system that one gets by first restricting \( \Sigma \) to its reachable subspace \( R_\Sigma \), and then projecting the resulting system onto \( X_\bullet \) along its unobservable subspace \( R_{\mathcal{R}_\Sigma} \cap U_\Sigma \). The main operator \( A_\bullet \) of \( \Sigma_\bullet \) is projection in \( L(X_\bullet) \) along \( R_{\mathcal{R}_\Sigma} \cap U_\Sigma \) of the restriction of \( A \) in \( L(R_\Sigma) \).

Proof. The proof is analogous to the proof of Theorem 3.2.31. □
4.2.36. **Lemma** (cf. Lemma 3.2.32). The minimal semi-bounded compression of a semi-bounded i/s/o system \( \Sigma = (A, B, C, D; X, U, Y) \) is unique if and only at least one of conditions (i) and (ii) below holds:

(i) \( \Sigma \) is observable, i.e., \( \mathcal{U}_\Sigma = \{0\} \), where \( \mathcal{U}_\Sigma \) is the unobservable subspace of \( \Sigma \),

(ii) the following equivalent conditions hold:
   (a) \( \Sigma \) has a compression with state space \( \{0\} \),
   (b) the i/o resolvent of \( \Sigma \) is a constant in \( \rho_\infty(\Sigma) \),
   (c) \( \mathcal{R}_\Sigma \subseteq \mathcal{U}_\Sigma \), where \( \mathcal{R}_\Sigma \) is the reachable subspace of \( \Sigma \).

In case (i) the unique minimal compression \( \Sigma_{\text{min}} \) is the restriction of \( \Sigma \) to \( \mathcal{R}_\Sigma \), i.e., \( \Sigma_{\text{min}} = (S_{\text{min}}; \mathcal{R}_\Sigma, U, Y) \) where

\[
S_{\text{min}} = \begin{bmatrix}
A_{\text{min}} & B_{\text{min}} \\
C_{\text{min}} & D_{\text{min}}
\end{bmatrix} = \begin{bmatrix}
A|_{\mathcal{R}_\Sigma} & B \\
C|_{\mathcal{R}_\Sigma} & D
\end{bmatrix}.
\]

In case (ii) the unique minimal compression is \( \Sigma_{\text{min}} = (\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}; \{0\}, U, Y) \). If neither (i) nor (ii) holds, then \( \Sigma \) has an infinite number of minimal compressions.

**Proof.** The proof is analogous to the proof of Lemma 3.2.32. \(\square\)
4.3. Semi-Bounded S/S Systems (Feb 02, 2016)

In this section we study semi-bounded s/s systems in the same spirit as in the previous section on semi-bounded i/s/o system. Many of the results are natural extensions of the corresponding i/s/o results, but some of the natural s/s formulations are less obvious. In some cases we are able to prove results directly for s/s system by using arguments analogous to those that we gave for i/s/o system, and in other cases the s/s results are reduced to the corresponding i/s/o result by use of i/s/o semi-bounded i/s/o representations of the semi-bounded s/s system.

4.3.1. Introduction to semi-bounded s/s systems.

4.3.1. Definition. Let \( \Sigma_{i/s/o} = (V; \mathcal{X}, \mathcal{W}) \) be a s/s system.

(i) We say that \( \Sigma \) is semi-bounded if \( \Sigma \) has at one semi-bounded i/s/o representation.

(ii) An i/o representation \((U, Y)\) is called semi-boundedly i/s/o-admissible for \( \Sigma \) if the corresponding i/s/o representation is semi-bounded (and hence \( \Sigma \) is semi-bounded).

4.3.2. Lemma (cf. Lemma 2.2.26 and Theorem 2.2.27). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a semi-bounded s/s system.

(i) If \( \Sigma_{i/s/o} = ([A \ B]; \mathcal{X}, U, Y) \) is a semi-bounded i/s/o representation of \( \Sigma \), then \( V \) has the following three equivalent representations:

\[
V = \begin{bmatrix}
1_x & 0 & 0 & 0 \\
0 & 0 & 1_x & 0 \\
0 & \mathcal{I}_y & 0 & \mathcal{I}_U \\
0 & 0 & \mathcal{I}_y & 0
\end{bmatrix}
gph\left(\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}\right), \tag{4.3.1a}
\]

\[
V = \text{rng}\left(\begin{bmatrix}
A & B \\
1_x & 0 \\
\mathcal{I}_y & \mathcal{I}_y D + \mathcal{I}_U
\end{bmatrix}\right), \tag{4.3.1b}
\]

\[
V = \ker\left(\begin{bmatrix}
-1_x & A & BP^Y_{U} \\
0 & C & DP^D_{U} - P^Y_{U}
\end{bmatrix}\right). \tag{4.3.1c}
\]

Here the block operator matrix in (4.3.1b) maps \([\text{dom}(A)]\) into \( \mathcal{R} \), and the block operator matrix in (4.3.1c) maps \([\text{dom}(A)]\) into \([\mathcal{X}]\).

(ii) The generating operator operator \([A \ B]_{C D}\) of the semi-bounded i/s/o system in (i) can be recovered from \( V \) as follows:

\[
gph\left(\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}\right) = \begin{bmatrix}
1_x & 0 & 0 & 0 \\
0 & 0 & P^D_{U} & P^Y_{U} \\
0 & 1_x & 0 & P^Y_{U} \\
0 & 0 & P^D_{U} & P^Y_{U}
\end{bmatrix} V. \tag{4.3.2}
\]

(iii) The subspace \( \mathcal{W}_0 \) in (1.1.7) is closed, and it is given by (2.2.20).

---

1In the formulas (4.3.1) and (4.3.2) we use the convention introduced in Notation 2.1.2 regarding the domain of a block matrix operator where the operators in the same column are allowed to have different domains. Note, in particular, that (4.3.2) defines directly only the restriction of \( C \) to \( \text{dom}(A) \), but since \( C \) is bounded and \( \text{dom}(A) \) is dense in \( \mathcal{X} \) this defines \( C \) uniquely.
4.3.3. Theorem (cf. Theorem 2.2.29). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a semi-bounded s/s node, let \((U, Y)\) be an i/o representation of \(\mathcal{W}\), and define \( W_0 \) as in (1.1.7). Then \( W_0 \) is closed and the following conditions are equivalent:

(i) the i/o representation \((U, Y)\) of \(\mathcal{W}\) is semi-boundedly i/s/o-admissible for \(\Sigma\);

(ii) \( Y \) is a direct complement to \( W_0 \) (and \( U \) is a direct complement to \( Y \));

(iii) \( P^Y_U|_{W_0} \) maps \( W_0 \) one-to-one onto \( U \) (and hence it has an inverse in \( B(U; W_0) \));

(iv) \( \text{rng} \left( \begin{bmatrix} P^Y_U|_{W_0} \\ P^Y_U|_{W_0} \end{bmatrix} \right) \) is the graph of an operator \( D \in B(U; Y) \).

Suppose that the equivalent conditions (i)–(iv) above hold. The following additional claims are true:

(v) The operator \( D \) in (iv) is equal to the feedthrough operator \( D \) of the semi-bounded i/s/o representation \( \Sigma_{i/s/o} = ([A, B] : \mathcal{X}, U, \mathcal{W}) \) corresponding to the i/o representation \((U, Y)\) of \(\Sigma\).

(vi) Let \( \Sigma_{i/s/o} = ([A, B] : \mathcal{X}, W_0, \mathcal{Y}) \) be the semi-bounded i/s/o representation corresponding to the i/o representation \((W_0, Y)\) of \(\Sigma\). Then \( D_0 = 0 \) and the quadruple \([A, B, C, D]\) in (v) is given by

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix} A_0 & B_0(P^Y_U|_{W_0})^{-1} \\
C_0 & P^Y_U|_{W_0}^{-1}
\end{bmatrix}.
\]

In particular, \( A \) and \( C \) do not depend on the choice of \( U \), and \( D = P^Y_U(P^Y_U|_{W_0})^{-1} \) depends only on \( W_0 \) (which depends on \( V \), \( U \), and \( Y \)). Moreover, \( D = 0 \) if and only if \( U = W_0 \).

Proof. The proof is analogous to the proof of Theorem 2.2.29. We therefore leave the some of the details to the reader and here concentrate on the differences between this proof and the proof of Theorem 2.2.29.

That \( W_0 \) is closed follows from Lemma 4.3.2.

(ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv): This follows from Lemma 2.2.12.

(i) \( \Rightarrow \) (ii), and the i/o representation \((W_0, Y)\) is semi-boundedly admissible: Denote the semi-bounded i/s/o representation corresponding to the i/o decomposition \((U, Y)\) by \( \Sigma = ([A, B] : \mathcal{X}, U, \mathcal{W}) \). By Lemma 4.3.2, \( W_0 = [I_Y, I_U] \text{gph}(D) \), and therefore by Lemma 2.2.12, \( W = W_0 + Y \). Thus (ii) holds. Let \( \Theta \) be the transition matrix from \((U, Y)\) to \((W_0, Y)\) (see Definition 2.2.9). Then \( \Theta = \begin{bmatrix} P^{W_0}_{W_0,U} |_{W_0} & 0 \\ P^{W_0}_{W_0,U} |_{W_0} & 1_Y \end{bmatrix} \), and by Lemma 2.2.12, \( P^{W_0}_{W_0,U} \) maps \( U \) one-to-one onto \( W_0 \). Define \( K \) and \( M \) by (2.2.28) and (2.2.29) with \( S_0 \) replaced by \([A, B]\). Then \( K = \begin{bmatrix} 0 & 0 \\ 0 & P^{W_0}_{W_0,U} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & P^{W_0}_{W_0,U} \end{bmatrix} \), and \( M = \begin{bmatrix} 1_{\text{dom}(A)} & 0 \\ 0 & P^{W_0}_{W_0,U} \end{bmatrix} \). Here \( M \) maps \([\text{dom}(A)] \) one-to-one onto \([\text{dom}(A)] \). By Theorem 2.2.25, the decomposition \((W_0, Y)\) is i/s/o admissible for \(\Sigma\), and the system operator \( S_0 \) of the corresponding i/s/o representation \( \Sigma_{i/s/o} = (S_0; \mathcal{X}, W_0, \mathcal{Y}) \) is given by \( S_0 = \cdots \).
This implies that $\Sigma^0$ is semi-bounded. By (2.2.20) the feedthrough operator $(P^Y_{\mathcal{W}_0}|_{\mathcal{U}})(P^{\mathcal{Y}}_{\mathcal{W}_0}|_{\mathcal{U}})^{-1}$ above must is zero, and thus

\begin{equation}
S_0 = \begin{bmatrix} A & B(P^Y_{\mathcal{W}_0}|_{\mathcal{U}})^{-1} \\ C & (P^{\mathcal{Y}}_{\mathcal{W}_0}|_{\mathcal{U}}) + (P^Y_{\mathcal{W}_0}|_{\mathcal{U}})^{-1} \end{bmatrix}.
\end{equation}

(ii) $\Rightarrow$ (i): Since $\Sigma$ is semi-bounded it has at least one semi-bounded i/s/o representation $\Sigma^1_{i/o} = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1 \right)$. As we saw in the proof above, the i/o representation $(\mathcal{W}_0, \mathcal{Y}_1)$ is semi-boundedly admissible for $\Sigma$. Denote the corresponding i/s/o representation of $\Sigma$ by $\Sigma^1_{i/o} = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix} : \mathcal{X}, \mathcal{W}_0, \mathcal{Y}_1 \right)$.

Let $\mathcal{Y}$ be a direct complement to $\mathcal{W}_0$, and let $\Theta_1$ be the transition matrix from $(\mathcal{W}_0, \mathcal{Y}_1)$ to $(\mathcal{W}_0, \mathcal{Y})$. Then $\Theta_1 = \begin{bmatrix} 1_{\mathcal{W}_0} & P^Y_{\mathcal{W}_0}|_{\mathcal{Y}_1} \\ 0 & P^{\mathcal{Y}}_{\mathcal{W}_0}|_{\mathcal{Y}_1} \end{bmatrix}$, and by Lemma 2.2.12 $P^{\mathcal{Y}}_{\mathcal{W}_0}|_{\mathcal{Y}_1}$ maps $\mathcal{Y}_1$ one-to-one onto $\mathcal{Y}$. Define $M$ and $K$ by (2.2.28) and (2.2.29) with $S_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \end{bmatrix}$.

Then

\begin{equation}
K = \begin{bmatrix} A_1 & B_1 \\ P^{\mathcal{Y}}_{\mathcal{W}_0}C_1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1_{\text{dom}(A_1)} & 1 \end{bmatrix}.
\end{equation}

Here $M$ maps $\begin{bmatrix} \text{dom}(A_1) \\ 0 \end{bmatrix}$ one-to-one onto itself. By Theorem 2.2.25 the decomposition $(\mathcal{W}_0, \mathcal{Y})$ is i/s/o admissible for $\Sigma$, and the system operator $S_2$ of the corresponding i/s/o representation $\Sigma^0_{i/o} = (S_0; \mathcal{X}, \mathcal{W}_0, \mathcal{Y})$ is given by

\begin{equation}
S_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 - B_1P^Y_{\mathcal{W}_0}C_1 & B_1 \\ P^{\mathcal{Y}}_{\mathcal{W}_0}C_1 & 0 \end{bmatrix}.
\end{equation}

By Lemma 4.1.16 the main operator $A_1 - B_1P^Y_{\mathcal{W}_0}C_1$ is the generator of a $C_0$ semi-group, and the control operator $B_1$ and observation operator $P^{\mathcal{Y}}_{\mathcal{W}_0}C_1$ are bounded. This means that $\Sigma^0_{i/o}$ is semi-bounded, and the decomposition $(\mathcal{W}_0, \mathcal{Y})$ is semi-boundedly i/s/o-admissible for $\Sigma$.

Let $\mathcal{U}$ be a direct complement to $\mathcal{Y}$, and let $\Theta_2$ be the transition matrix from $(\mathcal{W}_0, \mathcal{Y})$ to $(\mathcal{U}, \mathcal{Y})$. Then $\Theta = \begin{bmatrix} P^Y_{\mathcal{W}_0} & 0 \\ P^{\mathcal{Y}}_{\mathcal{W}_0} & 1 \end{bmatrix}$. Arguing as in the proof of the implication (i) $\Rightarrow$ (ii) (but interchanging the roles of $\mathcal{U}$ and $\mathcal{W}_0$ and replacing $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ by $\begin{bmatrix} A_0 & B_0 \\ C_0 & 0 \end{bmatrix}$) we find that the decomposition $(\mathcal{U}, \mathcal{Y})$ is semi-boundedly i/s/o-admissible for $\Sigma$, and that the system operator $S$ of the corresponding i/s/o representation $\Sigma^0_{i/o}$ is given by

\begin{equation}
S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_0 & B_0(P^Y_{\mathcal{U}})^{-1} \\ C_0 & P^{\mathcal{Y}}_{\mathcal{U}}(P^Y_{\mathcal{U}})^{-1} \end{bmatrix}.
\end{equation}

We have now proved that (i)–(iv) are equivalent. The proofs of claims (v) and (vi) are analogous to the proofs of the corresponding claims in Theorem 2.2.29. \qed

4.3.4. Theorem. Theorem 2.2.31 remains true also in the setting where $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a semi-bounded s/s node with the semi-bounded i/s/o representation $\Sigma^1_{i/o} = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} : \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1 \right)$, and the phrase “semi-boundedly i/s/o-admissible” in condition (ii)/(a) is replaced by “semi-boundedly i/s/o-admissible.”

Proof. The proof is essentially the same as the proof of Theorem 2.2.31. \qed
4.3.5. **Lemma.** Lemma 2.2.32 remains true in the setting where $\Sigma$ is a semi-bounded s/s system and $\Sigma^j_{i/s/o}$, $j = 1, 2$, are two semi-bounded i/s/o representations of $\Sigma$.

**Proof.** The proof is the same as the proof of Lemma 2.2.32. □

4.3.2. **The i/s/o-semi-bounded resolvent set of a semi-bounded s/s system.**

4.3.6. **Lemma.** Lemma 3.4.1 remains true in the setting where $\Sigma$ is a semi-bounded s/s node and let $\Sigma_{i/s/o} = ([A \ B] ; X, U, Y)$ be a semi-bounded i/s/o representation of $\Sigma$.

**Proof.** The proof is the same as the proof of Lemma 3.4.1. □

4.3.7. **Definition (cf. Definition 3.4.2).** Let $(V; \mathcal{X}, W)$ be a semi-bounded s/s node.

(i) The **i/s/o-semi-bounded resolvent set** of $\Sigma$ is the union of the resolvent sets of the main operators of all semi-bounded i/s/o representations of $\Sigma$. This set is denoted by $\rho_{\text{sbd}}(\Sigma)$.

(ii) The component of $\rho_{\text{sbd}}(\Sigma)$ which contains some right-half plane is denoted by $\rho_{\text{r.sbd}}(\Sigma)$.

Note that by Definition 4.3.1 and Theorem 4.1.10, $\rho_{\text{sbd}}(\Sigma) \neq \emptyset$ if (and only if) $\Sigma$ is semi-bounded, and that $\rho_{\text{r.sbd}}(\Sigma)$ always contains some right half-plane (if it is nonempty). □

4.3.8. **Remark.** In Definition 3.4.2 we introduced the notion of the i/s/o-semi-bounded resolvent set $\rho_{\text{sbd}}(\Sigma)$ of a bounded s/s system. Of course, it is also possible to apply Definition 4.3.7 in this case to get the i/s/o-semi-bounded resolvent set $\rho_{\text{sbd}}(\Sigma)$ of $\Sigma$. However, it follows from Theorems 2.2.29 and 4.3.3 that for a bounded s/s system it is always true that $\rho_{\text{sbd}}(\Sigma) = \rho_{\text{sbd}}(\Sigma)$.

4.3.9. **Lemma (cf. Lemma 3.4.3).** Let $\Sigma = (V; \mathcal{X}, W)$ be a semi-bounded s/s node, let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}$, and let $E \in B(\mathcal{X}; \mathcal{Z})$ for some $H$-space $\mathcal{Z}$.

(i) The bundle $\mathcal{G}_E$ in $[\mathcal{X}_1 \ Z \ W]$ whose fibers are given by (3.4.4) is analytic in $\rho_{\text{sbd}}(\Sigma)$.

(ii) The characteristic signal/state bundle $\hat{\mathcal{G}}$ of $\Sigma$ is analytic in $\rho_{\text{sbd}}(\Sigma)$.

(iii) The characteristic signal bundle $\hat{\mathcal{F}}$ of $\Sigma$ is analytic in $\rho_{\text{sbd}}(\Sigma)$.

(iv) $\hat{\mathcal{F}}(\infty) = \mathcal{W}_0$, where $\mathcal{W}_0$ is the canonical input space of $\Sigma$ (see Definition 2.2.30), and the values of the other bundles at infinity can be obtained from $\hat{\mathcal{F}}(\infty)$ as described in (3.4.5).

**Proof.** The proof is analogous to the proof of Lemma 3.4.3. □

4.3.3. **Strongly invariant and unobservably invariant subspaces.** At this point the reader may want to recall the notions of strongly invariant and unobservably invariant subspaces of a s/s system introduced in Definition 1.5.8.

4.3.10. **Lemma (cf. Lemma 3.4.4).** Let $\Sigma = (V; \mathcal{X}, W)$ be a semi-bounded s/s system.

(i) If $\mathcal{Z}$ is a strongly invariant or unobservable invariant subspace for $\Sigma$, then the closure of $\mathcal{Z}$ is also strongly invariant respectively unobservably invariant for $\Sigma$. □
If both $Z_1$ and $Z_2$ are strongly invariant for $\Sigma$, then $Z_1 + Z_2$ and $Z_1 \lor Z_2$ are strongly invariant for $\Sigma$.

(iii) If both $Z_1$ and $Z_2$ are unobservably invariant for $\Sigma$, then $Z_1 \cap Z_2$ is unobservably invariant for $\Sigma$.

Proof. By Definition 4.3.1, $\Sigma$ has a semi-bounded i/s/o representation $\Sigma_{i/s/o}$, and by Proposition 2.5.49, $Z$ is strongly invariant or unobservable invariant for $\Sigma$ if and only if $Z$ is strongly invariant or unobservable invariant for $\Sigma_{i/s/o}$. This combined Lemma 4.2.8 gives Lemma 4.3.10.

4.3.11. Lemma (cf. Lemma 3.4.6). Let $\Sigma = (V, X, W)$ be a semi-bounded s/s system with characteristic node bundle $\hat{E}$, let $Z$ be a closed subspace of $X$, and let $\lambda \in \rho_{sbd}(\Sigma)$. Then conditions (3.4.7a)–(3.4.7c) are equivalent.

Proof. The proof is analogous to the proof of Lemma 3.4.6.

4.3.12. Lemma (cf. Lemma 3.4.7). Let $\Sigma = (V, X, W)$ be a semi-bounded s/s system, and let $Z$ be a closed subspace of $X$. Then the following conditions are equivalent:

(i) $Z$ is a strongly invariant subspace for $\Sigma$;
(ii) at least one of conditions (3.4.7a)–(3.4.7c) holds for some $\lambda \in \rho_{sbd}(\Sigma)$;
(iii) conditions (3.4.7a)–(3.4.7c) holds for all $\lambda \in \rho_{sbd}(\Sigma)$.

Proof. The proof is analogous to the proof of Lemma 3.4.7.

4.3.13. Lemma (cf. Lemma 3.4.9). Let $\Sigma = (V, X, W)$ be a semi-bounded s/s system with characteristic node bundle $\hat{E}$, let $Z$ be a closed subspace of $X$, and let $\lambda \in \rho_{sbd}(\Sigma)$. Then conditions (3.4.9a)–(3.4.9c) are equivalent.

Proof. The proof is analogous to the proof of Lemma 3.4.9.

4.3.14. Lemma (cf. Lemma 3.4.10). Let $\Sigma = (V, X, W)$ be a bounded s/s system with characteristic node bundle $\hat{E}$, and let $Z$ be a closed subspace of $X$ with a direct complement $X_1$. Then the following conditions are equivalent:

(i) $Z$ is a unobservably invariant subspace for $\Sigma$;
(ii) at least one of conditions (3.4.9a)–(3.4.9c) holds for some $\lambda \in \rho_{sbd}(\Sigma)$;
(iii) conditions (3.4.9a)–(3.4.9c) holds for all $\lambda \in \rho_{sbd}(\Sigma)$.

Proof. The proof is analogous to the proof of Lemma 3.4.10.

At this point the reader may want to recall the notions of the reachable subspace and the unobservable subspace of an s/s system introduced in Definition 1.5.3.

4.3.15. Lemma (cf. Lemma 3.4.11). Let $\Sigma = (V, X, W)$ be a semi-bounded s/s system with characteristic node bundle $\hat{E}$. Let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\rho_{sbd}(\Sigma)$. Then the following claims are true:

(i) The reachable subspace $R_{\Omega'}$ of $\Sigma$ is the minimal closed strongly invariant subspace for $\Sigma$.
(ii) The unobservable subspace $U_{\Omega'}$ of $\Sigma$ is the maximal unobservably invariant subspace of $\Sigma$. 
(iii) $R_Σ$ can be computed in the following ways:

$$R_Σ = \bigvee_{\lambda \in \rho^\text{sb}(\Sigma)} \begin{bmatrix} 0 & 1 & 0 \\ \mathfrak{E}(\lambda) \cap \mathfrak{X} \end{bmatrix} \left( \begin{bmatrix} 0 \\ \mathfrak{W} \end{bmatrix} \right)$$

(4.3.5)

(iv) $U_Σ$ can be computed in the following ways:

$$U_Σ = \bigwedge_{\lambda \in \Omega'} \begin{bmatrix} 1 & 0 & 0 \\ \mathfrak{E}(\lambda) \cap \mathfrak{X} \end{bmatrix} \left( \begin{bmatrix} 0 \\ \mathfrak{W} \end{bmatrix} \right).$$

Proof. The proof is analogous to the proof of Lemma 3.4.11.

4.3.4. External equivalence of semi-bounded s/s system. At this point the reader may want to recall Definitions 1.5.21 and 1.5.43.

4.3.16. Lemma (cf. Lemma 3.4.12). Let $Σ_i = (V_i; X_i, W)$, $i = 1, 2$, be two semi-bounded s/s systems with the same signal space $W$. Denote the component of $\rho^\text{sb}(Σ_1) \cap \rho^\text{sb}(Σ_2)$ which contains some right half-plane by $Ω$, and let $Ω'$ be a subset of $Ω$ which contains a cluster point in $Ω$. Then the following conditions are equivalent:

(i) $Σ_1$ and $Σ_2$ are externally equivalent;
(ii) $Σ_1$ and $Σ_2$ have the same future behavior;
(iii) $\mathfrak{F}_1(\lambda) = \mathfrak{F}_2(\lambda)$ for all $λ \in Ω$;
(iv) $\mathfrak{F}_1(\lambda) = \mathfrak{F}_2(\lambda)$ for all $λ \in Ω'$.

Proof. The proof is analogous to the proof of Lemma 3.4.12.

4.3.17. Lemma. Let $Σ_i = (V_i; X_i, W)$, $i = 1, 2$, be two semi-bounded externally equivalent s/s systems. Then the i/o representation $(U, Y)$ of $W$ is semi-boundedly i/s/o-admissible for $Σ_1$ if and only if it is semi-boundedly i/s/o-admissible for $Σ_2$.

Proof. By Lemma 4.3.16, $\mathfrak{F}_1(\lambda) = \mathfrak{F}_2(\lambda)$ for all $λ \in Ω$, and hence by Lemma 3.4.3 the canonical input spaces $W^0_1$ and $W^0_2$ or $Σ_1$ respectively $Σ_2$ coincide. It therefore follows from Definition 4.3.1 that $Σ_1$ and $Σ_2$ have the same semi-boundedly i/s/o-admissible i/o representations.

4.3.5. Restrictions and projections of semi-bounded s/s systems. At this point the reader may want to recall Definitions 1.5.33 and 1.5.37 of what we mean by a restriction and a projection of an s/s system.

4.3.18. Theorem (cf. Theorem 3.4.14). Let $Σ = (V; X, W)$ be a semi-bounded s/s system, let $X_1$ be a closed subspace of $X$, and let $Σ_1 = (V_1; X_1, W)$ be the part of $Σ$ in $[X_1]$, i.e., $V_1$ is given by 3.4.13. Then the following conditions are equivalent:

(i) $X_1$ is a strongly invariant subspace for $Σ$;
(ii) $\Sigma$ has a restriction to $X_1$;
(iii) $\Sigma_1 = (V_1; X_1, W)$ is a semi-bounded s/s system, and the canonical input spaces of $\Sigma$ and $\Sigma_1$ coincide.

If these equivalent conditions hold then $\Sigma_1$ is the unique semi-bounded restriction of $\Sigma$ to $X_1$, and $\Sigma$ and $\Sigma_1$ are externally equivalent.

**Proof.** The proof is analogous to the proof of Theorem 3.4.14 \( \square \)

4.3.19. **Theorem** (cf. Theorem 3.4.15). Let $\Sigma = (V; X, W)$ be a bounded s/s system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. Let $\Sigma_1 = (V_1; X_1, W)$ be the static projection of $\Sigma$ onto $[X_1] \parallel [Z_1]$, i.e., $V_1$ is given by (3.4.14). Then the following conditions are equivalent:

(i) $Z_1$ is an unobservably invariant subspace for $\Sigma$;
(ii) $\Sigma$ has a projection onto $X_1$ along $Z_1$;
(iii) $\Sigma_1 = (V_1; X_1, W)$ is a semi-bounded s/s system, and the canonical input spaces of $\Sigma$ and $\Sigma_1$ coincide.

If these equivalent conditions hold then $\Sigma_1$ is the unique semi-bounded projection of $\Sigma$ onto $X_1$ along $Z_1$, and $\Sigma$ and $\Sigma_1$ are externally equivalent.

**Proof.** The proof is analogous to the proof of Theorem 3.4.15 \( \square \)

4.3.20. **Lemma** (cf. Lemma 3.4.16). Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two semi-bounded s/s systems (with the same signal space $W$), and let $P \in \mathcal{ML}(X_1; X_2)$ be closed. Let $\Sigma = (V; \text{gph}(P), W)$ be the gph $(P)$-short circuit of $\Sigma_2$ and $\Sigma_1$ (cf. Definition 1.2.29), i.e., $V$ is given by (3.4.15). Then the following conditions are equivalent:

(i) $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$;
(ii) $\Sigma$ is a semi-bounded s/s node, and the canonical input spaces of $\Sigma$, $\Sigma_1$, and $\Sigma_2$ coincide.
(iii) The following two conditions hold:
   (a) If $[z_1^i] \in \text{gph}(P)$ and $w \in W$, then $[z_1^i \bar{w}] \in V_1$ for some (unique) $z_1 \in X_1$ if and only if $[z_2^i \bar{w}] \in V_2$ for some (unique) $z_2 \in X_2$;
   (b) the vectors $z_1$ and $z_2$ in (a) satisfy $[z_i^i] \in \text{gph}(P)$.

If these equivalent conditions hold, then $\Sigma$ and $\Sigma_1$ are intertwined by the bounded operator $P_{X_1}^{X_2}|_{\text{gph}(P)}$, $\Sigma$ and $\Sigma_2$ are intertwined by the bounded operator $P_{X_2}^{X_1}|_{\text{gph}(P)}$, and $\Sigma$, $\Sigma_1$, and $\Sigma_2$ are externally equivalent.

**Proof.** The proof is analogous to the proof of Lemma 3.4.16 \( \square \)

4.3.21. **Theorem** (cf. Theorem 3.4.17). Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two semi-bounded s/s systems (with the same signal space). Let $\Sigma = (V; [X_2] \parallel [X_1], W)$ be the 1 short circuit of $\Sigma_2$ and $\Sigma_1$, i.e., $V$ is given by (3.4.16) (this s/s node will not be semi-bounded in general). Then the following claims are true.

(i) $\Sigma_1$ and $\Sigma_2$ are intertwined by some closed $P \in \mathcal{ML}(X_1; X_2)$ if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent.
(ii) Suppose that $\Sigma_1$ and $\Sigma_2$ are externally equivalent. Let $\Sigma_i = (A_i, B_i ; C_i, D_i)$ be i/s/o representations of $\Sigma_j$ (with the same input and output spaces; by Definition 4.3.7 and Lemma 4.3.17 such representations exist), and let $\Sigma_{i/o}$ be the difference connection of $\Sigma_i$ and $\Sigma_{i/o}$ (see Definition 2.3.38). (Note that $\Sigma_{i/o}$ is not an i/s/o representation of $\Sigma_i$.) Then the following claims are true.

(a) There exists a unique minimal closed $P_{\min} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique closed $P_{\min} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P_{\min}) \subset \text{gph}(P)$ for any other closed $P$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\min}$ can be described in the following two equivalent ways:

1. $\text{gph}(P_{\min})$ is equal to the reachable subspace of $\Sigma_i$;
2. $\text{gph}(P_{\min})$ is equal to the reachable subspace of $\Sigma_{i/o}$.

(b) There exists a unique maximal $P_{\max} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique $P_{\max} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P_{\max}) \subset \text{gph}(P)$ for any other closed $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\max}$ can be described in the following three equivalent ways:

1. $\text{gph}(P_{\max})$ coincides with the set of all possible initial states of all generalized future trajectories of $\Sigma_i$;
2. $\text{gph}(P_{\max})$ is equal to the unobservable subspace of $\Sigma_{i/o}$.

In particular, $P_{\max}$ is closed.

Thus, if $P$ is an arbitrary closed multi-valued operator which intertwines $\Sigma_1$ and $\Sigma_2$, then

$$\text{gph}(P_{\min}) \subset \text{gph}(P) \subset \text{gph}(P_{\max}).$$

**Proof.** The proof is analogous to the proof of Theorem 3.4.17. □

4.3.22. Corollary (cf. Corollary 3.4.18). Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two semi-bounded s/s systems (with the same input and output spaces). Moreover, suppose that both $\Sigma_1$ and $\Sigma_2$ are controllable and observable. (According to Theorem 4.3.30 below, this is equivalent to the assumption that both $\Sigma_1$ and $\Sigma_2$ is minimal.) Then $\Sigma_1$ and $\Sigma_2$ are pseudo-similar if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent. Among all the pseudo-similarities between $\Sigma_1$ and $\Sigma_2$ there is a (unique) minimal one $P_{\min}$ and a (unique) maximal one $P_{\max}$, namely those defined in Theorem 4.3.21 (both of which in this case are single-valued densely defined injective operators with dense range).

**Proof.** The proof is analogous to the proof of Corollary 3.4.18. □

4.3.7. Compressions of semi-bounded s/s systems. At this point the reader may want to recall Definition 1.5.28 of what we mean by a compression of a s/s system.

4.3.23. Lemma (cf. Lemma 3.4.19). Let $\Sigma = (V; X, W)$ be a semi-bounded s/s system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. If $\Sigma$ has a compression onto $X_1$ along $Z_1$, then $\Sigma$ also has a unique semi-bounded compression $\Sigma_1 = (V_1; X_1, W)$. The generating subspace $V_1$ of $\Sigma_1$ is given by 3.4.19, and $\Sigma$ and $\Sigma_1$ are externally equivalent.
4.3.24. **Lemma (cf. Lemma 3.4.20).** Let \( \Sigma = (V; \mathcal{X}, W) \) be a semi-bounded s/s system, and let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \), define \( V_1 \) by \( (3.4.19) \), and let \( \Sigma_1 = (V_1; \mathcal{X}_1, W) \) be the semi-bounded s/s system generated by \( V_1 \). Let \( \Omega' \) be a subset of \( \rho_{+\infty}^{\text{sbd}}(\Sigma) \) which contains a cluster point in \( \rho_{+\infty}^{\text{sbd}}(\Sigma) \). Then the following conditions are equivalent:

- (i) \( \Sigma_1 \) is a compression of \( \Sigma \) onto \( \mathcal{X}_1 \) along \( Z_1 \);
- (ii) for all \( \lambda \in \rho_{-\infty}^{\text{sbd}}(\Sigma) \) the node bundles \( \mathcal{E} \) and \( \mathcal{E}_1 \) of \( \Sigma \) respectively \( \Sigma_1 \) satisfy \( (3.4.20) \);
- (iii) for all \( \lambda \in \Omega' \) the node bundles \( \mathcal{E} \) and \( \mathcal{E}_1 \) of \( \Sigma \) respectively \( \Sigma_1 \) satisfy \( (3.4.20) \).

If these equivalent conditions hold, then \( \rho_{+\infty}^{\text{sbd}}(\Sigma) \subset \rho_{+\infty}^{\text{sbd}}(\Sigma_1) \).

**Proof.** The proof is analogous to the proof of Lemma 3.4.19. \( \Box \)

4.3.25. **Lemma (cf. Lemma 3.4.21).** Let \( \Sigma = (V; \mathcal{X}, W) \) be a semi-bounded s/s system, let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \), let \( W_{\text{ext}} = \begin{bmatrix} V_{\text{ext}} \\ \mathcal{X}_1 \end{bmatrix} \), and let \( \Sigma_{\text{ext}} = (V_{\text{ext}}; \mathcal{X}, W_{\text{ext}}) \) be the i/o extension of \( \Sigma \) with control operator equal to the embedding operator \( \mathcal{I}_{\mathcal{X}_1} : \mathcal{X}_1 \hookrightarrow \mathcal{X} \), observation operator \( P_{\mathcal{X}_1}^{\mathcal{E}_1} \), and feedthrough operator zero, i.e., \( V_{\text{ext}} \) is given by \( (3.4.21) \). Then the following claims are true.

(i) There exists a (unique) minimal closed strongly invariant subspace \( \mathcal{X}_{\text{min}} \) for \( \Sigma \) which contains \( \mathcal{X}_1 \) (i.e., \( \mathcal{X}_{\text{min}} \) is closed and strongly invariant for \( \Sigma \), and \( \mathcal{X}_{\text{min}} \) is contained in every other closed strongly invariant subspace of \( \Sigma \) which contains \( \mathcal{X}_1 \)). This subspace has the following alternative descriptions:

- (a) \( \mathcal{X}_{\text{min}} \) is the reachable subspace of \( \Sigma_{\text{ext}} \);
- (b) \( \mathcal{X}_{\text{min}} \) is equal to the subspace \( \mathcal{X}_{\text{min}} \) in Lemma 4.2.27 with \( \Sigma \) replaced by an arbitrary semi-bounded i/s/o representation \( \Sigma_{\text{i/s/o}} \) of \( \Sigma \).

(ii) The space \( \mathcal{X}_{\text{min}} \) has the direct sum decomposition \( \mathcal{X}_{\text{min}} = \mathcal{X}_1 + Z_{\text{min}} \), where

\[
Z_{\text{min}} = \mathcal{X}_{\text{min}} \cap Z_1 = P_{\mathcal{E}_1}^{\mathcal{X}_1} \mathcal{X}_{\text{min}}.
\]

(iii) There exists a (unique) maximal unobservably invariant subspace \( \mathcal{Z}_{\text{max}} \) for \( \Sigma \) which is contained in \( \Sigma_{\text{ext}} \) (i.e., \( \mathcal{Z}_{\text{max}} \) is observably invariant for \( \Sigma \), and \( \mathcal{Z}_{\text{max}} \) contains every other unobservably invariant subspace for \( \Sigma \) which is contained in \( \Sigma_{\text{ext}} \)). This subspace has the following alternative descriptions:

- (a) \( \mathcal{Z}_{\text{max}} \) is the unobservable subspace of \( \Sigma_{\text{ext}} \);
- (b) \( \mathcal{Z}_{\text{min}} \) is equal to the subspace \( \mathcal{Z}_{\text{min}} \) in Lemma 4.2.27 with \( \Sigma \) replaced by an arbitrary semi-bounded i/s/o representation \( \Sigma_{\text{i/s/o}} \) of \( \Sigma \).

In particular, \( \mathcal{Z}_{\text{max}} \) is closed.

**Proof.** The proof is analogous to the proof of Lemma 3.4.21. \( \Box \)

4.3.26. **Lemma (cf. Lemma 3.4.22).** Let \( \Sigma = (V; \mathcal{X}, W) \) be a semi-bounded s/s system, and let \( \mathcal{X} = \mathcal{X}_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \). Let \( \Omega' \) be a subspace of \( \rho_{+\infty}^{\text{sbd}}(\Sigma) \) which has a cluster point in \( \rho_{+\infty}^{\text{sbd}}(\Sigma) \), where \( \rho_{+\infty}^{\text{sbd}}(\Sigma) \) is the
unbounded component of $\rho(\Sigma)$. Denote the characteristic node bundle of $\Sigma$ by $\hat{\mathcal{E}}$.

Then the following claims are true.

(i) The subspace $\mathcal{X}_{\text{min}}$ in Lemma 4.3.25 can be computed in the following ways:

$$
\mathcal{Z}_{\text{min}} = \bigvee_{\lambda \in \rho_{sbd}^+(\Sigma)} \begin{bmatrix} 0 & 1 \times & 0 \end{bmatrix}\left(\hat{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{W} \end{bmatrix}\right)
$$

(ii) The subspace $\mathcal{Z}_{\text{min}}$ in Lemma 4.3.25 can be computed in the following ways:

$$
\mathcal{Z}_{\text{min}} = \bigvee_{\lambda \in \Omega'} \begin{bmatrix} 0 & 1 \times & 0 \end{bmatrix}\left(\hat{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{W} \end{bmatrix}\right).
$$

(iii) The subspace $\mathcal{Z}_{\text{max}}$ in Lemma 4.3.25 can be computed in the following ways:

$$
\mathcal{Z}_{\text{max}} = \bigcap_{\lambda \in \rho_{sbd}^+(\Sigma)} \begin{bmatrix} 1 \mathcal{Z}_1 & 0 & 0 \end{bmatrix}\left(\hat{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_1 \\ \{0\} \end{bmatrix}\right)
$$

PROOF. The proof is analogous to the proof of Lemma 3.4.22.

4.3.27. THEOREM (cf. Theorem 3.4.23). Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a semi-bounded s/s system, and let $\mathcal{X} = \mathcal{X}_1 \sqcup \mathcal{Z}_1$ be a direct sum decomposition of $\mathcal{X}$. Let $\mathcal{X}_{\text{min}}$ be the minimal closed strongly invariant subspace of $\Sigma$ which contains $\mathcal{X}_1$, let $\mathcal{Z}_{\text{max}}$ be the maximal unobservably invariant subspace of $\Sigma$ which is contained in $\mathcal{Z}_1$, and let $\mathcal{Z}_{\text{min}} = \mathcal{X}_{\text{min}} \cap \mathcal{Z}_1$ (cf. Lemma 4.3.25). Then the following conditions are equivalent:

(i) $\Sigma$ has a compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;

(ii) $\mathcal{Z}_1$ contains some closed unobservably invariant subspace $\mathcal{Z}$ for $\Sigma$ such that $\mathcal{X}_1 + \mathcal{Z}$ is strongly invariant for $\Sigma$;

(iii) $\mathcal{Z}_{\text{min}}$ in an unobservably invariant subspaces for $\Sigma$;

(iv) $\mathcal{X}_1 + \mathcal{Z}_{\text{max}}$ is a strongly invariant subspace for $\Sigma$;

(v) $\mathcal{Z}_{\text{min}} \subset \mathcal{Z}_{\text{max}}$.

Two possible choices of the subspace $\mathcal{Z}$ in (ii) are $\mathcal{Z} = \mathcal{Z}_{\text{min}}$ and $\mathcal{Z} = \mathcal{Z}_{\text{max}}$, and every possible subspace $\mathcal{Z}$ in (ii) satisfies $\mathcal{Z}_{\text{min}} \subset \mathcal{Z} \subset \mathcal{Z}_{\text{max}}$.

Suppose that the equivalent conditions (i)–(v) hold, and define $\mathcal{V}_1$ by (3.4.19). Then the semi-bounded s/s system $\Sigma_1 = (\mathcal{V}_1; \mathcal{X}, \mathcal{W})$ is a compression of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, and $\Sigma_1$ is the unique forward or backward compression of $\Sigma$ among all solvable forward or backward compressions of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$. 
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4.3.28. Theorem (cf. Theorem 3.4.24). Let \( \Sigma = (V; X, W) \) be a semi-bounded s/s system, and let \( X = X_1 \oplus Z_1 \) be a direct sum decomposition of \( X \), and suppose that \( \Sigma_1 = (V_1; X_1, W) \) is a semi-bounded compression of \( \Sigma \) onto \( X_1 \) along \( Z_1 \). Let \( Z \) satisfy the conditions listed in (ii) in Theorem 4.3.27, and let \( Z_c \) be an arbitrary direct complement to \( Z \) in \( Z_1 \).

(i) Let \( \Sigma_2 \) be the unique semi-bounded restriction of \( \Sigma \) to the strongly invariant subspace \( X_1 \oplus Z \) for \( \Sigma \) given by Theorem 4.3.18. Then \( Z \) is unobservably invariant for \( \Sigma_2 \), and \( \Sigma_1 \) is the unique semi-bounded projection onto \( X_1 \oplus Z \) of \( \Sigma_2 \) given by Theorem 4.3.19.

(ii) Let \( \Sigma_3 \) be the unique semi-bounded projection of \( \Sigma \) onto \( X_1 \oplus Z_c \) along \( Z \) given by Theorem 4.3.19. Then \( X_1 \) is strongly invariant for \( \Sigma_3 \), and \( \Sigma_1 \) is the unique semi-bounded restriction of \( \Sigma_3 \) to \( X_1 \) given by Theorem 4.3.18.

Proof. This follows from Proposition 2.5.50 and Theorem 4.2.31 applied to an arbitrary i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \).

4.3.29. Lemma (cf. Lemma 3.4.25). Let \( \Sigma = (V; X, W) \) and \( \Sigma_1 = (V; X_1, W) \) be two semi-bounded s/s systems with \( X = X_1 \oplus Z_1 \). Then the following two conditions are equivalent.

(i) \( \Sigma_1 \) is the compression of \( \Sigma \) onto \( X_1 \) along \( Z_1 \).

(ii) \( Z_1 \) contains some closed subspace \( Z \) such that \( \Sigma \) and \( \Sigma_1 \) are intertwined by the operator \( P_{X_1}^Z |_{X_1 + Z} \).

Condition (ii) above holds for some particular subspace \( Z \) if and only condition (ii) in Theorem 4.3.27 holds for the same subspace \( Z \). Thus, in particular, two possible choices of the subspace \( Z \) in (ii) are the subspaces \( Z = Z_{\min} \) and \( Z = Z_{\max} \) defined in Lemma 4.3.27, and every possible subspace \( Z \) satisfies \( Z_{\min} \subset Z \subset Z_{\max} \).

Proof. This follows from Proposition 2.5.50 and Lemma 4.2.32 applied to an arbitrary i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \).

4.3.30. Theorem (cf. Theorem 3.4.26). A semi-bounded s/s system \( \Sigma = (V; X, W) \) is minimal if and only if \( \Sigma \) is both controllable and observable.

Proof. This follows from Proposition 2.5.49 and Theorem 4.2.33 applied to an arbitrary i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \).

As the following theorem shows, every semi-bounded s/s system can be compressed into a minimal semi-bounded s/s system.

4.3.31. Theorem (cf. Theorem 3.4.27). Every semi-bounded i/s/o system \( \Sigma \) has a semi-bounded minimal compression. Two families of such compressions are described below, where we have denoted the reachable and unobservable subspaces of \( \Sigma \) by \( R_\Sigma \) respectively \( U_\Sigma \):

(i) Let \( X_1 \) be a direct complement to \( U_\Sigma \) in \( X \), and let \( X_o = \overline{P_{X_1}^U \cdot \Sigma} \). Define \( V_o \) by (3.4.26). Then \( \Sigma_o = (V_o; X_o, W) \) is a minimal semi-bounded compression of \( \Sigma \). This compression is the semi-bounded s/s system that
one gets by first projecting $\Sigma$ onto $X_1$ along its unobservable subspace $U_\Sigma$, and then restricting the resulting system to its reachable subspace $X_\circ$.

(ii) Let $X_\bullet$ be a direct complement to $R_{\Sigma} \cap U_\Sigma$ in $R_{\Sigma}$, and define $V_\bullet$ by (3.4.27). Then $\Sigma_\bullet = (V_\bullet; X_\bullet; W)$ is a semi-bounded minimal compression of $\Sigma$. This compression is the system that one gets by first restricting $\Sigma$ to its reachable subspace $R_{\Sigma}$, and then projecting the resulting system onto $X_\bullet$ along its unobservable subspace $R_{\Sigma} \cap U_\Sigma$.

Proof. This follows from Proposition 2.5.50 and Theorem 4.2.35 applied to an arbitrary i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$. □

4.3.32. Lemma (cf. Lemma 3.4.28). The minimal compression of a semi-bounded s/s system $\Sigma = ([A B]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is unique if and only if at least one of conditions (i) and (ii) below holds:

(i) $\Sigma$ is observable, i.e., $U_\Sigma = \{0\}$, where $U_\Sigma$ is the unobservable subspace of $\Sigma$,

(ii) the following equivalent conditions hold:
   (a) $\Sigma$ has a compression with state space $\{0\}$,
   (b) the characteristic node signal of $\Sigma$ is a constant in $\rho_{ab} + \infty(\Sigma)$,
   (c) $R_{\Sigma} \subset U_\Sigma$, where $R_{\Sigma}$ is the reachable subspace of $\Sigma$.

In case (i) the unique minimal compression $\Sigma_{\min}$ is the restriction of $\Sigma$ to $R_{\Sigma}$, i.e., $\Sigma_{\min} = (V_{\min}; R_{\Sigma}; W)$ where $V_{\min}$ is given by MinObsSSCompress. In case (ii) the unique minimal compression is $\Sigma_{\min} = (V_{\min}; \{0\}; W)$ where $V_{\min}$ is given by (3.4.29). If neither (i) nor (ii) holds, then $\Sigma$ has an infinite number of minimal compressions.

Proof. The proof is analogous to the proof of Lemma 3.4.28. □
4.4. Notes and Comments (Feb 02, 2016)

A the results in this chapter remain valid if we throughout replace $L^1$ by $L^p$ where $1 \leq p < \infty$.

All results in this chapter remain valid if we allow the state spaces of the i/s/o or s/s systems to be $B$-spaces instead of $H$-spaces, except for the results about existence of minimal compressions of a given i/s/o or s/s system, i.e., Theorems 4.2.33, 4.2.35, 4.3.30, and 4.3.31 and Lemmas 4.2.36 and 4.3.32.

All results in this chapter about i/s/o systems remain valid if we allow the input and output spaces of the i/s/o systems to be $B$-spaces instead of $H$-spaces. However, since Theorem 4.3.3 is not valid in a $B$-spaces setting unless $W_0$ is complemented, and since we make heavy use of semi-bounded i/s/o representations in Section 4.3, many of the proofs of the results about semi-bounded s/s systems in this chapter are not valid if the signal space is a $B$-space without the additional assumption that the canonical input space $W_0$ is complemented.
CHAPTER 5

The Input/State/Output Resolvent Matrix (Jan 02, 2016)

In this chapter we extend the notion of the resolvent set and the resolvent of a closed linear operator in a Hilbert space to the notion of the resolvent set and the i/s/o resolvent matrix of a closed i/s/o node. The i/s/o resolvent matrix $\hat{S}$ is an analytic $\mathcal{B}([X] ; [Y])$-valued function which satisfies a particular type of resolvent identity, the so called i/s/o resolvent identity. Since both the domain space and the range space of $\hat{S}$ are product spaces, the i/s/o resolvent matrix can be written in block operator form $\hat{S} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$.

In this chapter we also define what we mean by the resolvent set of a closed s/s node. This definition does not contain any explicit reference i/s/o representations of the s/s node, but it follows from the definition that the resolvent set of a s/s node is the union of the resolvent sets of all its i/s/o representations. The corresponding i/s/o resolvent matrices give graph representations of the characteristic node and signal bundles of the s/s node. The latter of these graph representation can be used to show that the characteristic signal bundle of a s/s node $\Sigma$ is analytic in the resolvent set of $\Sigma$. 

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5.1. Semi-Regular Resolvable Input/State/Output Nodes (Jan 02, 2016)

5.1.1. The resolvent set and the i/s/o resolvent matrix. As an introduction to the notion of the resolvent set and the i/s/o resolvent matrix of a regular i/s/o node, let us first take a closer look at the case where the i/s/o node is bounded.

Let \( \begin{bmatrix} x \\ y \end{bmatrix} \) be a classical future trajectory induced by a bounded i/s/o node \( \Sigma = ([A, B] ; X, U, Y) \) with initial state \( x(0) = x^0 \) and with a Laplace transformable input function \( u \). Then it follows from \( 2.1.19 \) that also \( x \) and \( y \) are Laplace transformable. By taking Laplace transforms in \( 2.1.12 \) (i.e., we multiply \( 2.1.12 \) by \( e^{-\lambda t} \), integrate over \( \mathbb{R}^+ \), and integrate the left-hand side by parts) we find that, for all \( \lambda \) in some right half-plane, the Laplace transforms \( \hat{x} \), \( \hat{u} \), and \( \hat{y} \) of \( x \), \( u \), and \( y \) satisfy

\[
\lambda \hat{x} - x^0 = A\hat{x} + B\hat{u}, \quad \hat{y} = C\hat{x} + D\hat{u}.
\]

For \( \Re \lambda \) large enough so that \( \lambda \in \rho(A) \) then we can solve \( \hat{x}(\lambda) \) and \( \hat{y}(\lambda) \) from \( 5.1.1 \) in terms of \( x^0 \) and \( \hat{u}(\lambda) \) to get

\[
\hat{x}(\lambda) = \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} B + D \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} B + D \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \rho(A),
\]

Denote

\[
\hat{S}(\lambda) = \begin{bmatrix} \hat{S}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} := \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} B + D \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} \begin{bmatrix} \lambda - A \end{bmatrix}^{-1} B + D \end{bmatrix}, \quad \lambda \in \rho(A),
\]

Then \( 5.1.2 \) can be written in the form

\[
\hat{x}(\lambda) = \hat{S}(\lambda) \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{S}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}
\]

for all those \( \lambda \in \rho(A) \) for which the Laplace transforms converge. In view of formula \( 5.1.4 \), the function \( \hat{S} \) defined by \( 5.1.3 \) for all \( \lambda \in \rho(A) \) is called the i/s/o resolvent matrix of the bounded i/s/o node \( \Sigma = ([A, B] ; X, U, Y) \).

The same argument can be extended to any regular i/s/o node. Before presenting this extension, let us first discuss the case where there is no input and no output (i.e., \( U = Y = \{0\} \)), and \( S \) coincides with its main operator \( A \).

We start by recalling the following definition.

5.1.1. Definition (cf. Definition 3.1.1). Let \( A \) be a linear operator in an \( H \)-space.

(i) A point \( \lambda \in \mathbb{C} \) belongs to the resolvent set of \( A \) if the operator \( (\lambda - A) \) has a bounded everywhere defined inverse.

(ii) The spectrum \( \sigma(A) \) of \( A \) is the complement of the resolvent set of \( A \).

(iii) For \( \lambda \in \rho(A) \) the operator \( (\lambda - A)^{-1} \) is called the resolvent of \( A \) (evaluated at \( \lambda \)).

Note that if \( \rho(A) \neq \emptyset \), then \( A \) is necessarily closed. This follows from the fact that for each \( \lambda \in \rho(A) \) the inverse \( (\lambda - A)^{-1} \) is bounded, and hence closed. Therefore also \( (\lambda - A) \) is closed, which is true if and only if \( A \) is closed.
One way to motivate the above definition is the following: Consider the linear stationary dynamical system (4.1.2) with no input and no output. We call \( x \) a classical trajectory of \( \Sigma \) if \( x \in C^1(\mathbb{R}^+; \mathcal{X}) \) and \( x \) satisfies (4.1.1). If \( x \) is Laplace transformable, then by taking Laplace transforms in (4.1.1) and using the fact that \( A \) is closed we get
\[
\lambda \hat{x}(\lambda) - x^0 = A \hat{x}(\lambda),
\]
for all \( \lambda \in \mathbb{C} \) for which the Laplace transform converges. After rearranging the terms we get the equivalent equation
\[
(\lambda - A) \hat{x}(\lambda) = x^0.
\]
Clearly \( \lambda \in \rho(A) \) if and only if it is true that for every \( x^0 \in \mathcal{X} \) the equation (5.1.6) has a unique solution \( \hat{x}(\lambda) \) which depends continuously on \( x^0 \). Moreover, this solution is given by
\[
\hat{x}(\lambda) = (\lambda - A)^{-1} x^0.
\]

We now return to the regular i/s/o system (2.1.1). If \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical Laplace transformable future trajectory of the i/s/o system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), then by taking Laplace transforms in (2.1.1) and using the fact that \( S \) is closed we get the equation (for all those \( \lambda \) for which all the Laplace transforms converge)
\[
\begin{bmatrix}
\lambda \hat{x}(\lambda) - x^0 \\
\hat{y}(\lambda)
\end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix},
\]
\( \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \text{dom} (S) \).

Above we arrived at formula (5.1.7) by using Laplace transforms, assuming that the i/s/o system (2.1.1) has a classical Laplace transformable future trajectory. In the sequel we shall study (5.1.7) directly, without assuming that \( \hat{x}(\lambda) \), \( \hat{u}(\lambda) \), and \( \hat{y}(\lambda) \) are Laplace transforms of some functions \( x \), \( u \), and \( y \) evaluated at the point \( \lambda \in \mathbb{C} \). To simplify the notations we replace \( x^0 \), \( \hat{x}(\lambda) \), \( \hat{u}(\lambda) \), and \( \hat{y}(\lambda) \) in (2.1.1) by \( x^0, x_\lambda, u_\lambda, \) and \( y_\lambda \) to get
\[
\begin{bmatrix}
\lambda x_\lambda - x^0 \\
y_\lambda
\end{bmatrix} = S \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix},
\]
\( \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix} \in \text{dom} (S) \).

After rearranging the terms we get the equivalent equation
\[
\begin{bmatrix}
S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \\
0 & 0
\end{bmatrix} \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix} = \begin{bmatrix} -x^0 \\ y_\lambda \end{bmatrix},
\]
\( \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix} \in \text{dom} (S) \).

By adding the redundant equation \( u_\lambda = u_\lambda \) to both sides of (5.1.8b) we may further rewrite (5.1.8b) into the equivalent form
\[
\begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix} \in \text{dom} (S),
\]
\[
\begin{bmatrix} x^0 \\ u_\lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1_X & 0 \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix},
\]
\( y_\lambda = \begin{bmatrix} 0 & 1_Y \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix} \).

5.1.2. Definition. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a semi-regular i/s/o node.

(i) A point \( \lambda \in \mathbb{C} \) belongs to the resolvent set of \( \Sigma \), or alternatively, to the i/s/o resolvent set of the system operator \( S \) of \( \Sigma \), if the following condition holds: for every every \( x^0 \in \mathcal{X} \) and for every \( u_\lambda \in \mathcal{U} \) there
is a unique pair of vectors \([\hat{v}_u, \hat{v}_y]\) \in \([\hat{X}, \hat{Y}]\) satisfying (5.1.8a), and \([\hat{v}_u, \hat{v}_y]\) depends continuously on \([\hat{x}_u, \hat{x}_y]\). This set of points is denoted by \(\rho(\Sigma)\) or by \(\rho_{i/s/o}(S)\).

(ii) The **spectrum** of \(\Sigma\) is the complement of the resolvent set of \(\Sigma\). This set is alternatively called the **i/s/o spectrum** of the system operator \(S\) of \(\Sigma\), and is denoted by \(\sigma(\Sigma)\) or by \(\sigma_{i/s/o}(S)\).

(iii) By a **resolvable regular i/s/o node** we mean a regular i/s/o node with a nonempty resolvent set.

5.1.3. **Lemma.** Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a semi-regular i/s/o node, and let \(\lambda \in \mathbb{C}\). Then the following conditions are equivalent:

(i) \(\lambda\) belongs to the resolvent set of \(\Sigma\) (and hence, in particular, \(\Sigma\) is resolvable);

(ii) there exists four bounded linear operators \(\hat{\mathcal{A}}(\lambda), \hat{\mathcal{B}}(\lambda), \hat{\mathcal{C}}(\lambda), \hat{\mathcal{D}}(\lambda)\) such that the four vectors \(x^0, u_\lambda, x_\lambda, y_\lambda\) satisfy (5.1.8a) if and only if

\[
\begin{bmatrix}
x_\lambda \\
y_\lambda
\end{bmatrix} = \begin{bmatrix}
\hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\
\hat{\mathcal{C}}(\lambda) & \hat{\mathcal{D}}(\lambda)
\end{bmatrix} \begin{bmatrix}
x^0 \\
u_\lambda
\end{bmatrix}, \quad \begin{bmatrix}
x^0 \\
u_\lambda
\end{bmatrix} \in \begin{bmatrix}
\hat{X} \\
\hat{U}
\end{bmatrix}.
\]

**Proof.** It is easy to see that (ii) \(\Rightarrow\) (i). Conversely, if (i) holds, then (5.1.8a) defines an operator \(\hat{\mathcal{S}}(\lambda)\) with \(\text{dom}(\hat{\mathcal{S}}(\lambda)) = \begin{bmatrix}\hat{X} \\ \hat{U}\end{bmatrix}\) such that \([\hat{v}_u, \hat{v}_y] = \hat{\mathcal{S}}(\lambda) [\hat{x}_u, \hat{x}_y]\). Since (5.1.8b) is a linear equation, the operator \(\hat{\mathcal{S}}(\lambda)\) is linear, and by assumption, it is also bounded. Since both the domain space and the range space of this operator are products of \(H\)-spaces, the operator \(\hat{\mathcal{S}}(\lambda)\) has a block matrix decomposition \(\hat{\mathcal{S}}(\lambda) = \begin{bmatrix}\hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\
\hat{\mathcal{C}}(\lambda) & \hat{\mathcal{D}}(\lambda)\end{bmatrix}\), and we arrive at (5.1.9).

5.1.4. **Definition.** Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a semi-regular resolvable i/s/o node. The \(\mathcal{B}([\hat{X}, \hat{Y}])\)-valued function \(\hat{\mathcal{S}} = \begin{bmatrix}\hat{\mathcal{A}} & \hat{\mathcal{B}} \\
\hat{\mathcal{C}} & \hat{\mathcal{D}}\end{bmatrix}\) with \(\text{dom}(\hat{\mathcal{S}}) = \rho(\Sigma)\) whose existence follows from Lemma (5.1.3) is called the **i/s/o resolvent matrix** of \(\Sigma\). The components \(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}}\) of the i/s/o resolvent matrix \(\hat{\mathcal{S}}\) are called as follows:

(i) \(\hat{\mathcal{A}}\) is the \(s/s\) (state/state) resolvent of \(\Sigma_{i/s/o}\);

(ii) \(\hat{\mathcal{B}}\) is the \(i/s\) (input/state) resolvent of \(\Sigma_{i/s/o}\);

(iii) \(\hat{\mathcal{C}}\) is the \(s/o\) (state/output) resolvent of \(\Sigma_{i/s/o}\);

(iv) \(\hat{\mathcal{D}}\) is the \(i/o\) (input/output) resolvent of \(\Sigma_{i/s/o}\).

Above we arrived at the equation (5.1.8b) by taking Laplace transforms of classical future trajectories of \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) with initial state \(x^0\). In the special case where \(x^0 = 0\) in (5.1.8b) it is also possible to arrive at this equation in a as follows.

5.1.5. **Lemma.** Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a semi-regular i/s/o node, let \(\begin{bmatrix}x_0 \\
u_0\end{bmatrix} \in \begin{bmatrix}\hat{X} \\
\hat{U}\end{bmatrix}\) and \(\lambda \in \mathbb{C}\), and define \(\begin{bmatrix}x(t) \\
u(t)\end{bmatrix} = e^{\lambda t} \begin{bmatrix}x_0 \\
u_0\end{bmatrix}, \quad t \in \mathbb{R}\). Then the following conditions are equivalent:

(i) \(\begin{bmatrix}x \\
u\end{bmatrix}\) is a classical two-sided trajectory of \(\Sigma\);

(ii) \(\begin{bmatrix}x_0 \\
u_0\end{bmatrix} \in \text{dom}(S)\) and \(\begin{bmatrix}\lambda x_0 \\
u_0\end{bmatrix} = S \begin{bmatrix}x_0 \\
u_0\end{bmatrix}\).

If, in addition, \(\lambda \in \rho(\Sigma)\), then conditions (i) and (ii) above are equivalent to the following condition:
(iii) \[
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
= \begin{bmatrix}
\hat{\mathbf{U}}(\lambda) \\
\hat{\mathbf{D}}(\lambda)
\end{bmatrix} u_0,
\] where \(\hat{\mathbf{U}}(\lambda)\) and \(\hat{\mathbf{D}}(\lambda)\) are the input/state and input/output resolvents of \(\Sigma\) evaluated at \(\lambda\).

**Proof.** (i) \(\Rightarrow\) (ii): If (i) holds, then \[
\begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix} \in \text{dom}(S) \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = S \begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix}.
\]
Here \[
\begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix} = \begin{bmatrix} x_0 \\
y_0
\end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and thus (ii) holds.}
\]

(ii) \(\Rightarrow\) (i): For all \(t \in \mathbb{R}\) we have \[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_0 \\
y_0
\end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
so if (ii) holds, then \[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} \in \text{dom}(S) \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = S \begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} \text{ for all } t \in \mathbb{R}. \] Thus \[
\begin{bmatrix} x \\ y
\end{bmatrix}
\] is a classical two-sided trajectory of \(\Sigma\).

If \(\lambda \in \rho(\Sigma)\), then it follows from Lemma 5.1.3 that (ii) and (iii) are equivalent. \(\square\)

The following theorem gives a necessary and sufficient condition for a point \(\lambda \in \mathbb{C}\) to belong to the resolvent set of \(\Sigma\), and it also gives an explicit formula for how to compute the i/s/o resolvent matrix of \(\Sigma\) from the system operator \(S\) of \(\Sigma\).

**5.1.6. Theorem.** Let \(\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a semi-regular i/s/o node, and let \(\lambda \in \mathbb{C}\). Then the following conditions are equivalent:

(i) \(\lambda\) belongs to the resolvent set of \(\Sigma\) (and hence, in particular, \(\Sigma\) is resolvable);

(ii) The operator \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_y & 0 \\ 0 & 0 \end{bmatrix} S \right)\) (with domain equal to \(\text{dom}(S)\)) has a bounded inverse in \(B(\mathcal{Y})\). (Equivalently, this operator is closed, injective and surjective.)

Suppose that these equivalent conditions hold, and denote the i/s/o resolvent matrix of \(\Sigma\) evaluated at \(\lambda\) by \(\hat{\mathbf{R}}(\lambda) = \begin{bmatrix} \hat{\mathbf{U}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ \hat{\mathbf{C}}(\lambda) & \hat{\mathbf{D}}(\lambda) \end{bmatrix}\). Then the inverse of the operator \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_y & 0 \\ 0 & 0 \end{bmatrix} S \right)\) in (ii) is given by

\[
(5.1.10) \quad \left(\begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_y & 0 \\ 0 & 0 \end{bmatrix} S \right)^{-1} = \begin{bmatrix} \hat{\mathbf{U}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ \hat{\mathbf{C}}(\lambda) & \hat{\mathbf{D}}(\lambda) \end{bmatrix},
\]

and \(\hat{\mathbf{R}}(\lambda)\) is equal to

\[
(5.1.11) \quad \hat{\mathbf{R}}(\lambda) := \hat{\mathbf{U}}(\lambda) \quad \hat{\mathbf{B}}(\lambda) = \begin{bmatrix} 1_y & 0 \\ 0 & 1_u \end{bmatrix} S \left(\begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_y & 0 \\ 0 & 0 \end{bmatrix} S \right)^{-1}.
\]

**Proof.** (i) \(\Rightarrow\) (ii): Suppose that \(\lambda\) is a regular point of \(\Sigma\). By ignoring the value of the variable \(y_0\) in (5.1.8c) and (5.1.9) we find that for all \(\begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \in \text{dom}(S)\), if we define \(x^0\) by the top row of the middle formula in (5.1.8c), then

\[
(5.1.12) \quad \begin{bmatrix} x^0 \\ u_\lambda
\end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_y & 0 \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u_\lambda
\end{bmatrix}
\]

and

\[
(5.1.13) \quad \begin{bmatrix} x_\lambda \\ u_\lambda
\end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} \begin{bmatrix} x^0 \\ u_\lambda
\end{bmatrix}.
\]

This shows that \(\begin{bmatrix} \hat{\mathbf{U}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ \hat{\mathbf{C}}(\lambda) & \hat{\mathbf{D}}(\lambda) \end{bmatrix}\) is a left inverse of \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_y & 0 \\ 0 & 0 \end{bmatrix} S \right)\). That it is also a right inverse follows from the fact that if \(\begin{bmatrix} x^0 \\ u_\lambda \end{bmatrix} \in \mathcal{Y}^\perp\), and if we define \(x_\lambda\) by the top row of (5.1.13), then (5.1.12) holds. This shows that (i) \(\Rightarrow\) (ii) and (5.1.10) holds.
(ii) ⇒ (i): Suppose that (ii) holds. It is easy to see that the inverse of
\[
\begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix} - \begin{bmatrix} 1_x & 0 \\ 0 & 0 \end{bmatrix} S
\]
must be an operator of the form (5.1.10) for some bounded linear operators \( \mathfrak{A}(\lambda) \) and \( \mathfrak{B}(\lambda) \). The first equation in (5.1.13) is then equivalent to the equation (5.1.13), and (5.1.8c) can be rewritten in the equivalent form
\[
(5.1.14)
\]
Thus, in particular, both \( x_\lambda \) and \( y_\lambda \) in (5.1.8a) are uniquely determined by \( x^0 \) and \( u_\lambda \) through (5.1.14).

We claim that the operator on the right-hand side of (5.1.14) is bounded. To show this it suffices to show that the operator \( S \begin{bmatrix} \mathfrak{A}(\lambda) & \mathfrak{B}(\lambda) \\ 0 & 1_u \end{bmatrix} \) is closed. That this is true can be seen as follows: The first factor \( S \) is closed, and the second factor \( \begin{bmatrix} \mathfrak{A}(\lambda) & \mathfrak{B}(\lambda) \\ 0 & 1_u \end{bmatrix} \) is bounded, and hence the product is closed. Therefore, by the closed graph theorem, \( S \begin{bmatrix} \mathfrak{A}(\lambda) & \mathfrak{B}(\lambda) \\ 0 & 1_u \end{bmatrix} \) is bounded. As we noticed above, this implies that the operator on the right-hand side of (5.1.14) is bounded. Thus (ii) ⇒ (i), and (5.1.11) holds.

5.1.7. Corollary. Let \( \Sigma = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a semi-regular resolvable i/s/o node with main operator \( A \) and observation operator \( C \). Denote the i/s/o resolvent matrix of \( \Sigma \) by \( \widehat{\mathcal{S}} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \). Then, for all \( \lambda \in \rho(\Sigma) \) the operators \( \mathfrak{A}(\lambda) \), \( \mathfrak{B}(\lambda) \), \( \mathfrak{C}(\lambda) \), and \( \widehat{\mathfrak{D}}(\lambda) \) are given by
\[
(5.1.15a) \quad \mathfrak{A}(\lambda) = (\lambda - A)^{-1},
\]
\[
(5.1.15b) \quad \mathfrak{B}(\lambda) = \begin{bmatrix} 1_x & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 1_u \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1_u \end{bmatrix},
\]
\[
(5.1.15c) \quad \mathfrak{C}(\lambda) = C(\lambda - A)^{-1},
\]
\[
(5.1.15d) \quad \widehat{\mathfrak{D}}(\lambda) = \begin{bmatrix} 0 & 1_y \end{bmatrix} S \begin{bmatrix} \mathfrak{B}(\lambda) \\ 1_u \end{bmatrix}.
\]

Proof. The formulas for \( \mathfrak{A}(\lambda) \) and \( \mathfrak{B}(\lambda) \) directly follow from (5.1.10), and the formulas for \( \mathfrak{C}(\lambda) \) and \( \mathfrak{D}(\lambda) \) follow from (5.1.11).

In the case of a semi-bounded i/s/o node \( \Sigma = (\hat{A} \ [\hat{B} \ C]) ; \mathcal{X}, \mathcal{U}, \mathcal{Y} \) Theorem 5.1.6 can be simplified as follows, and it is possible to add two more formulas for the computation of the i/s/o resolvent matrix.

5.1.8. Corollary. Let \( \Sigma = (\hat{A} \ [\hat{B} \ C]) ; \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be a semi-bounded i/s/o node.
Then \( \Sigma \) is regular and resolvable, \( \rho(\Sigma) = \rho(\hat{A}) \), and the i/s/o resolvent matrix of \( \Sigma \) is given by
\[
(5.1.16)
\]
\[
\widehat{\mathcal{S}}(\lambda) = \begin{bmatrix} \mathfrak{A}(\lambda) & \mathfrak{B}(\lambda) \\ \mathfrak{C}(\lambda) & \mathfrak{D}(\lambda) \end{bmatrix} = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1} B \\ C(\lambda - A)^{-1} & C(\lambda - A)^{-1} B + D \end{bmatrix}, \quad \lambda \in \rho(\hat{A}),
\]
This formula can alternatively be written in the forms
\[
(5.1.17a) \quad \widehat{\mathcal{S}}(\lambda) = \begin{bmatrix} \mathfrak{A}(\lambda) & \mathfrak{B}(\lambda) \\ \mathfrak{C}(\lambda) & \mathfrak{D}(\lambda) \end{bmatrix} = \begin{bmatrix} 1_x & 0 \\ C & D \end{bmatrix} \begin{bmatrix} \lambda - A & -B \\ 0 & 1_u \end{bmatrix}^{-1}, \quad \lambda \in \rho(\hat{A}),
\]
\( (5.1.17b) \quad \mathcal{G}(\lambda) = \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda - A & 0 \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \times & B \\ 0 & D \end{bmatrix}, \quad \lambda \in \rho(A). \)

**Proof.** In the case of a semi-bounded i/s/o node we have
\[
\left( \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \times & 0 \\ 0 & 0 \end{bmatrix} S \right) = \begin{bmatrix} \lambda - A & -B \\ 0 & 1 \end{bmatrix},
\]
and the right-hand side is invertible if and only if \( \lambda \in \rho(A) \). Formulas \((5.1.16)\) and \((5.1.17a)\) follows from \((5.1.11)\) since \( \left( \frac{\lambda - A - B}{1} \right)^{-1} = \left( \frac{(\lambda - A)^{-1} \times B}{1} \right) \). From \((5.1.16)\) one gets \((5.1.17b)\) by factoring out \( \left( \frac{\lambda - A}{0} \right)^{-1} = \left( \frac{(\lambda - A)^{-1} \times 1}{1} \right) \) to the left. \( \square \)

In the following lemma we take a closer look at the two operators \( \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix} \)
and \( \begin{bmatrix} \hat{\mathbf{A}}(\lambda) \\ \hat{\mathbf{B}}(\lambda) \end{bmatrix} \)
and their inverses.

5.1.9. **Lemma.** Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a semi-regular resolvable i/s/o node with main operator \( A \) and observation operator \( C \). Denote the i/s/o resolvent matrix of \( \Sigma \) by \( \mathcal{G} = \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ \hat{\mathbf{C}}(\lambda) & \hat{\mathbf{D}}(\lambda) \end{bmatrix} \), and let \( \lambda \in \rho(\Sigma) \). Then the following claims hold:

(i) The operator \( \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix} \) maps \( [\mathcal{X}] \) on-to-one onto \( \text{dom}(S) \),
\[
\begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda - A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \times & -\hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix} = \left( \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \times & 0 \\ 0 & 0 \end{bmatrix} S \right).
\]

(ii) The operator \( \begin{bmatrix} 1 \times \hat{\mathbf{A}}(\lambda) \\ 0 \end{bmatrix} \) is bounded and invertible on \( [\mathcal{X}] \), and it maps \( [\text{dom}(A)] \) one-to-one onto \( \text{dom}(S) \). The inverse of this operator (which maps \( [\mathcal{X}] \) one-to-one onto itself and \( S \) one-to-one onto \( [\text{dom}(A)] \)) is \( \begin{bmatrix} 1 \times & -\hat{\mathbf{A}}(\lambda) \\ 0 & 1 \end{bmatrix} \).

(iii) The operator \( \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & 0 \\ \hat{\mathbf{C}}(\lambda) & 1 \end{bmatrix} \) maps \( [\mathcal{X}] \) one-to-one onto \( \text{dom}(A) \), and
\[
\begin{bmatrix} \hat{\mathbf{A}}(\lambda) & 0 \\ \hat{\mathbf{C}}(\lambda) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 \times & 0 \\ -\hat{\mathbf{C}}(\lambda) & 1 \end{bmatrix} \begin{bmatrix} \lambda - A & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - A & 0 \\ -C & 1 \end{bmatrix}.
\]

**Proof.** (i) To prove the first identity in \((5.1.18)\) we factor \( \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \end{bmatrix} \) into
\[
\begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & 0 \\ 0 & 1 \end{bmatrix}.
\]
Here both the operators on the right-hand side are invertible, and thus
\[
\begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \times & \hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda - A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \times & -\hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix}.
\]
The second identity in \((5.1.18)\) is equivalent to \((5.1.16)\).

(ii) By \((5.1.18)\),
\[
\begin{bmatrix} 1 \times & -\hat{\mathbf{B}}(\lambda) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{A}}(\lambda) & \hat{\mathbf{B}}(\lambda) \end{bmatrix}^{-1}.
\]
Here the second factor maps \( \text{dom} (S) \) one-to-one onto \([X]_U \), and the first factor maps \([X]_U \) one-to-one onto \([\text{dom}(A)]_U \).

(iii) To prove the first identity in (5.1.19) we factor \( \hat{\mathfrak{A}}(\lambda) 0 \) into
\[
\begin{bmatrix}
\hat{\mathfrak{A}}(\lambda) & 0 \\
\mathfrak{C}(\lambda) & 1_y
\end{bmatrix}
= \begin{bmatrix}
1_x & 0 \\
0 & 1_{1_U}
\end{bmatrix}
\begin{bmatrix}
\hat{\mathfrak{A}}(\lambda) & 0 \\
\mathfrak{C}(\lambda) & 1_y
\end{bmatrix}.
\]

Here both the operators on the right-hand side are invertible, and thus
\[
\begin{bmatrix}
\hat{\mathfrak{A}}(\lambda) & 0 \\
\mathfrak{C}(\lambda) & 1_y
\end{bmatrix}^{-1} = \begin{bmatrix}
1_x & 0 \\
0 & 1_{1_U}
\end{bmatrix}^{-1}
\begin{bmatrix}
\hat{\mathfrak{A}}(\lambda) & 0 \\
\mathfrak{C}(\lambda) & 1_y
\end{bmatrix}^{-1} = \begin{bmatrix}
1_x & 0 \\
0 & 1_{1_U}
\end{bmatrix} \begin{bmatrix}
\hat{\mathfrak{A}}(\lambda) & 0 \\
\mathfrak{C}(\lambda) & 1_y
\end{bmatrix} = \begin{bmatrix}
\lambda - A & 0 \\
0 & 1_{1_U}
\end{bmatrix}.
\]

The second identity in (5.1.19) follows from (5.1.15). \(\square\)

5.1.10. REMARK. Looking at (5.1.18) it would be tempting to add one more formula by multiplying the two factors on the right-hand side of (5.1.18) to get the block matrix operator
\[
\begin{bmatrix}
\lambda - A & - (\lambda - A) \hat{\mathfrak{B}}(\lambda) \\
0 & 1_{1_U}
\end{bmatrix}.
\]

If \( S \) is bounded, then this is, indeed, possible, but for an unbounded \( S \) this is not possible in general, due to the fact that \([x^u] \in \text{dom} (S)\) does not imply that \( x \in \text{dom} (A) \) and \( \hat{\mathfrak{B}}(\lambda) u \in \text{dom} (A) \). It is, however, possible to interpretation the above formula in an “extended” sense, where \( A \) has been replace by an extension \( A_e \) which is defined on all of \( X \) and maps into a “extrapolation space” \( X' \). This will be explained in detail in Sections 5.1.2–5.1.4

5.1.11. THEOREM. Let \( \Sigma = (S; X, U, Y) \) be a semi-regular resolvable i/s/o node with main operator \( A \). Then \( A \) is closed and \( \rho(\Sigma) = \rho(A) \). In particular, \( \rho(A) \neq \emptyset \), and \( \rho(\Sigma) = \rho(A) \) is an open subset of \( \mathbb{C} \).

Below we present one proof of this theorem which is based on Theorem 5.1.6. The proof of Theorem 5.2.16 below (which instead is based on the notion of the graph of an operator) could also be used to prove this theorem.

PROOF OF THEOREM 5.1.11. It follows from (5.1.10) that for all \( \lambda \in \rho(\Sigma) \) we have \( \lambda \in \rho(A) \) and \( (\lambda - A)^{-1} = \hat{\mathfrak{A}}(\lambda) \). Conversely, suppose that \( \lambda \in \rho(A) \). Then \( \lambda - A \) is both injective and surjective. It is easy to see that the injectivity of \( \lambda - A \) implies that the operator \( \left( \begin{bmatrix} x^u \\ 0 \end{bmatrix} \in [X]_U \right) \) is injective. We claim that this operator is also surjective, i.e., that for every \( \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] \in [X]_U \) there exists some \( x_\lambda \in X \) such that \( \left[ x^u \right] \in \text{dom} (S) \) and \( \left[ x^u \right] = \left( \left[ \hat{\mathfrak{A}} \right] \right)^{-1} \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] \). Below we shall not only prove this, but we shall actually give a formula for how to find this \( x_\lambda \). (Note that \( x_\lambda \) is unique since \( \left( \begin{bmatrix} x^u \\ 0 \end{bmatrix} \in [X]_U \right) \) is injective.)

Let \( \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] \in \text{dom} (S) \) and denote \( \lambda x_\lambda - \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] S \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] = x^0 \). Choose some arbitrary \( \mu \in \rho(\Sigma) \). Then by Lemma 5.1.9, the equation \( \left[ x^u \right] = \left( \left[ \hat{\mathfrak{B}}(\mu) \right] \right)^{-1} \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] \) has the unique solution \( x_\mu \) (equal to \( \hat{\mathfrak{B}}(\mu) x^0 + \hat{\mathfrak{B}}(\mu) u \)). Since both \( \left[ x^u \right] \in \text{dom} (S) \) and \( \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] \in \text{dom}(S) \) we have \( \left[ x^u - x_\mu \right] \in \text{dom} (S) \), i.e., \( x_\lambda - x_\mu \in \text{dom} (A) \). Moreover,
\[
A(x_\lambda - x_\mu) = \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] S \left[ \begin{bmatrix} x^u - x_\mu \\ 0 \end{bmatrix} \right] = \left[ \begin{bmatrix} x^0 \\ 0 \end{bmatrix} \right] S \left( \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] - \left[ \begin{bmatrix} x^u \\ 0 \end{bmatrix} \right] \right) = \lambda x_\lambda - \mu x_\mu.
\]
From here we can solve

\[(5.1.20) \quad x_\lambda = x_\mu + (\mu - \lambda)(\lambda - A)^{-1}x_\mu.\]

We claim that indeed, if we define \(x_\lambda\) by \((5.1.20)\), then \(\lambda x_\lambda - \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u \end{bmatrix} = x^0\). Clearly \((5.1.20)\) implies that \(x_\lambda - x_\mu \in \text{dom}(A)\), and that

\[
\lambda x_\lambda - \mu x_\mu = A(x_\lambda - x_\mu) = \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u \end{bmatrix} - \begin{bmatrix} x_\mu \\ u \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u \end{bmatrix} - (\mu x_\mu - x^0).
\]

This gives the expected identity \(\lambda x_\lambda - \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S \begin{bmatrix} x_\lambda \\ u \end{bmatrix} = x^0\) and completes the proof of the surjectivity of the operator \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S\right)\).

The operator \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S\right)\) is closed since it is the inverse of a bounded operator, and therefore also \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S\right)\) is closed. By the closed graph theorem, \(\left(\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} S\right)\) has a bounded inverse. By Theorem 5.1.6 \(\lambda\) is a regular point for \(\Sigma\).

As is well-known (and proved in Theorem 5.2.3 below), the resolvent set of a closed operator \(A\) in an \(H\)-space is open. Thus, \(\rho(\Sigma) = \rho(A)\) is open.

Our following lemma presents different ways how we can recover the generating operator of an i/s/o node from its i/s/o resolvent matrix.

5.1.12. Lemma. Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a semi-regular resolvable i/s/o node with main operator \(A\). Denote the i/s/o resolvent matrix of \(\Sigma\) by \(\widehat{S} = \begin{bmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{bmatrix}\), and let \(\lambda \in \rho(\Sigma)\). Then \(\text{dom}(S)\) and \(S\) itself can be recovered from \(\widehat{S}(\lambda)\) by means of the following formulas:

\[
\begin{align*}
(5.1.21a) \quad & \text{dom}(S) = \text{rng} \left( \begin{bmatrix} \widehat{A}(\lambda) & \widehat{B}(\lambda) \\ \widehat{C}(\lambda) & \widehat{D}(\lambda) \end{bmatrix} \right), \\
(5.1.21b) \quad & \text{dom}(S) = \text{dom} \left( \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & \widehat{B}(\lambda) \\ 0 & \widehat{D}(\lambda) \end{bmatrix} \right), \\
(5.1.21c) \quad & \text{dom}(S) = \text{dom} \left( \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\widehat{A}(\lambda) & 0 \\ 0 & \widehat{D}(\lambda) \end{bmatrix} \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1} \right), \\
(5.1.21d) \quad & \text{dom}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \mid \widehat{B}(\lambda)u - x \in \text{dom}(A) \right\}.
\end{align*}
\]

\[
\begin{align*}
(5.1.22a) \quad & S = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & \widehat{C}(\lambda) \\ \widehat{A}(\lambda) & 0 \end{bmatrix} \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1}, \\
(5.1.22b) \quad & S = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & \widehat{B}(\lambda) \\ 0 & \widehat{D}(\lambda) \end{bmatrix}, \\
(5.1.22c) \quad & S = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\widehat{A}(\lambda) & 0 \\ 0 & \widehat{D}(\lambda) \end{bmatrix} \begin{bmatrix} \widehat{A}(\lambda) & 0 \\ \widehat{C}(\lambda) & 1 \end{bmatrix}^{-1}, \\
(5.1.22d) \quad & S \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} A(x - \widehat{B}(\lambda)u) + \lambda \widehat{B}(\lambda)u \\ C(x - \widehat{B}(\lambda)u) + \widehat{D}(\lambda)u \end{bmatrix}, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom}(S).
\end{align*}
\]
Note that according to formulas (5.1.21a) and (5.1.21d), already the two operators \( \hat{\mathfrak{A}}(\lambda) \) and \( \hat{\mathfrak{B}}(\lambda) \) determine \( \text{dom}(S) \) uniquely (the set \( \text{dom}(A) \) which appears in (5.1.21d) can be replaced by \( \text{rng}(\hat{\mathfrak{A}}(\lambda)) \)), and that the operators \( A \) and \( C \) in (5.1.22d) may be recovered from \( \hat{\mathfrak{A}}(\lambda) \) and \( \hat{\mathfrak{C}}(\lambda) \) by \( A = \lambda - \hat{\mathfrak{A}}(\lambda)^{-1} \) and \( C = \hat{\mathfrak{C}}(\lambda)\hat{\mathfrak{A}}(\lambda)^{-1} \).

**Proof of Lemma 5.1.12** We begin by observing that (5.1.21a) and (5.1.21d) follow from parts (i) and (iii) of Lemma 5.1.9.

By (5.1.10) and (5.1.11),

\[
\begin{bmatrix} 1_X & 0 \\ 0 & 1_Y \end{bmatrix} \begin{bmatrix} S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1_X & 0 \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) & \hat{\mathfrak{B}}(\lambda) \\ 0 & 1_U \end{bmatrix}^{-1}
\]

This gives (5.1.22a).

To prove (5.1.22b) and (5.1.22c) we recall from Lemma 5.1.3 that (5.1.8a) holds for some \( [\frac{x}{u}] \) if and only if (5.1.9) holds for some \( [\frac{x}{u}] \). The latter equation can be rewritten in the form

\[
\begin{bmatrix} \hat{\mathfrak{A}}(\lambda) \\ \hat{\mathfrak{C}}(\lambda) \end{bmatrix} \begin{bmatrix} -x^0 \\ y_\lambda \end{bmatrix} = \begin{bmatrix} -1_X & 0 \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) & \hat{\mathfrak{B}}(\lambda) \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix}.
\]

Here \( \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) \\ \hat{\mathfrak{C}}(\lambda) \end{bmatrix} \) maps \( [\frac{x}{y}] \) one-to-one onto \( [\text{dom}(A) \ y] \), and it follows from (5.1.21d) that \( \begin{bmatrix} -1_X & 0 \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) & \hat{\mathfrak{B}}(\lambda) \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix} \) maps \( [\frac{x}{y}] \) into \( [\text{dom}(A) \ y] \). After multiplying the above identity by \( \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) \\ \hat{\mathfrak{C}}(\lambda) \end{bmatrix}^{-1} \) to the left we get (5.1.22b) and (5.1.22c).

The equality (5.1.21c) and the identity (5.1.22d) follow from the earlier equalities and identities, since

\[
\begin{bmatrix} \hat{\mathfrak{A}}(\lambda) \\ \hat{\mathfrak{C}}(\lambda) \end{bmatrix} \begin{bmatrix} -x^0 \\ y_\lambda \end{bmatrix} = \begin{bmatrix} -1_X & 0 \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) & \hat{\mathfrak{B}}(\lambda) \\ 0 & 1_U \end{bmatrix} \begin{bmatrix} x_\lambda \\ u_\lambda \end{bmatrix}.
\]

Recall, in particular, that \( \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) \\ \hat{\mathfrak{C}}(\lambda) \end{bmatrix}^{-1} \) maps \( \text{dom}(S) \) one-to-one onto \( [\frac{x}{y}] \), and that \( \begin{bmatrix} \hat{\mathfrak{A}}(\lambda) \\ \hat{\mathfrak{C}}(\lambda) \end{bmatrix}^{-1} \) maps \( [\text{dom}(A) \ y] \) one-to-one onto \( [\frac{x}{y}] \). Clearly

\[
S \begin{bmatrix} x \\ u \end{bmatrix} = S \left( \begin{bmatrix} x - \hat{\mathfrak{B}}(\lambda)u \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{\mathfrak{B}}(\lambda)u \\ u \end{bmatrix} \right).
\]

Here \( S \begin{bmatrix} x - \hat{\mathfrak{B}}(\lambda)u \\ 0 \end{bmatrix} = \frac{A(x - \hat{\mathfrak{B}}(\lambda)u)}{C(x - \hat{\mathfrak{B}}(\lambda)u)} \), and it follows from (5.1.22d) that \( S \begin{bmatrix} \hat{\mathfrak{B}}(\lambda)u \\ u \end{bmatrix} = \begin{bmatrix} \lambda \hat{\mathfrak{B}}(\lambda) \\ \hat{\mathfrak{B}}(\lambda) \end{bmatrix} \). This gives (5.1.22d). □
In the following theorem we give a set of necessary and sufficient conditions on the system operator \( S \) of a semi-regular or regular i/s/o node for this node to have a nonempty resolvent set.

**5.1.13. Theorem.** A linear operator \( S : [X] \supset \text{dom}(S) \rightarrow [Y] \) is the system operator of a semi-regular resolvable i/s/o node \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) if and only if the following four conditions hold:

(i) \( S \) is closed.

(ii) The operator \([1X \ 0] S \) is closed.

(iii) The main operator \( A \) of \( S \) has a nonempty resolvent.

(iv) For every \( u \in \mathcal{U} \) there exists some \( x \in \mathcal{X} \) such that \([x_u] \in \text{dom}(S)\).

Moreover,

(v) \( \Sigma \) is regular if and only if \( \text{dom}(A) \) is dense in \( X \).

**Proof.** We begin by assuming that \( \Sigma \) is a semi-regular i/s/o node with \( \rho(\Sigma) \neq \emptyset \). Then by Definition 2.1.1, \( S \) is closed, i.e., condition (i) holds. The invertibility condition in Definition 5.1.2 implies that the operator \([\lambda 0 \ 0 \ 1u] - [1X 0 0] S \) is closed. Since the operator \([\lambda 0 \ 0 \ 1u] - [1X 0 0] S \) is bounded, this implies that \([1X \ 0] S \) is closed. Thus condition (ii) holds. That also condition (iii) holds follows from Theorem 5.1.11. Finally, it is clear that (iv) holds, since \( \text{dom}(S) \) is the range of the operator \([\lambda(\lambda)u_n \ 0 1u]\). Thus, if \( \Sigma \) is semi-regular and has a nonempty resolvent set, then conditions (i)–(iv) above hold.

We next show that if \( \Sigma \) is regular, i.e., if, in addition, \( \text{dom}(S) \) is dense in \([X]_U\), then \( \text{dom}(A) \) is dense in \( X \). Let \( x \in X \). Since \( \text{dom}(S) \) is dense in \([X]_U\), there exists a sequence \([x_n] \in \text{dom}(S)\) such that \( \lim_{n \rightarrow \infty} [x_n] = [x] \). Fix \( \lambda \in \rho_{i/s/o}(S) \), and define \( x_n' = x_n - \hat{B}(\lambda)u_n \). Since \([\hat{B}(\lambda)u_n] \in \text{dom}(S)\), we have \([x_n'] \in \text{dom}(S)\), and consequently \( x_n' \in \text{dom}(A) \). In addition \( x_n' \rightarrow x \) as \( n \rightarrow \infty \). Thus, \( \text{dom}(A) \) is dense in \( X \), and we have proved one half of (v).

Conversely, suppose that (i)–(iv) hold. By assumption (i), \( S \) is closed, so according to Theorem 5.1.6 we only need to show that the operator \([\lambda 0 \ 0 \ 1u] - [1X 0 0] S \) has a bounded inverse for some \( \lambda \in \mathbb{C} \). This can be seen as follows. It follows from (ii) that this operator is closed. It is easy to see that \([\lambda 0 \ 0 \ 1u] - [1X 0 0] S \) is injective (since both \( 1u \) and \( \lambda - A \) are injective). That this operator is also surjective follows from condition (iv) and the surjectivity of \( \lambda - A \) in the following way: Let \([a] \in [X]_U \). Since we assume condition (iv) to hold, there exists some \( x_0 \in \mathcal{X} \) such that \([x_0_u] \in \text{dom}(S)\). Let \( z_0 = [1X \ 0] S [x_0_u] \). Since \( \lambda - A \) is surjective there exists some \( x_1 \in \text{dom}(A) \) such that \( (\lambda - A)x_1 = z - \lambda x_0 + z_0 \). Define \( x = x_0 + x_1 \). Then

\[
\left(\begin{array}{c}
\lambda \\
0
\end{array}\right) - \left(\begin{array}{c}
1X \\
0
\end{array}\right) S \left(\begin{array}{c}
x \\
u
\end{array}\right) = \left(\begin{array}{c}
\lambda \\
0
\end{array}\right) \left(\begin{array}{c}
0 \\
1u
\end{array}\right) - \left(\begin{array}{c}
1X \\
0
\end{array}\right) \left(\begin{array}{c}
x_0 \\
u
\end{array}\right)
+ \left(\begin{array}{c}
\lambda \\
0
\end{array}\right) \left(\begin{array}{c}
0 \\
1u
\end{array}\right) - \left(\begin{array}{c}
1X \\
0
\end{array}\right) S \left(\begin{array}{c}
x_1 \\
0
\end{array}\right)
= \left(\begin{array}{c}
\lambda x_0 - z_0 \\
u
\end{array}\right) + \left(\begin{array}{c}
(\lambda - A)x_1 \\
0
\end{array}\right) = \left(\begin{array}{c}
z \\
u
\end{array}\right).
\]

This shows that the operator \([\lambda 0 \ 0 \ 1u] - [1X 0 0] S \) is a bijection from \( \text{dom}(S) \) onto \([X]_U \). By the closed graph theorem it has a bounded inverse. By Theorem 5.1.6, \( \lambda \)
is a regular point for \( \Sigma \), and hence \( \Sigma \) has a nonempty resolvent set. This proves that if (i)–(iv) hold, then \( \Sigma \) has a nonempty resolvent set.

Finally, let us prove that \( \text{dom}(S) \) is dense in \( \hat{\mathcal{X}} \) whenever \( \text{dom}(A) \) is dense in \( \chi \). We shall actually prove a slightly stronger statement, namely that for every \( u \in \mathcal{U} \) the set \( \{ x \in \chi \mid [x_u] \in \text{dom}(S) \} \) is dense in \( \chi \). Indeed, this set is dense because \( \text{dom}(A) \) is dense in \( \chi \) and the above set be parameterized as \( \{ x_1 + x_0 \mid x_1 \in \text{dom}(A) \} \), where \( [x_0_u] \) is some (arbitrarily chosen) fixed vector in \( \text{dom}(S) \) (that such a vector exists follows from assumption (iv)). \( \square \)

In our next lemma we describe some of the continuity properties of a system operator of a semi-regular i/s/o node with \( \rho(\Sigma) \neq \emptyset \). That lemma uses the notion of the graph topology induced by a closed linear operator on its domain. This topology is explained in more detail in Appendix A.1.4.

5.1.14. Lemma. Let \( \Sigma = (S; \chi, \mathcal{U}, \mathcal{Y}) \) be a semi-regular resolvable i/s/o node. Denote the i/s/o resolvent matrix of \( \Sigma \) by \( \hat{\mathcal{S}} = \left[ \begin{array}{cc} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{array} \right] \). Then the following statements are true:

(i) The operator \( \left[ \begin{array}{cc} 0 & 1_{\mathcal{Y}} \end{array} \right] S \) is continuous \( \text{dom}(S) \rightarrow \mathcal{Y} \) with respect to the graph topology in \( \text{dom}(S) \) (induced by \( S \)).

(ii) The observation operator \( C \) is continuous \( \text{dom}(A) \rightarrow \mathcal{Y} \) with respect to the graph topology in \( \text{dom}(A) \) (induced by \( A \)).

(iii) If \( \Sigma \) is regular, then for every \( u \in \mathcal{U} \), the set \( \{ x \in \chi \mid [x_u] \in \text{dom}(S) \} \) is dense in \( \chi \).

(iv) The graph topology in \( \text{dom}(S) \) induced by \( S \) is equivalent to the graph topology in \( \text{dom}(S) \) induced by \( \left[ \begin{array}{cc} 1_{\chi} & 0 \end{array} \right] S \).

Proof. (i) Every closed operator is continuous from its domain with the graph topology into its range space, so \( S \) is continuous from \( \text{dom}(S) \rightarrow \mathcal{Y} \). Therefore also the product \( \left[ \begin{array}{cc} 0 & 1_{\mathcal{Y}} \end{array} \right] S \) is continuous from \( \text{dom}(S) \rightarrow \mathcal{Y} \).

(ii) The space \( \left[ \begin{array}{cc} \text{dom}(A) \end{array} \right] \) is a closed subspace of \( \text{dom}(S) \), and \( C \) is the restriction of \( \left[ \begin{array}{cc} 0 & 1_{\mathcal{Y}} \end{array} \right] S \) to this subspace, so \( C \) is continuous from \( \text{dom}(A) \rightarrow \mathcal{Y} \).

(iii) Let \( u \in \mathcal{U} \), and let \( \lambda \in \rho(A) \). Then \( \left[ \hat{\mathcal{S}}(\lambda)u \right] \in \text{dom}(S) \), and consequently, \( \left[ \hat{\mathcal{A}}(\lambda)u \right] + \left[ \hat{\mathcal{B}} \right] \in \text{dom}(S) \) for every \( x_1 \in \text{dom}(A) \). By Theorem 5.1.13 if \( \Sigma \) is regular, then \( \text{dom}(A) \) is dense in \( \chi \), and (iii) follows.

(iv) Since the graph topology is the weakest topology on \( \text{dom}(S) \) which makes both the embedding map \( \text{dom}(S) \rightarrow \left[ \begin{array}{cc} \mathcal{X} \end{array} \right] \) and the operator \( S: \text{dom}(S) \rightarrow \left[ \begin{array}{cc} \hat{\mathcal{X}} \end{array} \right] \) respectively \( [1_{\chi} \ 0] S: \text{dom}(S) \rightarrow \chi \) continuous, and since the continuity of \( S \) implies the continuity of \( [1_{\chi} \ 0] S \), the graph topology in \( \text{dom}(S) \) induced by \( S \) is stronger than the graph topology in \( \text{dom}(S) \) induced by \( [1_{\chi} \ 0] S \). By the open mapping theorem (see, e.g., [Rudin 1973, Theorem 2.11, p. 47]), these two topologies must therefore be equivalent. \( \square \)

5.1.15. Lemma. Let \( \Sigma = (S; \chi, \mathcal{U}, \mathcal{Y}) \) be a semi-regular resolvable i/s/o node. Denote the i/s/o resolvent matrix of \( \Sigma \) by \( \hat{\mathcal{S}} := \left[ \begin{array}{cc} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{array} \right] \).
(i) If \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical future Laplace transformable trajectory of \( \Sigma \) with initial state \( x(0) = x^0 \), then the Laplace transform \( \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \) of \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) satisfies

\[
\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \hat{S}(\lambda) \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix},
\]

for all those \( \lambda \in \rho(\Sigma) \) for which the Laplace transform converges.

(ii) If instead \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical past time Laplace transformable trajectory of \( \Sigma \) with final state \( x(0) = x^0 \), then the past time Laplace transform \( \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \) of \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) satisfies

\[
\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \hat{S}(\lambda) \begin{bmatrix} -x^0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} -\hat{A}(\lambda) & \hat{B}(\lambda) \\ -\hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix},
\]

for all those \( \lambda \in \rho(\Sigma) \) for which the past time Laplace transform converges.

(iii) Finally, if \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is classical two-sided bilaterally Laplace transformable trajectory of \( \Sigma \) on \( \mathbb{R} \), except for a possible jump discontinuity of size \( x^0 \) at the origin, then the bilateral time Laplace transform \( \begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} \) of \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) satisfies

\[
\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \hat{S}(\lambda) \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix},
\]

for all those \( \lambda \in \rho(\Sigma) \) for which the bilateral Laplace transform converges.

Above, when we say “Laplace transformable trajectory” we mean that the Laplace transforms of \( x, \dot{x}, u, \) and \( y \) all exist (as absolutely converging integrals) for the given value of \( \lambda \). (In part (iii) the distribution derivative of \( x \) contains a point mass of size zero at the origin.)

**Proof of Lemma 5.1.15.** The proof of (i) is the given in the argument leading to (5.1.7). The proofs of the claims (ii)–(iv) are analogous to the proof of (i). \( \square \)

5.1.16. **Theorem.** Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a regular resolvable. Then the following conditions are equivalent:

(i) \( \Sigma \) is bounded.

(ii) The s/s resolvent of \( \Sigma \) is analytic at infinity.

(iii) The i/s/o resolvent matrix of \( \Sigma \) is analytic at infinity.

(iv) The main operator \( A \) of \( \Sigma \) is bounded.

**Proof.** (i) \( \Leftrightarrow \) (iv): See the argument before Definition 2.1.9

(ii) \( \Leftrightarrow \) (iv): See Lemma A.3.1

(iii) \( \Leftrightarrow \) (ii): Trivially (iii) implies (ii). That also the converse is true follows from (5.1.3). \( \square \)

5.1.2. **The interpolation space of a semi-regular resolvable i/s/o node.**

In the special case where \( A \) is the main operator of a semi-regular resolvable i/s/o node we use a special name and a special notation for \( \text{dom}(A) \) equipped with the graph norm.
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5.1.17. DEFINITION. Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-regular resolvable i/s/o node with main operator $A$ (so that, in particular, $A$ is closed). By the interpolation space of $\Sigma$ we mean $\text{dom}(A)$ equipped with the graph topology in Definition A.1.23. We denote the interpolation space of $\Sigma$ by $\mathcal{X}_\bullet$.

According to Lemma A.1.24, $\mathcal{X}_\bullet$ is an $H$-space, and if we fix an arbitrary admissible norm $\| \cdot \|_\mathcal{X}$ in $\mathcal{X}$, then the norm

\[
\|x\|_{\mathcal{X}_\bullet} = (\|x\|_\mathcal{X}^2 + \|Ax\|_\mathcal{X}^2)^{1/2},
\]

is an admissible norm in $\mathcal{X}_\bullet$. Moreover, $\mathcal{X}_\bullet$ is continuously embedded in $\mathcal{X}$. (This embedding is dense if $\text{dom}(A)$ is dense in $\mathcal{X}$.)

5.1.18. LEMMA. Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-regular resolvable i/s/o node with main operator $A$ and observation operator $C$. Then $A \in \mathcal{B}(\mathcal{X}_\bullet; \mathcal{X})$ and $C \in \mathcal{B}(\mathcal{X}_\bullet; \mathcal{Y})$.

Proof. The continuity of $A$ follows from Lemma A.1.24, and the continuity of $C$ from Lemma 5.1.14. □

As the following lemma shows, due to the fact that $A$ has a nonempty resolvent set it is possible to define another set of admissible norms in $\mathcal{X}_\bullet$ in addition to those defined in (5.1.25).

5.1.19. LEMMA. Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a semi-regular resolvable i/s/o node with main operator $A$, and let $\alpha \in \rho(A)$.

(i) The operator $\alpha - A$ is a bicontinuous map from the interpolation space $\mathcal{X}_\bullet$ onto $\mathcal{X}$.

(ii) For each admissible norm $\| \cdot \|_\mathcal{X}$ in $\mathcal{X}$ the norm

\[
\|x\|_{\mathcal{X}_\bullet, \alpha} = \| (\alpha - A)x \|_\mathcal{X}, \quad x \in \text{dom}(A),
\]

is an admissible norm in $\mathcal{X}_\bullet$. In particular, this norm is equivalent to the norm $\| \cdot \|_{\mathcal{X}_\bullet}$ defined in (5.1.25) (for an arbitrary choice of Hilbert space inner product $(\cdot, \cdot)_\mathcal{X}$ in $\mathcal{X}$).

Proof. (i) The operator $(\alpha - A)^{-1}$ is continuous in $\mathcal{X}$ with $\text{rng}((\alpha - A)^{-1}) = \mathcal{X}_\bullet$, and by Lemma A.1.17 it is therefore also continuous as an operator $\mathcal{X} \rightarrow \mathcal{X}_\bullet$. By Lemma A.1.16 the inverse $(\alpha - A)$ of this operator is therefore continuous $\mathcal{X}_\bullet \rightarrow \mathcal{X}$.

(ii) It follows from (i) that $\text{dom}(A)$ is a Hilbert space if we equip it with the norm defined in (5.1.25). By the triangle inequality, for all $x \in \text{dom}(A)$,

\[
\|x\|_{\mathcal{X}_\bullet, \alpha} = \|(\alpha - A)x\|_\mathcal{X} \leq |\alpha| \|x\|_\mathcal{X} + \|Ax\|_\mathcal{X} \leq (1 + |\alpha|) \|x\|_{\mathcal{X}_\bullet}.
\]

Thus $\| \cdot \|_{\mathcal{X}_\bullet, \alpha}$ is weaker than $\| \cdot \|_{\mathcal{X}_\bullet}$. By Lemma A.1.3 they are therefore equivalent. □

5.1.20. REMARK. In the above construction of $\mathcal{X}_\bullet$ the assumption that $A$ is the main operator of a regular resolvable i/s/o node set was used only to guarantee that $\rho(A) \neq \emptyset$, except in the proof of the claim $C \in \mathcal{B}(\mathcal{X}_\bullet; \mathcal{Y})$ in Lemma 5.1.18. Thus, with the exception of that specific claim, all the results in this subsection are valid for an arbitrary operator $A$ in a $H$-space with a nonempty resolvent set.
5.1.3. The extrapolation space of a regular resolvable i/s/o node. We
next turn to the definition of the extrapolation space $\mathcal{X}_0$ of a regular resolvable i/s/o
node $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, which is a larger $H$-space in which the original state space $\mathcal{X}$
is continuously and densely embedded. To construct this space we start by defining
a new norm on $\mathcal{X}$, namely the norm
\begin{equation}
\|x\|_{\alpha, \mu} = \|(\alpha - A)^{-1}x\|_{\mathcal{X}}, \quad x \in \mathcal{X},
\end{equation}
where $\alpha$ is an arbitrary point in $\rho(A)$ and $\|\cdot\|_{\mathcal{X}}$ is an arbitrary admissible norm in $\mathcal{X}$. Note that this norm is not an admissible norm for $\mathcal{X}$, unless $\alpha - A$ is bounded, i.e., $A$ itself is bounded (with dom($A$) = $\mathcal{X}$).

5.1.21. Lemma. The different norms defined in (5.1.27) that one gets by varying $\alpha \in \rho(A)$ and varying $\|\cdot\|_{\mathcal{X}}$ over all admissible Hilbert space inner products in $\mathcal{X}$
are equivalent to each other.

Proof. It is easy to see that different choices of the admissible norm $\|\cdot\|_{\mathcal{X}}$ lead to equivalent norms in (5.1.27) (see Definition A.1.1). We claim that the same is true for different choices of $\alpha$ as well. By the resolvent identity (cf. (5.2.4) below), if $\alpha, \beta \in \rho(A)$, then
\begin{equation}
(\alpha - A)^{-1} = (\mu - A)^{-1} + (\mu - \lambda)(\alpha - A)^{-1}(\mu - A)^{-1},
\end{equation}
and therefore, for all $x \in \mathcal{X}$,
\begin{align*}
\|(\alpha - A)^{-1}x\|_{\mathcal{X}} &= \|(\mu - A)^{-1}x + (\mu - \lambda)(\alpha - A)^{-1}(\mu - A)^{-1}x\|_{\mathcal{X}} \\
&\leq (1 + |\alpha - \mu|\|\alpha^{-1}\|_{\mathcal{X}})\|(\mu - A)^{-1}x\|_{\mathcal{X}},
\end{align*}
where $\|\alpha^{-1}\|_{\mathcal{X}}$ is the operator norm of $\alpha^{-1}$ with respect to the norm $\|\cdot\|_{\mathcal{X}}$ in $\mathcal{X}$. This shows that the norm $\|\cdot\|_{\alpha, \mu}$ is weaker than the norm $\|\cdot\|_{\alpha, \mu}$. By interchanging $\alpha$ and $\mu$ with each other we find that $\|\cdot\|_{\alpha, \mu}$ is weaker than $\|\cdot\|_{\alpha, \mu}$. Thus, $\|\cdot\|_{\alpha, \mu}$ and $\|\cdot\|_{\alpha, \mu}$ are equivalent. \qed

5.1.22. Definition. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular resolvable i/s/o node
with main operator $A$ (so that, in particular, $\rho(A) \neq \emptyset$ and dom($A$) is dense in $\mathcal{X}$).
By the extrapolation space of $\Sigma$ we mean the completion of $\mathcal{X}$ with respect to the
topology induced by the norm $\|\cdot\|_{\alpha, \mu}$ in (5.1.27) for some $\alpha \in \rho(A)$ and some
admissible norm $\|\cdot\|_{\mathcal{X}}$ in $\mathcal{X}$. (By Lemma 5.1.21, this completion does not depend
on the particular choice of $\alpha$ and $\|\cdot\|_{\mathcal{X}}$ in (5.1.27).)

As the following lemma shows, it is possible to extend $A$ into a bounded linear
operator $\mathcal{X} \to \mathcal{X}_0$.

5.1.23. Lemma. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a regular resolvable i/s/o node.
\begin{enumerate}
\item[(i)] The main operator $A$ of $\Sigma$ has a unique extension $A_0$ to a bounded linear
operator $\mathcal{X} \to \mathcal{X}_0$, where $\mathcal{X}_0 \supset \mathcal{X}$ is the extrapolation space of $\Sigma$.
\item[(ii)] For each $\alpha \in \rho(A)$ the operator $\alpha - A_0$ is a continuous bijection from $\mathcal{X}$
onto $\mathcal{X}_0$.
\item[(iii)] $A$ and $A_0$ are similar to each other. More precisely, for each $\alpha \in \rho(A) = \rho(A_0)$ we have
\begin{equation}
A_0 = (\alpha - A_0)A(\alpha - A_0)^{-1}, \quad A = (\alpha - A_0)^{-1}A_0(\alpha - A_0).
\end{equation}
\end{enumerate}
In particular, $\rho(A_0) = \rho(A)$.
(iv) If we interpret $A_o$ as an operator in $X_o$ with domain $\text{dom}(A_o)$, then $A_o$ is a closed operator with a nonempty resolvent set, and the graph topology induced by $A_o$ on its domain $X$ is equivalent to the original topology of $X$.

**Proof.** (i) We start by re-interpreting $A$ as an operator from $X$ into $X_o$ with domain $\text{dom}(A) \subset X_o$. This operator will not be closed (unless $A$ is bounded), but it follows from (5.1.27) that this operator is continuous (on its domain). Let $A_o$ be the closure of this operator. Then $A_o \in B(X; X_o)$.

(ii) Let $\alpha \in \rho(A)$. For all $x \in X$,

$$
\|(\alpha - A)^{-1}x\|_{o, \alpha} = \|(\alpha - A)^{-2}x\|_X \leq \|(\alpha - A)^{-1}\| \|(\alpha - A)^{-1}x\|_X = \|(\alpha - A)^{-1}\| \|x\|_{o, \alpha},
$$

and hence the operator $(\alpha - A)^{-1}$ can be reinterpreted as a continuous operator in $X_o$ with domain $X \subset X_o$. Denote the closure of this operator by $B \in B(X, X_o)$. This operator maps $X_o$ continuously into $X$ since $\|(\alpha - A)^{-1}x\|_X = \|x\|_{o, \alpha}$, and hence also $\|Bx\|_X = \|x\|_{o, \alpha}$.

(iii) By taking the closure in the identity

$$(\alpha - A)^{-1}Ax = A(\alpha - A)^{-1}x, \quad x \in \text{dom}(A),$$

we get the identity

(5.1.29) \hspace{1cm} (\alpha - A_o)^{-1}A_o = A(\alpha - A)^{-1}.

From here we get the first identity in (5.1.28) by multiplying (5.1.29) to the left by $(\alpha - A_o)^{-1}$ and observing that $A(\alpha - A)^{-1} = A(\alpha - A_o)^{-1}$ (because by the definition of the product of two operators, $\text{dom}(A(\alpha - A)^{-1}) = (\alpha - A_o)^{-1}\text{dom}(A) = X$ and $(\alpha - A_o)^{-1}\text{dom}(A) = (\alpha - A)^{-1}$). To get the second identity in (5.1.28) we multiply (5.1.29) to the right by $(\alpha - A)$ and observe that $A_o(\alpha - A) = A_o(\alpha - A_o)$ (since $\text{dom}(A_o(\alpha - A_o)) = (\alpha - A_o)^{-1}\text{dom}(A_o) = \text{dom}(A)$ and $(\alpha - A_o)|_{\text{dom}(A)} = (\alpha - A)$).

(iv) This is true because $A$ is a closed operator in $X$ and, according to (iii), $A_o$ is similar to $A$. □

5.1.24. REMARK. In the above construction of $X_o$, the assumption that $A$ is the main operator of a regular resolvable i/s/o node was used only to guarantee that $\rho(A) \neq \emptyset$ and that $\text{dom}(A)$ is dense in $X$. Thus, all the results in this subsection are valid for an arbitrary operator $A$ in a $H$-space with a nonempty resolvent set and a dense domain.

5.1.4. The control operator of a regular resolvable i/s/o node. With the help of the extrapolation space $X_o$ introduced above it is possible to introduce the notion of a control operator of a regular i/s/o node with a nonempty resolvent set.

5.1.25. Theorem. Let $\Sigma = (S; X, U, Y)$ be a regular resolvable i/s/o node.

(i) The formula

(5.1.30) \hspace{1cm} Bu := [1_X \ 0] S [\vec{z}] - A_o x, \quad [\vec{z}] \in \text{dom}(S)

defines an operator $B \in B(U, X_o)$ (in particular, the right-hand side depends only on $u$ and not on the particular value of $x$).
(ii) The i/s resolvent $\widehat{B}$ of $\Sigma$ satisfies
\begin{equation}
\widehat{B}(\lambda) = (\lambda - A_o)^{-1} B, \quad \lambda \in \rho(A).
\end{equation}

(iii) The operator $[A_o \ B]$ is the unique extension of $[1_X \ 0]$ to an operator in $B([\lambda \ U]; X_o)$ and $\text{dom}(S) = \{ [\begin{smallmatrix} u \\ \gamma \end{smallmatrix}] \in [\lambda \ U] \mid A_o x + Bu \in X \}$. 

**Proof.** (i)–(ii) The right-hand side of (5.1.30) does not depend on $x$ since $[1_X \ 0] S [\gamma] - A_o x = Ax - Ax = 0$ if $[\gamma] \in \text{dom}(S)$, i.e., $x \in \text{dom}(A)$.

To show that $B$ is a linear operator in $B(U; X_o)$ it suffices to prove (5.1.31), since this implies that $(\lambda - A_o)^{-1} B \in B(X)$ for every $\lambda \in \rho(A)$. Multiplying the right-hand side of (5.1.30) by $(\lambda - A_o)^{-1}$ we get
\begin{align*}
(\lambda - A_o)^{-1} ([1_X \ 0] S [\gamma] - A_o x) &= (\lambda - A)^{-1} [1_X \ 0] S [\gamma] - (\lambda - A_o)^{-1} A_o x \\
&= (\lambda - A)^{-1} [1_X \ 0] S [\gamma] - A(\lambda - A)^{-1} x.
\end{align*}

Recall that $(\lambda - A)^{-1} = \widehat{A}(\lambda)$. By (5.1.10),
\begin{equation}
\begin{bmatrix} \lambda - A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_X & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1_X & 0 \\ 0 & 1 \end{bmatrix} S \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix}, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \text{dom}(S).
\end{equation}

From here we can solve for $(\lambda - A)^{-1} [1_X \ 0] S [\gamma]$ to get
\begin{equation}
(\lambda - A)^{-1} [1_X \ 0] S [\gamma] = \lambda(\lambda - A)^{-1} x - x + \widehat{A}(\lambda) u = A(\lambda - A)^{-1} x + \widehat{A}(\lambda) u.
\end{equation}

Thus we get
\begin{equation}
(\lambda - A_o)^{-1} ([1_X \ 0] S [\gamma] - A_o x) = \widehat{B}(\lambda) u.
\end{equation}

This shows that the right-hand side of (5.1.30) multiplied by $(\lambda - A_o)^{-1}$ is equal to $\widehat{B}(\lambda) u$, where $\widehat{B}(\lambda) \in B(\lambda; X)$. Thus, $B$ defined in (5.1.30) is a linear operator in $B(U; X_o)$, and (5.1.31) holds for all $\lambda \in \rho(A)$. This finishes the proof of (i) and (ii).

(iii) Clearly, it follows from (5.1.30) that $[A_o \ B] |_{\text{dom}(S)} = [1_X \ 0] S$. Since $\text{dom}(S)$ is dense in $[\lambda \ U]$, this means that $[A_o \ B]$ is the unique extension of $[1_X \ 0] S$ to an operator in $B([\lambda \ U]; X_o)$.

We still need to show that $\text{dom}(S) = \{ [\begin{smallmatrix} u \\ \gamma \end{smallmatrix}] \in [\lambda \ U] \mid A_o x + Bu \in X \}$. By construction, $A_o x + Bu = [1_X \ 0] S [\gamma] \in X$ for all $[\gamma] \in \text{dom}(S)$, and hence $\text{dom}(S) \subset \{ [\begin{smallmatrix} u \\ \gamma \end{smallmatrix}] \in [\lambda \ U] \mid A_o x + Bu \in X \}$. Conversely, suppose that $[\begin{smallmatrix} u \\ \gamma \end{smallmatrix}] \in [\lambda \ U]$ and that $A_o x + Bu \in X$. Then, by condition (iv) in Theorem 5.1.13, there exists $x^0 \in X$ such that $[\begin{smallmatrix} u \\ x^0 \end{smallmatrix}] \in \text{dom}(S)$, hence $A_o x^0 + Bu \in X$. This implies that $A_o (x - x^0) \in X$, hence $[\begin{smallmatrix} x - x^0 \\ u \end{smallmatrix}] \in \text{dom}(S)$. This gives $[\begin{smallmatrix} u \\ \gamma \end{smallmatrix}] \in \text{dom}(S)$. Thus $\text{dom}(S) = \{ [\begin{smallmatrix} u \\ \gamma \end{smallmatrix}] \in [\lambda \ U] \mid A_o x + Bu \in X \}$.

5.1.26. Definition. The operator $B$ in (5.1.30) is called the control operator of the regular resolvable i/s/o node $\Sigma$.

5.1.27. Lemma. The system operator $S$ of a regular resolvable i/s/o node $\Sigma = (S; X, U, Y)$ with $\rho(\Sigma) \neq \emptyset$ is determined uniquely by its main operator $A$, control operator $B$, observation operator $C$, and i/o resolvent $\widehat{D}(\lambda)$ evaluated at some point $\lambda \in \rho(A)$. One way to recover $S$ from $A$, $B$, $C$, and $\widehat{D}(\lambda)$ is to first define the i/s/o resolvent matrix $\widehat{S}$ of $\Sigma$ at the point $\lambda$ via the formula
\begin{equation}
\widehat{S}(\lambda) = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A_o)^{-1} B \\ C(\lambda - A)^{-1} & \widehat{D}(\lambda) \end{bmatrix}, \quad \lambda \in \rho(A),
\end{equation}

where $\lambda$ is the resolvent of $A$.
and then recover $S$ from $\hat{S}(\lambda)$ as described in Theorem 5.1.12. A more explicit way of recovering $S$ from $A$, $B$, $C$, and $\hat{S}(\lambda)$ is to define
\[
\text{dom}(S) := \text{dom} \left( \left[ \begin{array}{c} 1 \cr \lambda \end{array} \right] S \right) = \left\{ \left[ \begin{array}{c} x \\ u \end{array} \right] \in [\mathcal{X} / \mathcal{U}] : A_0 x + Bu \in \mathcal{X} \right\},
\]
\[
\left[ \begin{array}{c} 1 \\ \lambda \end{array} \right] S := \left[ \begin{array}{c} A_0 \\ B \end{array} \right] |_{\text{dom}(S)},
\]
\[
\left[ \begin{array}{c} 0 \\ 1 \end{array} \right] S \left[ \begin{array}{c} x \\ u \end{array} \right] := C(x - (\lambda - A_0)^{-1}Bu) + \hat{S}(\lambda)u,
\]
\[
\left[ \begin{array}{c} x \\ u \end{array} \right] \in \text{dom}(S).
\]

**Proof.** That the first method described above to recover $S$ is valid follows from Theorems 5.1.12, 5.1.13, and 5.1.25. By Theorem 5.1.25, $\text{dom}(S)$ and $\left[ \begin{array}{c} 1 \\ \lambda \end{array} \right] S$ are the ones given in (5.1.33). The formula for $\left[ \begin{array}{c} 0 \\ 1 \end{array} \right] S$ can be derived from (5.1.15) as follows: For all $\left[ \begin{array}{c} x \\ u \end{array} \right] \in \text{dom}(S)$ we have
\[
\left[ \begin{array}{c} 0 \\ 1 \end{array} \right] S \left[ \begin{array}{c} x \\ u \end{array} \right] = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] S \left[ \begin{array}{c} x - \hat{S}(\lambda)u \\ 0 \end{array} \right] + \left[ \begin{array}{c} \hat{S}(\lambda)u \\ u \end{array} \right] = C(x - \hat{S}(\lambda)u) + \hat{S}(\lambda)u \quad \square
\]

Above we started with a regular resolvable i/s/o node $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and then extracted the main operator $A$, the control operator $B$, the observation operator $C$, and the transfer function $\hat{S}$. Conversely, it is possible to start with $A$, $B$, $C$, and the transfer function $\hat{S}$ evaluated at one point in $\rho(A)$, and from this set of data construct a regular resolvable i/s/o node $\Sigma$ as follows.

5.1.28. **Lemma.** Let $A$ be a closed densely defined operator on a Hilbert space $X$ satisfying $\rho(A) \neq \emptyset$ with interpolation space $\mathcal{X}_\bullet$, extrapolation space $\mathcal{X}_\circ$, and extension $A_\circ : \mathcal{X} \to \mathcal{X}_\circ$ of $A$. Let $\alpha \in \rho(A)$, and let $B \in \mathcal{B}(\mathcal{U}; \mathcal{X}_\circ)$, $C \in \mathcal{B}(\mathcal{X}_\bullet; \mathcal{Y})$, and $D \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are two more Hilbert spaces. Define the operator $S$ from $\text{dom}(S) \subset \mathcal{X} \cap \mathcal{Y}$ to $\mathcal{Y}$ by
\[
\text{dom}(S) := \left\{ \left[ \begin{array}{c} x \\ u \end{array} \right] \in [\mathcal{X} / \mathcal{U}] : A_0 x + Bu \in \mathcal{X} \right\},
\]
\[
\left[ \begin{array}{c} 1 \\ \lambda \end{array} \right] S := \left[ \begin{array}{c} A_0 \\ B \end{array} \right] |_{\text{dom}(S)},
\]
\[
\left[ \begin{array}{c} 0 \\ 1 \end{array} \right] S \left[ \begin{array}{c} x \\ u \end{array} \right] := C(x - (\alpha - A_0)^{-1}Bu) + Du,
\]
\[
\left[ \begin{array}{c} x \\ u \end{array} \right] \in \text{dom}(S).
\]

Then $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a regular resolvable i/s/o node with $\rho(\Sigma) = \rho(A)$. The main operator of $\Sigma$ is $A$, the observation operator is $B$, the control operator $C$, and the i/o resolvent $\hat{S}$ satisfies $\hat{S}(\alpha) = D$.

We remark that
\[
x - (\alpha - A_0)^{-1}Bu = (\alpha - A_0)^{-1}(\alpha x - (A_0 x + Bu)), \quad \left[ \begin{array}{c} x \end{array} \right] \in [\mathcal{X} / \mathcal{U}],
\]
and hence $x - (\alpha - A_0)^{-1}Bu \in \text{dom}(A)$ for all $\left[ \begin{array}{c} x \end{array} \right] \in [\mathcal{X} / \mathcal{U}]$, so that the term $C(x - (\alpha - A_0)^{-1}Bu)$ in (5.1.34) is well-defined.

**Proof of Lemma 5.1.28** The proof of this lemmas given below is based on Theorem 5.1.13. We shall postpone the proof that condition (i) in that theorem holds, i.e., that $S$ is closed, to the very end of this proof, and instead start by checking conditions (ii)–(iv).

Condition (ii) in Theorem 5.1.13, which says that $\left[ \begin{array}{c} 1 \\ 0 \end{array} \right] S$ is closed, follows from Lemma A.1.17. Condition (iii) on the main operator $A$ follows directly from the assumption (it is easy to see that the operator $A$ given in the statement of the
lemma is the main operator of $S$). To prove that also condition (iv) holds we take some arbitrary $u \in \mathcal{U}$, and define $x = (\alpha - A_o)^{-1} Bu$. Then

$$[A_o \quad B] \begin{bmatrix} x \\ u \end{bmatrix} = A_o (\alpha - A_o)^{-1} Bu + Bu = \alpha (\alpha - A_o)^{-1} Bu \in \mathcal{X}.$$ 

This means that $[x_n] \in \text{dom}(S)$, and it shows that condition (iv) in Theorem 5.1.13 is also satisfied.

It remains to prove that $S$ is closed. Let $[\tilde{x}_n] \in \text{dom}(S)$, $[\tilde{z}_n] \in \text{dom}(\tilde{S})$, and suppose that also $[\tilde{y}_n] := S [\tilde{x}_n] \rightarrow \tilde{z}$ in $[\tilde{S}]$. Since $[1 \quad 0] S$ is closed this implies that $[\tilde{z}] \in \text{dom}(S)$ and $z = [1 \quad 0] S [\tilde{z}]$. Since $z_n = A_o x_n + Bu_n \rightarrow A_o x + Bu = z$ in $\mathcal{X}$, it follows from (5.1.35) that

$$x_n - (\alpha - A_o)^{-1} Bu_n \rightarrow x - (\alpha - A_o)^{-1} Bu \text{ in } \mathcal{X},$$

and by the continuity assumptions on $C$ and $D$,

$$\begin{align*}
0 \quad 1_y \quad S [\tilde{x}_n] &= C (x_n - (\alpha - A_o)^{-1} Bu_n) + Du_n \\
&\rightarrow C (x - (\alpha - A_o)^{-1} Bu) + Du \\
&= [0 \quad 1_y] S [\tilde{z}].
\end{align*}$$

This proves that $S$ is closed.

We already mentioned above that the operator $A$ given in the statement of the lemma is equal to the main operator of $S$. It is also easy to see that $B$ is the control operator and $C$ the observation operator. Finally, that the i/o resolvent satisfies $\mathcal{D}(\alpha) = D$ follows from (5.1.33) and (5.1.34). \hfill \square

Theorem 5.1.25 makes it possible to rewrite the equation (2.1.1) for classical trajectories or a regular resolvable i/s/o system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ in the form

$$\begin{align*}
\dot{x}(t) &= A_o x(t) + Bu(t), \\
y(t) &= [0 \quad 1_y] S [z(t)]_u(t), \quad t \in I, \\
[z(t)]_u(t) &\in \text{dom}(S),
\end{align*}$$

(5.1.36) 

where $A_o$ is the extension of the main operator $A$ to an operator in $\mathcal{B}(\mathcal{X}; \mathcal{X}_o)$ and $B$ is the control operator of $\Sigma$. Here $\dot{x}(t)$ is the derivative of $x$ at the point $t$ computed in the state space $\mathcal{X}$, so it belongs to $\mathcal{X}$. Also the right-hand side of the first equation in (5.1.36) belongs to $\mathcal{X}$ since $A_o x(t) + Bu(t) = [1 \quad 0] S [z(t)]_u(t)$ for all $[z(t)]_u(t) \in \text{dom}(S)$. However, each of two two terms $A_o x(t)$ and $Bu(t)$ in this equation are allowed to lie in the larger extrapolation space $\mathcal{X}_o$ (only their sum must belong to $\mathcal{X}$). When $u(t) = 0$ the system (5.1.36) can be rewritten in the more familiar form

$$\begin{align*}
\dot{x}(t) &= Ax(t), \\
y(t) &= Cx(t), \quad t \in I, \\
x(t) &\in \text{dom}(A),
\end{align*}$$

(5.1.37)

5.1.29. Example (cf. Example 2.4.12). The main operator $A$ of the i/s/o node $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ in Example 2.4.12 with $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{U} = \mathcal{Y} = \{0\}$ is the system operator of a $C_0$ semigroup, so it has a nonempty resolvent set. More precisely, according to [Staffans 2005 Examples 3.3.1 and 3.3.2], the resolvent set
Example 5.1.29. Therefore by Lemma 5.2.28, \[ \Sigma \] has a nonempty resolvent set, and \( \rho(\Sigma_{i/s/o}) = \rho(A) = \mathbb{C}^+ \). Since \( \mathcal{U} = \mathcal{Y} = \{0\} \) the i/s/o resolvent matrix of \( \Sigma \) is \( \hat{\mathbf{S}}(\lambda) = \begin{bmatrix} (\lambda-A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) for \( \lambda \in \mathbb{C}^+ \).

5.1.30. Example (cf. Example 2.4.13). Also the main operator \( A \) of the i/s/o node \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) in Example 2.4.13 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{U} = \mathcal{Y} = \{0\} \) is the system operator of a \( C_0 \) semigroup, so it has a nonempty resolvent set. More precisely, as explained in the analysis of Example 2.4.14 with \( (\Sigma_{i/s/o}) \) is the open right-half plane \( \mathbb{C}^+ \). Since \( \mathcal{U} = \mathcal{Y} = \{0\} \) the i/s/o resolvent matrix of \( \Sigma \) is \( \hat{\mathbf{S}}(\lambda) = \begin{bmatrix} (\lambda-A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) for \( \lambda \in \mathbb{C}^+ \).

Example 5.1.30. Therefore by Lemma 5.2.28, \[ \Sigma \] has a nonempty resolvent set, and \( \rho(\Sigma_{i/s/o}) = \rho(A) = \mathbb{C}^+ \). Since \( \mathcal{U} = \mathcal{Y} = \{0\} \) the i/s/o resolvent matrix of \( \Sigma \) is \( \hat{\mathbf{S}}(\lambda) = \begin{bmatrix} (\lambda-A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) for \( \lambda \in \mathbb{C}^+ \).

5.1.31. Example (cf. Example 2.4.14). The i/s/o node \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) in Example 2.4.14 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{U} = \mathcal{Y} = \{0\} \) is the time reflection of Example 5.1.30. Therefore by Lemma 5.2.28, \( \Sigma_{i/s/o} \) has a nonempty resolvent set, \( \rho(\Sigma_{i/s/o}) = \mathbb{C}^- \), and the resolvent \( (\lambda-A)^{-1} \) is the operator

\[
(\lambda - A)^{-1} \varphi = \left( \xi \mapsto \int_{\xi}^{\infty} e^{\lambda(\xi-\zeta)} \varphi(\zeta) \, d\zeta \right), \quad \varphi \in L^2(\mathbb{R}^+), \quad \lambda \in \mathbb{C}^-.
\]

By Theorem 5.1.13, \( \Sigma_{i/s/o} \) has a nonempty resolvent set, and \( \rho(\Sigma_{i/s/o}) = \rho(A) = \mathbb{C}^- \). Since \( \mathcal{U} = \mathcal{Y} = \{0\} \) the i/s/o resolvent matrix of \( \Sigma \) is \( \hat{\mathbf{S}}(\lambda) = \begin{bmatrix} (\lambda-A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) for \( \lambda \in \mathbb{C}^- \).

5.1.32. Example (cf. Example 2.4.15). The i/s/o node \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) in Example 2.4.15 with \( \mathcal{X} = L^2(\mathbb{R}^+) \) and \( \mathcal{U} = \mathcal{Y} = \{0\} \) is the time reflection of Example 5.1.29. Therefore by Lemma 5.2.28, \( \Sigma_{i/s/o} \) has a nonempty resolvent set, \( \rho(\Sigma_{i/s/o}) = \mathbb{C}^- \), and the resolvent \( (\lambda-A)^{-1} \) is the operator

\[
(\lambda - A)^{-1} \varphi = \left( \xi \mapsto -\int_{\xi}^{\infty} e^{\lambda(\xi-\zeta)} \varphi(\zeta) \, d\zeta \right), \quad \varphi \in L^2(\mathbb{R}^+), \quad \lambda \in \mathbb{C}^-.
\]

By Theorem 5.1.13, \( \Sigma_{i/s/o} \) has a nonempty resolvent set, and \( \rho(\Sigma_{i/s/o}) = \rho(A) = \mathbb{C}^- \). Since \( \mathcal{U} = \mathcal{Y} = \{0\} \) the i/s/o resolvent matrix of \( \Sigma \) is \( \hat{\mathbf{S}}(\lambda) = \begin{bmatrix} (\lambda-A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) for \( \lambda \in \mathbb{C}^- \).
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5.1.33. Example (cf. Example 2.4.16). The i/s/o node $\Sigma_{\text{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ in Example 2.4.16 with $\mathcal{X} = L^2(\mathbb{R}^+)$, $\mathcal{U} = \{0\}$ and $\mathcal{Y} = \mathbb{C}$ has the same main operator $A$ as Example 5.1.29 with $\rho(A) = \mathbb{C}^+$. By Theorem 5.1.13 $\Sigma_{\text{i/s/o}}$ has a nonempty resolvent set, and $\rho(\Sigma_{\text{i/s/o}}) = \rho(A) = \mathbb{C}^+$. Thus, the s/s resolvent $(\lambda - A)^{-1}$ is given by (5.1.38). Since $\mathcal{Y} = \mathbb{C}$ this system also has a nontrivial i/s/o resolvent, which is given by

$$\hat{\mathcal{C}}(\lambda)\varphi = \int_0^\infty e^{-\lambda \zeta} \varphi(\zeta) \, d\zeta, \quad \varphi \in L^2(\mathbb{R}^+), \quad \lambda \in \mathbb{C}^+.$$  

Note that the right-hand side is equal to the Laplace transform of $\varphi$ evaluated at $\lambda \in \mathbb{C}^+$, and that the observation operator is point evaluation at zero. Since $\mathcal{U} = \{0\}$ the i/s/o resolvent matrix of $\Sigma$ is $\hat{\mathcal{S}}(\lambda) = \left[ \begin{array}{cc} \hat{\mathcal{A}}(\lambda) & 0 \\ 0 & \hat{\mathcal{E}}(\lambda) \end{array} \right]$ for $\lambda \in \mathbb{C}^+$ where $\hat{\mathcal{A}}(\lambda) = (\lambda - A)^{-1}$.

5.1.34. Example (cf. Example 2.4.17). The i/s/o node $\Sigma_{\text{i/s/o}} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ in Example 2.4.17 with $\mathcal{X} = L^2(\mathbb{R}^+)$, $\mathcal{U} = \mathbb{C}$ and $\mathcal{Y} = \{0\}$ has the same main operator $A$ as Example 5.1.30 with $\rho(A) = \mathbb{C}^+$. By Theorem 5.1.13 $\Sigma_{\text{i/s/o}}$ has a nonempty resolvent set, and $\rho(\Sigma_{\text{i/s/o}}) = \rho(A) = \mathbb{C}^+$. Thus, the s/o resolvent $(\lambda - A)^{-1}$ is given by (5.1.39). Since $\mathcal{Y} = \mathbb{C}$ this system also has a nontrivial i/s resolvent, which is given by

$$\hat{\mathcal{B}}(\lambda)u = (\xi \mapsto e^{-\lambda \xi} u), \quad u \in \mathbb{C}, \quad \lambda \in \mathbb{C}^+.$$  

Note that the right-hand side is kernel of the Laplace transform, and that formally the control operator is the Dirac delta at zero (or the adjoint of the point evaluation operator at zero). Since $\mathcal{Y} = \{0\}$ the i/s/o resolvent matrix of $\Sigma$ is $\hat{\mathcal{S}}(\lambda) = \left[ \begin{array}{cc} \hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\ 0 & 0 \end{array} \right]$ for $\lambda \in \mathbb{C}^+$ where $\hat{\mathcal{A}}(\lambda) = (\lambda - A)^{-1}$.

The purpose of the following example is to illustrate the necessary and sufficient conditions given in Theorem 5.1.13 and Lemmas 5.1.27 5.1.28 for an operator $S$ to be the system operator of a regular i/s/o node with a nonempty spectrum.

5.1.35. Example. Let $\mathcal{X}$ be an $H$-space, and let $T$ be a closed unbounded operator in $\mathcal{X}$ with a dense domain $\text{dom}(T) \neq \mathcal{X}$ and a nonempty resolvent set, and define the operators $S_i : [X_0] \to [X_0]$, $i = 1, 2, \ldots, 16$, by

$$S_1 = \left[ \begin{array}{c} T \\ 0 \end{array} \right], \quad S_2 = \left[ \begin{array}{c} 0 \\ T \end{array} \right], \quad S_3 = \left[ \begin{array}{c} T \\ 0 \end{array} \right],$$

$$S_4 = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \quad S_5 = \left[ \begin{array}{c} 0 \\ T \end{array} \right], \quad S_6 = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right],$$

$$S_7 = \left[ \begin{array}{c} T \\ 0 \end{array} \right], \quad S_8 = \left[ \begin{array}{c} 0 \\ T \end{array} \right], \quad S_9 = \left[ \begin{array}{c} T \\ T \end{array} \right],$$

$$S_{10} = \left[ \begin{array}{c} T \\ 0 \end{array} \right], \quad S_{11} = \left[ \begin{array}{c} 0 \\ T \end{array} \right], \quad S_{12} = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right],$$

$$S_{13} = \left[ \begin{array}{c} T \\ T \end{array} \right], \quad S_{14} = \left[ \begin{array}{c} 0 \\ T \end{array} \right], \quad S_{15} = \left[ \begin{array}{c} T \\ 0 \end{array} \right],$$

$$S_{16} = \left[ \begin{array}{c} 0 \\ T \end{array} \right].$$

with their natural domains

$$\text{dom}(S_1) = \text{dom}(S_2) = \text{dom}(S_2) = \left[ \begin{array}{c} \text{dom}(T) \\ X \end{array} \right],$$

$$\text{dom}(S_4) = \text{dom}(S_5) = \text{dom}(S_6) = \left[ \begin{array}{c} X \\ \text{dom}(T) \end{array} \right],$$

$$\text{dom}(S_7) = \text{dom}(S_8) = \text{dom}(S_9) = \{ [z_1, z_2] \in [X] | z_1 + z_2 \in \text{dom}(T) \},$$

$$\text{dom}(S_i) = \left[ \begin{array}{c} \text{dom}(T) \\ \text{dom}(T) \end{array} \right], \quad 10 \leq i \leq 16.$$
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(In the above formulas $S_7$ should be interpreted as the operator which maps $[\begin{array}{c} x_1 \\ x_2 \end{array}]$ into $[\begin{array}{c} T(x_1+x_2) \\ 0 \end{array}]$, and $S_8$ and $S_9$ are interpreted analogously.) Then $\Sigma_{i/s/o} = (S_j, X', X, X')$ is a regular i/s/o node for each $j = 1, 2, \ldots, 17$. Out of these regular i/s/o nodes only the systems $\Sigma_1^{i/s/o}$, $\Sigma_3^{i/s/o}$, $\Sigma_7^{i/s/o}$, and $\Sigma_9^{i/s/o}$ have nonempty resolvent sets. These four nodes all have the same main operator $T$, and $\rho(\Sigma_1^{i/s/o}) = \rho(\Sigma_3^{i/s/o}) = \rho(\Sigma_7^{i/s/o}) = \rho(\Sigma_9^{i/s/o}) = \rho(T)$, and their i/s/o resolvent matrices are given by

$$
\begin{align*}
\hat{S}_{i/s/o}^1(\lambda) &= \begin{bmatrix} (\lambda-T)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\
\hat{S}_{i/s/o}^3(\lambda) &= \begin{bmatrix} (\lambda-T)^{-1} & 0 \\ T(\lambda-T)^{-1} & 0 \end{bmatrix}, \\
\hat{S}_{i/s/o}^7(\lambda) &= \begin{bmatrix} (\lambda-T)^{-1} & T(\lambda-T)^{-1} \\ 0 & 0 \end{bmatrix}, \\
\hat{S}_{i/s/o}^9(\lambda) &= \begin{bmatrix} (\lambda-T)^{-1} & T(\lambda-T)^{-1} \\ T(\lambda-T)^{-1} & \lambda T(\lambda-T)^{-1} \end{bmatrix},
\end{align*}
$$

(5.1.46)

respectively, for $\lambda \in \rho(T)$.

Indeed, it is not difficult to show that all the operators $S_j$ above are closed and densely defined. To see which of the i/s/o nodes $\Sigma_{i/s/o}^j$ above have a nonempty resolvent set it suffices to check which of them satisfy conditions (ii)–(iv) in Theorem 5.1.13 (of course, all of the examples satisfy condition (i)). As we mentioned above, all of conditions (ii)–(iv) hold for $j = 1, 3, 7, \text{ and } 9$. In all the other cases at least one of the conditions (ii)–(iv) is violated. We leave it to the reader to check that the i/s/o resolvent matrices for $j = 1, 3, 7, \text{ and } 9$ are given by (5.1.46), and that the following list is correct:

(i) Condition (ii) in Theorem 5.1.13 holds for $j = 1, 3, 4, 6, 7, \text{ and } 9$;

(ii) Condition (iii) in Theorem 5.1.13 holds for $j = 1, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, \text{ and } 15$;

(iii) Condition (iv) in Theorem 5.1.13 holds for $j = 1, 2, 3, 7, 8, \text{ and } 9$.

Note, in particular that $S_4$ and $S_6$ satisfy all conditions but (iv), that $S_{11}$ and $S_{16}$ do not satisfy any of (ii)–(iv), and that the remaining examples $S_2, S_5, S_8, S_{10}, S_{12}, S_{13}, S_{14}, S_{15}$ violate two of the three conditions (ii)–(iv).

In Example 5.3.5 we shall return to the above example, and show that in spite of the fact that most of the above i/s/o nodes $\Sigma_{i/s/o}^j$ have an empty resolvent set, it is actually true that all but two of the s/s systems induced by these i/s/o systems do have a nonempty resolvent set.
5.2. General Resolvable Input/State/Output Nodes (Jan 02, 2016)

In this section we extend the notion of the i/s/o resolvent matrix of a semi-regular i/s/o node and define what we mean by the i/s/o resolvent matrix of an arbitrary (closed) i/s/o node. We then show that the i/s/o resolvent matrix satisfies a generalization of the standard resolvent identity, the so called i/s/o resolvent identity.

Before defining what we mean by an i/s/o resolvent matrix of a general i/s/o node we need to first recall the notion of the resolvent of a closed linear multi-valued operator. See Appendix A.4 for a short review of the basic properties of multi-valued operators.

5.2.1. Operator resolvents and pseudo-resolvents.

5.2.1. Definition. Let \( A \in \mathcal{ML}(\mathcal{X}) \) be closed with domain \( \supset \text{dom} (A) \).

(i) A point \( \lambda \in \mathbb{C} \) belongs to the resolvent set of \( A \) if \( \ker (\lambda - A) = \{0\} \), \( \text{rng}(\lambda - A) = \mathcal{X} \) and \( (\lambda - A)^{-1} \in \mathcal{B}(\mathcal{X}) \). This set of points is denoted by \( \rho(A) \).

(ii) The spectrum \( \sigma(A) \) of \( A \) is the complement of the resolvent set of \( A \), and it is denoted by \( \sigma(A) \).

(iii) For \( \lambda \in \rho(A) \) the operator \( (\lambda - A)^{-1} \) is called the resolvent of \( A \) (evaluated at \( \lambda \)).

The assumption that \( A \) is closed in Definition 5.2.1 is redundant in the sense that \( A \) cannot have a nonempty resolvent set unless \( A \) is closed. This is true because for all \( \lambda \in \rho(A) \) the operator \( (\lambda - A)^{-1} \) is closed (since it is assumed to be bounded), and hence \( \lambda - A \) is closed, which is true if and only if \( A \) is closed.

5.2.2. Lemma. For each closed \( A \in \mathcal{ML}(\mathcal{X}) \) the following conditions are equivalent:

(i) \( \lambda \) belongs to the resolvent set of \( A \).

(ii) There exists an operator \( \hat{A}(\lambda) \in \mathcal{B}(\mathcal{X}) \) such that the graph of \( (\lambda - A) \) has the equivalent representations

\[
\text{gph}(\lambda - A) = \text{rng} \left( \begin{bmatrix} 1_X & 0 \\ \hat{A}(\lambda) \\ 1_X \end{bmatrix} \right),
\]

\[
(5.2.1a)
\]

\[
\text{gph}(\lambda - A) = \ker \left( \begin{bmatrix} \hat{A}(\lambda) \\ -1_X \end{bmatrix} \right).
\]

\[
(5.2.1b)
\]

(iii) There exists an operator \( \hat{A}(\lambda) \in \mathcal{B}(\mathcal{X}) \) such that the graph of \( A \) has the equivalent representations

\[
\text{gph}(A) = \text{rng} \left( \begin{bmatrix} -1_X & \lambda \\ 0 & 1_X \\ \hat{A}(\lambda) \end{bmatrix} \right),
\]

\[
(5.2.2a)
\]

\[
\text{gph}(A) = \text{rng} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) - 1_X \\ \hat{A}(\lambda) \end{bmatrix} \right),
\]

\[
(5.2.2b)
\]

\[
\text{gph}(A) = \ker \left( \begin{bmatrix} \hat{A}(\lambda) \\ 1_X - \lambda \hat{A}(\lambda) \end{bmatrix} \right).
\]

\[
(5.2.2c)
\]

When these equivalent conditions hold, then the operators \( \hat{A}(\lambda) \) in (ii) and (iii) coincide with the resolvent of \( A \) evaluated at \( \lambda \).

Proof. (i) \( \Leftrightarrow \) (ii): That \( \lambda \in \rho(A) \) if and only if there exists a bounded operator \( \hat{A}(\lambda) \) such that \( (5.2.1a) \) holds follows directly from Definition 5.2.1 and the definition
of the inverse of a multi-valued operator. At the same time we see that \( \hat{A}(\lambda) = (\alpha - A)^{-1} \). It is also easy to show that \((5.2.1b)\) is equivalent to \((5.2.1a)\).

(ii) \(\equiv\) (iii): Clearly \((5.2.1a)\) is equivalent to \((5.2.2a)\) since \(\text{gph}(\lambda - A) = \begin{bmatrix} -1^x & \lambda^x \\ 0 & 1^x \end{bmatrix} \text{gph}(A)\), where the operator \(\begin{bmatrix} -1^x & \lambda^x \\ 0 & 1^x \end{bmatrix}\) is its own inverse. Trivially, \((5.2.2a)\) and \((5.2.2b)\) are equivalent.

To prove that \((5.2.2b)\) and \((5.2.2c)\) are equivalent to each other can argue as follows. Let \(\hat{z} \in \text{rng} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) - 1^x \\ \hat{A}(\lambda) \end{bmatrix} \right)\). Then there exists some \(x^0 \in \mathcal{X}\) such that
\[
\hat{z} = \lambda \hat{A}(\lambda)x^0 - x^0 \quad \text{and} \quad x = \lambda \hat{A}(\lambda)x^0.
\]
From here we can solve \(x^0 = \lambda x - z\), and therefore
\[
x = \hat{A}(\lambda)x^0 = \hat{A}(\lambda)(\lambda x - z) = \lambda \hat{A}(\lambda)x - \hat{A}(\lambda)z.
\]
This is equivalent to the condition \(\hat{z} \in \text{ker} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) & 1^x - \lambda \hat{A}(\lambda) \end{bmatrix} \right)\). Conversely, suppose that \(\hat{z} \in \text{ker} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) & 1^x - \lambda \hat{A}(\lambda) \end{bmatrix} \right)\), i.e., suppose that \(\hat{A}(\lambda)z = \lambda \hat{A}(\lambda)x - x\).

Define \(x^0 := \lambda x - z\). Then
\[
\hat{A}(\lambda)x^0 = \lambda \hat{A}(\lambda)x - \hat{A}(\lambda)z = x, \quad \lambda \hat{A}(\lambda)x^0 - x^0 = \lambda x - x^0 = z.
\]
This is equivalent to \(\hat{z} \in \text{rng} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) - 1^x \\ \hat{A}(\lambda) \end{bmatrix} \right)\). Thus, \((5.2.2b)\) and \((5.2.2c)\) are equivalent.

5.2.3. Theorem. Let \(A \in \mathcal{ML}(\mathcal{X})\) be closed and have a nonempty resolvent set. We denote the resolvent of \(A\) by \(\hat{A}(\lambda) := (\lambda - A)^{-1}, \lambda \in \rho(A)\).

(i) For all \(\lambda, \mu \in \rho(A)\),
\[
\text{ker}(\hat{A}(\lambda)) = \text{ker}(\hat{A}(\mu)) = \text{mul}(A),
\]
\[
\text{rng}(\hat{A}(\lambda)) = \text{rng}(\hat{A}(\mu)) = \text{dom}(A).
\]
In particular, \(A\) is single-valued if and only if \(\hat{A}(\lambda)\) is injective for some (and hence for all) \(\lambda \in \rho(A)\), and \(\text{dom}(A)\) is dense in \(\mathcal{X}\) if and only if \(\text{rng}(\hat{A}(\lambda))\) is dense in \(\mathcal{X}\) for some (and hence for all) \(\lambda \in \rho(A)\).

(ii) The resolvent \(\hat{A}\) satisfies
\[
\hat{A}(\lambda) - \hat{A}(\mu) = (\mu - \lambda)\hat{A}(\lambda)\hat{A}(\mu) = (\mu - \lambda)\hat{A}(\mu)\hat{A}(\lambda), \quad \lambda, \mu \in \rho(A).
\]
In particular, \(\hat{A}(\lambda)\hat{A}(\mu) = \hat{A}(\mu)\hat{A}(\lambda)\) for all \(\lambda, \mu \in \rho(A)\).

(iii) Let \(\lambda \in \rho(A)\). Then \(\mu \in \rho(A)\) if and only if the operator \(1^x + (\mu - \lambda)\hat{A}(\lambda)\) has a bounded inverse. In this case
\[
\left(1^x + (\mu - \lambda)\hat{A}(\lambda)\right)^{-1} = 1^x + (\lambda - \mu)\hat{A}(\mu), \quad \lambda, \mu \in \rho(A),
\]
and \(\hat{A}(\mu)\) can be solved in terms of \(\hat{A}(\lambda)\) from \((5.2.4)\) to get
\[
\hat{A}(\mu) = \hat{A}(\lambda)\left(1^x + (\mu - \lambda)\hat{A}(\lambda)\right)^{-1}, \quad \lambda, \mu \in \rho(A).
\]

(iv) The resolvent set \(\rho(A)\) is open, and hence the spectrum \(\sigma(A)\) is closed. More precisely, if we fix some arbitrary norm in \(\mathcal{X}\) and let \(\|\hat{A}(\lambda)\|\) be the corresponding operator norm of \(\hat{A}(\lambda)\), then for each \(\lambda \in \rho(A)\), the distance from \(\lambda\) to \(\sigma(A)\) is at least \(1/\|\hat{A}(\lambda)\|\).
\begin{equation}
\frac{d^n}{d\lambda^n} \tilde{A}(\lambda) = (-1)^n n! \tilde{A}(\lambda)^{n+1}, \quad \lambda \in \rho(A), \quad n \in \mathbb{Z}^+.
\end{equation}

The identity \([5.2.4]\) in (i) is usually called the \textit{resolvent identity}.

**Proof of Theorem 5.2.3.**

(i) This is true because \(\ker (\lambda - A)^{-1} = \text{mul} (\lambda - A) = \text{mul} (A)\) and \(\text{rng} (\lambda - A)^{-1} = \text{dom} (\lambda - A) = \text{dom} (A)\).

(ii) Let \(\lambda, \mu \in \rho(A)\). Then we get two different representations of \(\text{gph}(A)\) from \([5.2.2\text{a}]\), namely

\[
\text{gph}(A) = \text{rng} \left( \begin{bmatrix} -1 \lambda \\ 0 \\ 1 \lambda \end{bmatrix} \tilde{A}(\lambda) \right) = \text{rng} \left( \begin{bmatrix} -1 \lambda \\ 0 \\ 1 \lambda \end{bmatrix} \tilde{A}(\mu) \right).
\]

Here \(\begin{bmatrix} -1 \lambda \\ 0 \\ 1 \lambda \end{bmatrix}\) is its own inverse, and hence

\[
\text{rng} \left( \begin{bmatrix} 1 \lambda \\ \tilde{A}(\lambda) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} -1 \lambda \\ 0 \\ 1 \lambda \end{bmatrix} \begin{bmatrix} -1 \lambda \\ 0 \\ 1 \lambda \end{bmatrix} \tilde{A}(\mu) \right) = \text{rng} \left( \begin{bmatrix} 1 \lambda \\ \tilde{A}(\mu) \end{bmatrix} \right).
\]

This implies, in particular, that the operator \(1_X + (\lambda - \mu)\tilde{A}(\mu)\) is surjective. We claim that it is also injective. If \(x + (\lambda - \mu)\tilde{A}(\mu)x = 0\), then necessarily \(x \in \text{dom} (A)\) (since \(x = (\mu - \lambda)\tilde{A}(\mu)x\)). For all \(x \in \text{dom} (A)\) we have

\[
\tilde{A}(\mu)(\lambda - A)x = \tilde{A}(\mu)(\mu - A + \lambda - \mu)x = (1_X + (\lambda - \mu)\tilde{A}(\mu))x.
\]

Thus, \((1_X + (\lambda - \mu)\tilde{A}(\mu))x = 0\) if and only if \(x \in \text{dom} (A)\) and \(\tilde{A}(\mu)(\lambda - A)x = 0\). But the operator \(\tilde{A}(\mu)(\lambda - A)\) is injective since \((\lambda - A)\) is injective and \(\ker (\tilde{A}(\mu)) = \text{mul} (A) = \text{mul} (\lambda - A)\). This proves our claim that the operator \(1_X + (\lambda - \mu)\tilde{A}(\mu)\) is injective (whenever \(\lambda \in \rho(A)\)).

By the closed graph theorem, the operator \(1_X + (\lambda - \mu)\tilde{A}(\mu)\) has a bounded inverse. We can therefore continue the computation started above to get

\[
\text{rng} \left( \begin{bmatrix} 1 \lambda \\ \tilde{A}(\lambda) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} 1_X + (\lambda - \mu)\tilde{A}(\mu) \\ \tilde{A}(\mu) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} 1_X + (\lambda - \mu)\tilde{A}(\mu) \\ \tilde{A}(\mu) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} 1_X + (\lambda - \mu)\tilde{A}(\mu) \\ \tilde{A}(\mu) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} 1_X + (\lambda - \mu)\tilde{A}(\mu) \\ \tilde{A}(\mu) \end{bmatrix} \right).
\]

Thus \(\tilde{A}(\lambda) = \tilde{A}(\mu)(1_X + (\lambda - \mu)\tilde{A}(\mu))^{-1}\). This implies \([5.2.4]\).

(iii) To show that the two operators in (iii) are inverses of each other whenever both \(\lambda \in \rho(A)\) and \(\mu \in \rho(A)\) it suffices to multiply these operators (in either order) and to use \([5.2.4]\) to show that the product is equal to \(1_X\).

Conversely, suppose that \(\lambda \in \rho(A)\) and that the operator \(1_X + (\mu - \lambda)\tilde{A}(\lambda)\) has a bounded inverse. We may then define \(\tilde{A}(\mu)\) by

\begin{equation}
\tilde{A}(\mu) = \tilde{A}(\lambda)(1_X + (\mu - \lambda)\tilde{A}(\lambda))^{-1}.
\end{equation}
By (5.2.2a),
\[
gph(\mu - A) = \begin{bmatrix} -1 & \mu \\ 0 & 1 \end{bmatrix} \text{gph}(A)
\]
\[
= \operatorname{rng} \left( \begin{bmatrix} -1 & \mu \\ 0 & 1 \end{bmatrix} \left[ \mathfrak{A}(\lambda) - 1_X \right] \right)
\]
\[
= \operatorname{rng} \left( \begin{bmatrix} 1_X + (\mu - \lambda)\mathfrak{A}(\lambda) \\ \mathfrak{A}(\lambda) \end{bmatrix} \right)
\]
\[
= \operatorname{rng} \left( \begin{bmatrix} 1_X + (\mu - \lambda)\mathfrak{A}(\lambda) \\ \mathfrak{A}(\mu) \end{bmatrix} \right). 
\]

This implies that \( \mu \in \rho(A) \) and that \( (\mu - A)^{-1} = \mathfrak{A}(\mu) \).

(iv) Let \( \lambda \in \rho(A) \). Fix some admissible norm in \( \mathcal{X} \), and denote the corresponding operator norm of \( \mathfrak{A}(\lambda) \) by \( \|\mathfrak{A}(\lambda)\| \). Finally, take \( \mu \) sufficiently close to \( \lambda \) so that \( |\mu - \lambda| < 1/\|\mathfrak{A}(\lambda)\| \). Then the operator \( 1_X - (\mu - \lambda)\mathfrak{A}(\lambda) \) has a bounded inverse, and by (i), \( \mu \in \rho(A) \). This implies that \( \rho(A) \) is open and that the distance from \( \lambda \) to \( \sigma(A) \) is at least \( 1/\|\mathfrak{A}(\lambda)\| \).

(v) By (i), for all \( \lambda, \mu \in \rho(A) \) the operator \( 1_X - (\mu - \lambda)\mathfrak{A}(\lambda) \) has a bounded inverse, and hence we can solve \( \mathfrak{A}(\mu) \) from (5.2.4) to get (5.2.8). The right-hand side of (5.2.8) is an analytic function of \( \mu \) (this follows, e.g., from the implicit function theorem), and therefore \( \mathfrak{A} \) is analytic in \( \rho(A) \). That (5.2.7) holds for \( n = 1 \) follows from (5.2.4) by dividing by \( (\mu - \lambda) \) and letting \( \mu \to \lambda \), after which the general case can be proved by induction over \( n \).

5.2.4. DEFINITION. By a pseudo-resolvent \( \mathfrak{A} \) in \( (\mathcal{X}; \Omega) \), where \( \mathcal{X} \) is an \( H \)-space and \( \Omega \) is an open set in \( \mathbb{C} \), we mean a \( \mathcal{B}(\mathcal{X}) \)-valued function \( \mathfrak{A} \) defined on \( \Omega \) which satisfies the resolvent identity
\[
\mathfrak{A}(\lambda) - \mathfrak{A}(\mu) = (\mu - \lambda)\mathfrak{A}(\lambda)\mathfrak{A}(\mu), \quad \lambda, \mu \in \Omega.
\]

5.2.5. COROLLARY. The resolvent of a closed multi-valued operator \( A \) in an \( H \)-space \( \mathcal{X} \) with \( \rho(A) \neq \emptyset \) is a pseudo-resolvent in \( (\mathcal{X}; \rho(A)) \).

PROOF. This follows from Theorem 5.2.3 and definition 5.2.4.

5.2.6. THEOREM. Let \( \mathfrak{A} \) be a pseudo-resolvent in \( (\mathcal{X}; \Omega) \), where \( \mathcal{X} \) is an \( H \)-space and \( \Omega \) an open set in \( \mathbb{C} \). Then the following claims are true.

(i) \( \mathfrak{A} \) is the restriction to \( \Omega \) of the resolvent of a closed multi-valued operator \( A \) in \( \mathcal{X} \) satisfying \( \Omega \subseteq \rho(A) \). Thus in particular, all the conclusions listed in Theorem 5.2.3 are valid for a resolvent \( \mathfrak{A} \) in \( (\mathcal{X}; \Omega) \) with \( \rho(A) \) replaced by \( \Omega \).

(ii) The multi-valued operator \( A \) in (i), and hence also the resolvent \( \mathfrak{A} \) itself, is determined uniquely by the value of \( \mathfrak{A}(\lambda) \) at some point \( \lambda \in \Omega \).

(iii) The set \( \rho(A) \), where \( A \) is the multi-valued operator in (i), is the maximal open set in \( \mathbb{C} \) to which \( \mathfrak{A} \) can be extended as a pseudo-resolvent. More precisely, the resolvent of \( A \) is a pseudo-resolvent in \( (\mathcal{X}; \rho(A)) \) whose restriction to \( \Omega \) is equal to \( \mathfrak{A} \), and there does not exist any pseudo-resolvent in \( (\mathcal{X}; \Omega') \) with \( \Omega' \cap \sigma(A) \neq \emptyset \) whose restriction to \( \Omega \) coincides with \( \mathfrak{A} \).
PROOF. (i) Fix $\lambda \in \Omega$, and let $A_\lambda$ be the closed multi-valued operator whose graph is given by

$$
gph(A_\lambda) = \text{rng} \left( \begin{bmatrix} -1_x & \lambda \\ 0 & 1_x \end{bmatrix} \begin{bmatrix} 1_x \\ \widehat{A}(\lambda) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} \lambda \widehat{A}(\lambda) - 1_x \\ \widehat{A}(\lambda) \end{bmatrix} \right).
$$

Clearly $gph(A_\lambda)$ is closed since $gph(\widehat{A}(\lambda)^{-1}) = \text{rng} \left( \begin{bmatrix} 1_x \\ \widehat{A}(\lambda) \end{bmatrix} \right)$ is closed and $\left[ \begin{bmatrix} 1_x & \lambda \\ 0 & 1_x \end{bmatrix} \right]$ is invertible. Moreover,

$$
gph(\lambda - A_\lambda) = \begin{bmatrix} -1_x & \lambda \\ 0 & 1_x \end{bmatrix} gph(A_\lambda) = \text{rng} \left( \begin{bmatrix} 1_x \\ \widehat{A}(\lambda) \end{bmatrix} \right),
$$

so $\lambda - A_\lambda = \widehat{A}(\lambda)^{-1}$. Thus, $\lambda \in \rho(A_\lambda)$ and $(\lambda - A_\lambda)^{-1} = \widehat{A}(\lambda)$.

We can repeat the same construction with $\lambda \in \Omega$ replaced by some other $\mu \in \Omega$ to get another closed multi-valued operator $A_\mu$ such that $\mu \in \rho(A_\mu)$ and $(\mu - A_\mu)^{-1} = \widehat{A}(\mu)$. In order to complete the proof it suffices to show that $A_\mu = A_\lambda$, or equivalently, that $gph(A_\mu) = gph(A_\lambda)$.

The proof of (5.2.3) contained in the proof of Theorem 5.2.3 was based entirely on the resolvent identity (5.2.4), so by repeating the same proof we find that $1_x + (\lambda - \mu)\widehat{A}(\mu)$ has a bounded inverse for every $\lambda, \mu \in \Omega$. We can then solve $\widehat{A}(\lambda)$ in terms of $\widehat{A}(\mu)$ from (5.2.4) to get $\widehat{A}(\lambda) = \widehat{A}(\mu)(1_x + (\lambda - \mu)\widehat{A}(\mu))^{-1}$. Using this identity and the definitions of $A_\lambda$ and $A_\mu$ we get

$$
gph(A_\lambda) = \text{rng} \left( \begin{bmatrix} -1_x & \lambda \\ 0 & 1_x \end{bmatrix} \begin{bmatrix} 1_x \\ \widehat{A}(\lambda) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} -1_x & \lambda \\ 0 & 1_x \end{bmatrix} \begin{bmatrix} 1_x \\ \widehat{A}(\mu)(1_x + (\lambda - \mu)\widehat{A}(\mu))^{-1} \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} -1_x & \lambda \\ 0 & 1_x \end{bmatrix} \begin{bmatrix} 1_x + (\lambda - \mu)\widehat{A}(\mu) \\ \widehat{A}(\mu) \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} \mu \widehat{A}(\mu) - 1_x \\ \widehat{A}(\mu) \end{bmatrix} \right) = gph(A_\mu).
$$

Thus $A_\lambda = A_\mu$ for all $\lambda, \mu \in \Omega$.

(ii) That (ii) holds follows from the fact that a multi-valued operator $A$ is determined uniquely by the value of $(\alpha - A)^{-1}$ at any point $\alpha \in \rho(A)$.

(iii) By (i), the resolvent $\alpha \mapsto (\alpha - A)^{-1}$ of $A$ is a resolvent extension of $\widehat{A}$ to $\rho(A)$. Conversely, if $\widehat{A}$ can be extended as a resolvent to some open set $\Omega' \supset \Omega$, then it follows from (i) that $\Omega' \subset \rho(A)$. Thus, $\rho(A)$ is the maximal set to which $\widehat{A}$ can be extended as a resolvent. \hfill \Box

5.2.7. Corollary. If $\widehat{A}_1$ and $\widehat{A}_2$ are two pseudo-resolvents in $(\mathcal{X}; \Omega_1)$ respectively $(\mathcal{X}; \Omega_2)$ with $\Omega_1 \cap \Omega_2 \neq \emptyset$, and if $\widehat{A}_1(\lambda) = \widehat{A}_2(\lambda)$ at some point $\lambda \in \Omega_1 \cap \Omega_2$, then $\widehat{A}_1(\lambda) = \widehat{A}_2(\lambda)$ for all $\lambda \in \Omega_1 \cap \Omega_2$. Moreover, if we define $\widehat{A}$ on $\Omega := \Omega_1 \cup \Omega_2$ by

$$
\widehat{A}(\lambda) = \begin{cases} 
\widehat{A}_1(\lambda), & \lambda \in \Omega_1, \\
\widehat{A}_2(\lambda), & \lambda \in \Omega_2,
\end{cases}
$$

then $\widehat{A}$ is a pseudo-resolvent in $(\mathcal{X}; \Omega)$. 


Proof. This follows from Theorem 5.2.6.

5.2.2. The i/s/o resolvent of a closed i/s/o node. We now go back to the original theme of this section, and investigate what can be said about the existence of an i/s/o resolvent matrix of a (general) i/s/o node. It is possible to approach this problem in the same way as we did in the regular case discussed in Section 5.1. We now replace the equation (2.1.1) describing the evolution of a classical trajectory of a regular i/s/o node by the equation (2.1.10) describing the evolution of a classical trajectory of a general i/s/o node. The same argument that before gave us the (5.1.8a) and (5.1.8b) now leads instead to

\[
\begin{bmatrix}
\lambda x - x^0 \\
y
\end{bmatrix} 
\in S \begin{bmatrix}
x \\
u
\end{bmatrix}, \quad \begin{bmatrix}
x \\
u
\end{bmatrix} \in \text{dom}(S), \quad (5.2.10a)
\]

\[
\begin{bmatrix}
-x^0 \\
y
\end{bmatrix} 
\in \left(S - \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}\right) \begin{bmatrix}
x \\
u
\end{bmatrix}, \quad \begin{bmatrix}
x \\
u
\end{bmatrix} \in \text{dom}(S). \quad (5.2.10b)
\]

Thus, we may repeat Definition 5.1.2, Lemma 5.1.3, and Definition 5.1.4 in this more general setting.

5.2.8. Definition. Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a closed i/s/o node.

(i) A point \(\lambda \in \mathbb{C}\) belongs to the resolvent set of \(\Sigma\) (and hence, in particular, \(\Sigma\) is resolvable); if and only if

\[
\begin{bmatrix}
\lambda x - x^0 \\
y
\end{bmatrix} 
\in S \begin{bmatrix}
x \\
u
\end{bmatrix}, \quad \begin{bmatrix}
x \\
u
\end{bmatrix} \in \text{dom}(S), \quad (5.2.10a)
\]

\[
\begin{bmatrix}
-x^0 \\
y
\end{bmatrix} 
\in \left(S - \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}\right) \begin{bmatrix}
x \\
u
\end{bmatrix}, \quad \begin{bmatrix}
x \\
u
\end{bmatrix} \in \text{dom}(S). \quad (5.2.10b)
\]

Thus, we may repeat Definition 5.1.2, Lemma 5.1.3, and Definition 5.1.4 in this more general setting.

5.2.9. Remark. The assumption in Definition 5.2.8 that \(\Sigma\) is closed is redundant in the sense the closedness of \(\Sigma\) is a necessary condition for \(\Sigma\) to have a nonempty resolvent set. This follows from the representations (5.2.12) of \(\text{gph}(S)\) in Lemma 5.2.13. (The proof of Lemmas 5.2.10 and 5.2.13 remain valid even for a non-closed system \(\Sigma\), if we simply drop the word “closed” in Definition 5.2.8.)

5.2.10. Lemma. Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a closed i/s/o node, and let \(\lambda \in \mathbb{C}\).

Then the following conditions are equivalent:

(i) \(\lambda\) belongs to the resolvent set of \(\Sigma\) (and hence, in particular, \(\Sigma\) is resolvable);

(ii) there exists four bounded linear operators \(\hat{A}(\lambda), \hat{B}(\lambda), \hat{C}(\lambda), \text{ and } \hat{D}(\lambda)\) such that the four vectors \(x^0, u_\lambda, x_\lambda, \text{ and } y_\lambda\) satisfy (5.2.10a) if and only if (5.1.9) holds.

Proof. The proof is the same as the proof of Lemma 5.1.3.

5.2.11. Definition. Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a resolvable i/s/o node. The bounded operator-valued function \(\hat{S} = \begin{bmatrix}
\hat{S} & \hat{B} \\
\hat{C} & \hat{D}
\end{bmatrix}\) with \(\text{dom}(\hat{S}) = \rho(\Sigma)\) in (5.1.9) is

\[\text{multi-valued}^{1}\]
called the \emph{i/s/o resolvent matrix} of \( \Sigma \). The components \( \hat{A}, \hat{B}, \hat{C}, \) and \( \hat{D} \) of the i/s/o resolvent matrix \( \hat{S} \) are called as follows:

(i) \( \hat{A} \) is the \( s/s \) (state/state) resolvent of \( \Sigma_{i/s/o} \),
(ii) \( \hat{B} \) is the i/s (input/state) resolvent of \( \Sigma_{i/s/o} \),
(iii) \( \hat{C} \) is the s/o (state/output) resolvent of \( \Sigma_{i/s/o} \),
(iv) \( \hat{D} \) is the i/o (input/output) resolvent of \( \Sigma_{i/s/o} \).

As in the single-valued case, in the case where \( x^0 = 0 \) it is also possible to arrive at (5.2.10a) in a different way.

5.2.12. **Lemma** (cf. Lemma 5.1.5). Let \( \Sigma = (S; X, U, Y) \) be a closed i/s/o node,
let \( \begin{bmatrix} x_0 \\ u_0 \\ y_0 \end{bmatrix} \in \begin{bmatrix} X \\ U \\ Y \end{bmatrix} \) and \( \lambda \in \mathbb{C} \), and define \( \begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_0 \\ u_0 \\ y_0 \end{bmatrix}, t \in \mathbb{R} \). Then the following conditions are equivalent:

(i) \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a classical two-sided trajectory of \( \Sigma \);
(ii) \( \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \text{dom}(S) \) and \( \begin{bmatrix} \lambda x_0 \\ y_0 \end{bmatrix} \in S \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \).

If, in addition, \( \lambda \in \rho(\Sigma) \), then conditions (i) and (ii) above are equivalent to the following condition:

(iii) \( \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \hat{B}(\lambda) \\ \hat{D}(\lambda) \end{bmatrix} u_0 \), where \( \hat{B}(\lambda) \) and \( \hat{D}(\lambda) \) are the input/state and input/output resolvents of \( \Sigma \) evaluated at \( \lambda \).

**Proof.** The proof is the same as the proof of Lemma 5.1.5. \( \square \)

The following Lemma gives a graph interpretation of Lemma 5.2.10.

5.2.13. **Lemma.** Let \( \Sigma = (S; X, U, Y) \) be a closed i/s/o node, and let \( \lambda \in \mathbb{C} \). Then the following conditions are equivalent:

(i) \( \lambda \) belongs to the resolvent set of \( \Sigma \) (and hence, in particular, \( \Sigma \) is resolvable).
(ii) The graph of \( (S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}) \) has the following equivalent representations

\begin{align*}
\text{(5.2.11a)} & \quad \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & -1_x & 0 & 0 \\ 0 & 1_y & 0 & 0 & 0 \\ 1_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{gph} \left( \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \right), \\
\text{(5.2.11b)} & \quad \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{rng} \begin{bmatrix} -1_x & 0 \\ \hat{C}(\lambda) & \hat{D}(\lambda) \\ \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_u \end{bmatrix}, \\
\text{(5.2.11c)} & \quad \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{ker} \begin{bmatrix} \hat{A}(\lambda) & 0 & -1_x & \hat{B}(\lambda) \\ \hat{C}(\lambda) & 0 & 1_y & -\hat{D}(\lambda) \\ \hat{C}(\lambda) & 0 & 1_y & -\hat{D}(\lambda) \end{bmatrix}
\end{align*}

for some bounded linear operator \( \hat{S}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \in \mathcal{B}([X] ; [Y]) \).
(iii) The graph of $S$ has the following equivalent representations

(5.2.12a) \[ \text{gph}(S) = \begin{bmatrix} \lambda & 0 & -1_x & 0 \\ 0 & 1_y & 0 & 0 \\ 1_x & 0 & 0 & 1_t \\ 0 & 0 & 0 & 1_t \end{bmatrix} \text{gph} \left( \begin{bmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix} \right) \]

(5.2.12b) \[ \text{gph}(S) = \text{rng} \left( \begin{bmatrix} \lambda \mathcal{A}(\lambda) - 1_x \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix} \right) \]

(5.2.12c) \[ \text{gph}(S) = \text{ker} \left( \begin{bmatrix} \mathcal{A}(\lambda) & 0 \\ \mathcal{C}(\lambda) & 1_y \end{bmatrix} \left[ \begin{bmatrix} 1_x - \lambda \mathcal{A}(\lambda) \\ -\lambda \mathcal{C}(\lambda) \end{bmatrix} \right] \right) \]

for some bounded linear operator $\hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix} \in \mathcal{B}([U] ; [Y]).$

When these equivalent conditions hold, then the operators $\hat{\mathcal{S}}(\lambda)$ in (ii) and (iii) coincide with the $i/s/o$ resolvent of $\mathcal{S}$ evaluated at $\lambda$.

**Proof.** (i) $\Leftrightarrow$ (ii): By Lemma 5.2.10, $\lambda \in \rho(\mathcal{S})$ if and only if there exist four bounded linear operators $\hat{\mathcal{A}}(\lambda)$, $\hat{\mathcal{B}}(\lambda)$, $\hat{\mathcal{C}}(\lambda)$, and $\hat{\mathcal{D}}(\lambda)$ such that the first equivalence listed below holds (the second and third equivalences are obvious):

\[
\begin{bmatrix} -x^0 \\ y_{\lambda} \\ x_{\lambda} \\ u_{\lambda} \end{bmatrix} \in \text{gph} \left( \begin{bmatrix} \lambda & 0 \\ 0 & 0 \\ 1_x & 1_y \end{bmatrix} S \right) \Leftrightarrow \begin{bmatrix} x_{\lambda} \\ y_{\lambda} \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\ \hat{\mathcal{C}}(\lambda) & \hat{\mathcal{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ u_{\lambda} \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_{\lambda} \\ y_{\lambda} \\ x^0 \\ u_{\lambda} \end{bmatrix} \in \text{gph} \left( \begin{bmatrix} \hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\ \hat{\mathcal{C}}(\lambda) & \hat{\mathcal{D}}(\lambda) \end{bmatrix} \right) \Leftrightarrow \begin{bmatrix} -x^0 \\ y_{\lambda} \\ x_{\lambda} \\ u_{\lambda} \end{bmatrix} \in \text{gph} \left( \begin{bmatrix} 0 & 0 & -1_x & 0 \\ 0 & 1_y & 0 & 0 \\ 1_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_t \end{bmatrix} \begin{bmatrix} \lambda & 0 & -\lambda & 0 \\ 0 & 1_y & 0 & 0 \\ 0 & 0 & 1_x & 0 \\ 0 & 0 & 0 & 1_t \end{bmatrix} \text{gph}(S) \right), \quad \lambda \in \rho(\mathcal{S}).
\]

Thus (i) holds if and only if there exist four bounded linear operators $\hat{\mathcal{A}}(\lambda)$, $\hat{\mathcal{B}}(\lambda)$, $\hat{\mathcal{C}}(\lambda)$, and $\hat{\mathcal{D}}(\lambda)$ such that (5.2.11a) holds. It is easy to see that the three versions (5.2.11a)–(5.2.11c) of (5.2.11) are equivalent to each other. That the operator $\hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix}$ in (ii) is equal to the $i/s/o$ resolvent matrix of $\mathcal{S}$ follows from Definition 5.2.11.

(ii) $\Leftrightarrow$ (iii): That (5.2.11a) and (5.2.12a) are equivalent follows from the fact that

(5.2.13) \[ \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1_x & 0 & -\lambda & 0 \\ 0 & 1_y & 0 & 0 \\ 0 & 0 & 1_x & 0 \\ 0 & 0 & 0 & 1_t \end{bmatrix} \text{gph}(S), \quad \lambda \in \rho(\mathcal{S}), \]

where the block matrix operator on the right-hand side of (5.2.13) is its own inverse. Trivially (5.2.12a) and (5.2.12b) are equivalent.
To prove that (5.2.12b) and (5.2.12c) are equivalent to each other can argue as follows. Let

$$\begin{bmatrix} z_x \\ y_x \\ x_x \\ u_x \end{bmatrix} \in \text{rng} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) - I \mathcal{X} & \lambda \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \\ \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_{1U} \end{bmatrix} \right).$$  \hspace{1cm} (5.2.14)

Then there exists some $x_0 \in \mathcal{X}$ such that

$$z_x = \lambda \hat{A}(\lambda)x_0 - x_0 + \lambda \hat{B}(\lambda)u_x,$$

$$x_x = \hat{A}(\lambda)x_0 + \hat{B}(\lambda)u_x,$$

$$y_x = \hat{C}(\lambda)x_0 + \hat{D}(\lambda)u_x.$$  

From here we can solve $x_0 = \lambda x_x - z_x$, and therefore

$$x_x = \lambda \hat{A}(\lambda)x_x - \hat{A}(\lambda)z_x + \hat{B}(\lambda)u_x,$$

$$y_x = \lambda \hat{C}(\lambda)x_x - \hat{C}(\lambda)z_x + \hat{D}(\lambda)u_x.$$  

This is equivalent to the condition

$$\begin{bmatrix} z_x \\ y_x \\ x_x \\ u_x \end{bmatrix} \in \ker \left( \begin{bmatrix} \hat{A}(\lambda) & 0 & 1_{1X} - \lambda \hat{A}(\lambda) & -\hat{B}(\lambda) \\ \hat{C}(\lambda) & 1_{1Y} & -\lambda \hat{C}(\lambda) & -\hat{D}(\lambda) \end{bmatrix} \right).$$  \hspace{1cm} (5.2.15)

Conversely, suppose that (5.2.15) holds. Define $x_0 := \lambda x_x - z_x$. Then

$$\hat{A}(\lambda)x_0 + \hat{B}(\lambda)u_x = \lambda \hat{A}(\lambda)x_x - \hat{A}(\lambda)z_x + \hat{B}(\lambda)u_x = x_x,$$

$$\lambda \hat{A}(\lambda)x_0 - x_0 + \lambda \hat{B}(\lambda)u_x = \lambda x_x - x_0 = z_x,$$

$$\hat{C}(\lambda)x_0 + \hat{D}(\lambda)u_x = \lambda \hat{C}(\lambda)x_x - \hat{C}(\lambda)z_x + \hat{D}(\lambda)u_x = y_x.$$  

This is equivalent to (5.2.14). Thus, (5.2.12b) and (5.2.12c) are equivalent. \hspace{1cm} \Box

5.2.14. Lemma. Every resolvable i/s/o node $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ has a continuously determined output (see Definition \textit{2.3.40}). The operator $H$ is Definition \textit{2.3.40} is given by

$$\text{dom} \ (H) = \text{rng} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) - I \mathcal{X} & \lambda \hat{B}(\lambda) \\ \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_{1U} \end{bmatrix} \right),$$  \hspace{1cm} (5.2.16)

$$H \begin{bmatrix} x \\ z_x \\ u_x \end{bmatrix} = \hat{C}(\lambda)(\lambda x - z) + \hat{D}(\lambda)u_x, \quad \begin{bmatrix} x \\ z_x \\ u_x \end{bmatrix} \in \text{dom} \ (H),$$  

where $\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$ is the i/s/o resolvent matrix of $\Sigma$ and $\lambda$ is an arbitrary point in $\rho(\Sigma)$.

\text{Proof.} This follows from the representation (5.2.12b) for gph $(S)$. \hspace{1cm} \Box

5.2.15. Corollary. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a resolvable i/s/o node with main operator $A$ and i/s/o resolvent matrix $\hat{S} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$. Then the following formulas
are valid for all \( \lambda \in \rho(\Sigma) \):

(5.2.17a) \( \text{rng} (\hat{\mathcal{A}}(\lambda)) = \text{dom} (A) \),

(5.2.17b) \( \text{rng} ([\hat{\mathcal{A}}(\lambda) \ \hat{\mathcal{B}}(\lambda)]) = [1_X \ 0] \text{dom} (S) \)
\[
\quad = \left\{ x \in X \mid [x \ u] \in \text{dom} (S) \text{ for some } u \in U \right\},
\]

(5.2.17c) \( \text{rng} ([\hat{\mathcal{A}}(\lambda) \ \hat{\mathcal{B}}(\lambda)]) = \text{dom} (S) \),

(5.2.17d) \( \ker (\hat{\mathcal{A}}(\lambda)) = \text{mul} (A) \),

(5.2.17e) \( \ker ([\hat{\mathcal{A}}(\lambda) \ \hat{\mathcal{C}}(\lambda)]) = [1_X \ 0] (\text{mul} (S) \cap [X_0]) \)
\[
\quad = \left\{ z \in X \mid [z \ 0] \in \text{mul} (S) \right\},
\]

(5.2.17f) \( [1_Y] \ker (\hat{\mathcal{A}}(\lambda)) = \text{mul} (S) \).

Thus in particular, the left-hand sides of all the formulas in (5.2.17) are independent of \( \lambda \), as long as \( \lambda \in \rho(\Sigma) \).

**Proof.** All of these formulas follow from the representations (5.2.12) of \( \text{gph} (S) \) and the definition 2.1.11 of the main operator \( A \) of \( \Sigma \). \( \Box \)

5.2.16. Theorem. Let \( \Sigma = (S; X, U, Y) \) be a resolvable i/s/o node with main operator \( A \in \mathcal{ML}(X) \). Then \( A \) is closed, \( \rho(\Sigma) = \rho(A) \), and the s/s resolvent \( \hat{\mathcal{A}} \) of \( \Sigma \) satisfies \( \hat{\mathcal{A}}(\lambda) = (\lambda - A)^{-1} \) for all \( \lambda \in \rho(\Sigma) = \rho(A) \). In particular, \( \rho(A) \neq \emptyset \), and \( \rho(\Sigma) = \rho(A) \) is an open subset of \( C \).

**Proof.** It follows from (5.2.12b) (by dropping the second column and the second and fourth row on the right-hand side) that \( \rho(\Sigma) \subset \rho(A) \) and that \( \hat{\mathcal{A}}(\lambda) = (\lambda - A)^{-1} \) for all \( \lambda \in \rho(\Sigma) \). It remains to prove the opposite inclusion.

Let \( \mu \in \rho(A) \) and \( \lambda \in \rho(\Sigma) \). By (5.2.12b)

\[
\text{gph} \left( S - \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1_X & 0 & -\mu & 0 \\ 0 & 1_Y & 0 & 0 \\ 0 & 0 & 1_X & 0 \\ 0 & 0 & 0 & 1_U \end{bmatrix} \text{gph} (S)
\]

(5.2.18)
\[
= \text{rng} \left( \begin{bmatrix} (\lambda - \mu)\hat{\mathcal{A}}(\lambda) - 1_X & (\lambda - \mu)\hat{\mathcal{B}}(\lambda) \\ \hat{\mathcal{C}}(\lambda) & \hat{\mathcal{D}}(\lambda) \end{bmatrix} \right).
\]

By Theorem 5.2.3 the operator \( 1_X + (\mu - \lambda)\hat{\mathcal{A}}(\lambda) \) has the bounded inverse \( K := 1_X + (\lambda - \mu)\hat{\mathcal{A}}(\mu) \), and consequently also the operator

\[
\begin{bmatrix} 1_X + (\mu - \lambda)\hat{\mathcal{A}}(\lambda) & (\mu - \lambda)\hat{\mathcal{B}}(\lambda) \\ 0 & 1_U \end{bmatrix}.
\]
has the inverse \([ \begin{bmatrix} K \\ 0 \\ L \end{bmatrix} ]\), where \(L = -(\mu - \lambda)K\hat{\mathbf{B}}(\lambda)\). By multiplying the block matrix operator on the right-hand side of (5.2.18) by \([ \begin{bmatrix} K \\ 0 \\ L \end{bmatrix} ]\) we get

\[
gph \left( S - \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} -1 & 0 & 0 \\ \hat{E}(\lambda)K & \hat{E}(\lambda)L + \hat{\mathbf{B}}(\lambda) \\ \hat{A}(\lambda)K & \hat{A}(\lambda)L + \hat{\mathbf{B}}(\lambda) \end{bmatrix} \right) .
\]

By Lemma 5.2.13, \(\mu \in \rho(\Sigma)\) and

(5.2.19) \[\begin{bmatrix} \hat{A}(\mu) \\ \hat{B}(\mu) \\ \hat{C}(\mu) \\ \hat{D}(\mu) \end{bmatrix} = \begin{bmatrix} \hat{A}(\lambda) \\ \hat{B}(\lambda) \\ \hat{C}(\lambda) \\ \hat{D}(\lambda) \end{bmatrix} \begin{bmatrix} 1 & \mu - \lambda \hat{A}(\lambda) & (\mu - \lambda)\hat{B}(\lambda) \end{bmatrix}^{-1} . \]

The next theorem is a multi-valued version of Theorem 5.1.6.

5.2.17. Theorem. For each resolvable i/s/o node \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) the following conditions are equivalent:

(i) \(\lambda\) belongs to the resolvent set of \(\Sigma\).

(ii) The following two conditions hold:

(a) \(\text{mul}(S) \cap \begin{bmatrix} 0 \\ \mathcal{Y} \end{bmatrix} = \{0\}\),

(b) The operator \(\left( \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} S \right)\) (with domain equal to \(\text{dom}(S)\)) has an inverse in \(\mathcal{B}(\mathcal{X})\). (Equivalently, \(\left( \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} S \right)\) is closed, injective and surjective.)

When these equivalent conditions hold, then the inverse of the multi-valued operator \(\left( \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} S \right)\) in (ii) is given by (5.1.10), and the graph of \(\hat{\mathbf{S}}(\lambda)\) has the representation

(5.2.20) \[\gph \left( \begin{bmatrix} \hat{A}(\lambda) \\ \hat{B}(\lambda) \\ \hat{C}(\lambda) \\ \hat{D}(\lambda) \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \gph \left( S - \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} \right) .\]

If \(S\) is single-valued, then \(\hat{\mathbf{S}}(\lambda)\) is given by (5.1.11).

If \(S\) is single-valued, then all the above claims follow from Theorem 5.2.17, but in the multi-valued case the proof of that theorem needs to be slightly modified. Note, in particular, that the above theorem does not claim that \(\hat{\mathbf{S}}(\lambda)\) is given by (5.1.11) in the multi-valued case (the multi-valued parts of the left-hand and right-hand sides of (5.1.11) need not be the same).

Proof of Theorem 5.2.17: (i) \(\Rightarrow\) (ii): Let \(\lambda \in \rho(\Sigma)\). By Corollary 5.2.15, \(\text{mul}(S - \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ -\hat{E}(\lambda) \end{bmatrix} \text{mul}(A)\), and therefore \(\text{mul}(S) \cap \begin{bmatrix} 0 \\ \mathcal{Y} \end{bmatrix} = \{0\}\). Another consequence of the (5.2.11) is that \(\gph \left( \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} S \right)\) has a representation of the type

(5.2.21) \[\gph \left( \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} S \right) = \text{rng} \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hat{A}(\lambda) & \hat{B}(\lambda) & 0 \end{bmatrix} \right) .\]

This means that \(\left( \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} S \right)\) has a bounded inverse which is given by (5.1.10).
(ii) $\Rightarrow$ (i): Suppose that condition (ii) holds, and let \( \begin{bmatrix} x_0^\lambda \\ u_0^\lambda \\ v_0^\lambda \\ y_0^\lambda \end{bmatrix} \in \text{gph} (S - [\lambda 0 0 0]) \).

Then \( \begin{bmatrix} x_0^\lambda \\ u_0^\lambda \\ v_0^\lambda \\ y_0^\lambda \end{bmatrix} \in \text{gph} (\left([\lambda 0 0 0] - [1_x 0 0 0]\right) S) \). By the invertibility condition in (i) the vector \( x_\lambda \) is determined uniquely by \( \begin{bmatrix} x_0^\lambda \\ u_0^\lambda \\ v_0^\lambda \end{bmatrix} \). Thus, if in addition \( \begin{bmatrix} x_0^\lambda \\ u_0^\lambda \\ v_0^\lambda \end{bmatrix} = 0 \), then \( x_\lambda = 0 \), which implies that \( \begin{bmatrix} 0 \\ v_0^\lambda \end{bmatrix} \in \text{mul} (S - [\lambda 0 0 0]) = \text{mul} (S) \). By assumption \( \text{mul}(S) \cap \{0\} = \{0\} \), and consequently \( y_\lambda = 0 \). In other words, the condition \( \begin{bmatrix} x_0^\lambda \\ u_0^\lambda \\ v_0^\lambda \end{bmatrix} \in \text{gph} (S - [\lambda 0 0 0]) \) defines \( x_\lambda \) as a single-valued function of \( x_0^\lambda \). Since \( S \) is closed and the operator \( [0 0 0] \) is bounded, the graph of the operator \( S - [\lambda 0 0 0] \) is closed, and therefore the graph of the operator that maps \( \begin{bmatrix} x_0^\lambda \\ u_0^\lambda \end{bmatrix} \) into \( \begin{bmatrix} v_0^\lambda \end{bmatrix} \) is closed as well. By the closed graph theorem the graph of \( S - [\lambda 0 0 0] \) has a representation of the form (5.2.11) for some bounded linear operator \( \hat{S}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \in B([1_{U'}]; [Y']) \), and by Lemma 5.2.13 this is equivalent to the condition \( \lambda \in \rho(S) \).

We have now shown that (i) and (ii) are equivalent, and continue with the proofs of the additional claims, assuming that \( \lambda \in \rho(S) \). We already established the validity of (5.1.10) in the proof of the implication (i) $\Rightarrow$ (ii) above. The proof of the fact that (5.1.11) holds when \( S \) is single-valued is identical to the proof of (5.1.11) contained in the proof of Theorem 5.1.6. Finally, (5.2.20) follows from (5.2.11).

5.2.18. COROLLARY. Let \( \Sigma = (S; X, U, Y) \) be a resolvable i/s/o node with main operator \( A \in \mathcal{ML}(X) \) and observation operator \( C \in \mathcal{ML}(X; Y) \), and with i/s/o resolvent matrix \( \hat{S}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \). Then, for all \( \lambda \in \rho(S) \) the operators \( \hat{A}(\lambda) \) and \( \hat{D}(\lambda) \) are given by (5.1.15a) respectively (5.1.15b).

PROOF. This follows from (5.1.10).

Note that the two formulas (5.1.15c) and (5.1.15d) for \( \hat{C}(\lambda) \) respectively \( \hat{D}(\lambda) \) are not valid in the multi-valued case (the left-hand sides are single-valued, whereas the right-hand sides are multi-valued).

5.2.19. LEMMA. Let \( \Sigma = (S; X, U, Y) \) be a resolvable i/s/o node. Denote the i/s/o resolvent matrix of \( \Sigma \) by \( \hat{S} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \), and let \( \lambda \in \rho(S) \).

(i) The operator \( \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_{U'} \end{bmatrix} \) maps \( [X'] \) onto \( \text{dom} (S) \) and (5.1.18) holds.

The kernel of \( \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_{U'} \end{bmatrix} \) is \( \text{mul}(A(\lambda)) \).

(ii) The operator \( \begin{bmatrix} 1_x & \hat{B}(\lambda) \\ 0 & 1_{U'} \end{bmatrix} \) is bounded and invertible on \( [X'] \), and it maps \( \left[ \text{dom}(A) \right] \) one-to-one onto \( \text{dom} (S) \). The inverse of this operator (which maps \( [X'] \) one-to-one onto itself and \( \text{dom} (S) \) one-to-one onto \( \left[ \text{dom}(A) \right] \)) is \( \begin{bmatrix} 1_x & -\hat{B}(\lambda) \\ 0 & 1_{U'} \end{bmatrix} \).

(iii) The operator \( \begin{bmatrix} \hat{A}(\lambda) & 0 \\ \hat{C}(\lambda) & 1_y \end{bmatrix} \) maps \( [X'] \) onto \( \left[ \text{dom}(A) \right] \), and

\[
(5.2.22) \begin{bmatrix} \hat{A}(\lambda) & 0 \\ \hat{C}(\lambda) & 1_y \end{bmatrix}^{-1} = \begin{bmatrix} 1_x & 0 \\ -\hat{C}(\lambda) & 1_y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 1_y \end{bmatrix} - S \begin{bmatrix} 1_x & 0 \\ 0 & 0 \end{bmatrix}.
\]
The kernel of \( \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ \hat{\Sigma}(\lambda) & 1 \end{bmatrix} \) is \( \begin{bmatrix} 1_x \\ -\hat{\Sigma}(\lambda) \end{bmatrix} \) \( \text{mul} \) \( \begin{bmatrix} A \end{bmatrix} \).

**Proof.** (i)–(ii) It follows from \((5.1.10)\) that the operator \( \begin{bmatrix} \hat{\Phi}(\lambda) & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \) maps \( \mathcal{X} \) onto \( \text{dom} \begin{bmatrix} S \end{bmatrix} \). Clearly the kernel of this operator is \( \begin{bmatrix} \ker(\hat{\Phi}(\lambda)) \\ \{0\} \end{bmatrix} = \begin{bmatrix} \text{mul}(A) \\ \{0\} \end{bmatrix} \). The second identity in \((5.1.18)\) is a rewritten version of \((5.1.10)\), and it only remains to prove the first identity in \((5.1.18)\). To do this we factor \( \begin{bmatrix} \hat{\Phi}(\lambda) & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \) into

\[
(5.2.23) \begin{bmatrix} \hat{\Phi}(\lambda) & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1_x & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ 0 & 1 \end{bmatrix}.
\]

In this factorization \( \begin{bmatrix} \hat{\Phi}(\lambda) & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \) and \( \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ 0 & 1 \end{bmatrix} \) have the same kernel \( \begin{bmatrix} \text{mul}(A) \\ \{0\} \end{bmatrix} \), \( \begin{bmatrix} \hat{\Phi}(\lambda) & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \) maps \( \mathcal{X} \) onto \( \text{dom} \begin{bmatrix} S \end{bmatrix} \), and \( \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ 0 & 1 \end{bmatrix} \) maps \( \mathcal{X} \) onto \( \text{dom} \begin{bmatrix} \text{mul}(A) \end{bmatrix} \). This implies that the operator \( \begin{bmatrix} 1_x & \hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \) maps \( \text{dom}(A) \) onto \( \text{dom} \begin{bmatrix} \text{mul}(A) \end{bmatrix} \). Clearly \( \begin{bmatrix} 1_x & -\hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \) has the bounded inverse \( \begin{bmatrix} 1_x & -\hat{\Sigma}(\lambda) \\ 0 & 1 \end{bmatrix} \), which then maps \( \text{dom}(A) \) one-to-one onto \( \text{dom} \begin{bmatrix} S \end{bmatrix} \). By inverting \((5.2.23)\) we get \((5.1.18)\).

(iii) To prove \((5.1.19)\) we factor \( \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ \hat{\Sigma}(\lambda) & 1 \end{bmatrix} \) into

\[
(5.2.24) \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ \hat{\Sigma}(\lambda) & 1 \end{bmatrix} = \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1_x & 0 \\ \hat{\Sigma}(\lambda) & 1 \end{bmatrix}.
\]

Here the kernel of the right-hand side is equal to

\[
\begin{bmatrix} 1_x \\ -\hat{\Sigma}(\lambda) \\ 1 \end{bmatrix} \ker \left( \begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1_x \\ -\hat{\Sigma}(\lambda) \end{bmatrix} \text{mul} \begin{bmatrix} A \end{bmatrix},
\]

so this is also the kernel of the left-hand side. By inverting \((5.2.24)\) we get the first identity in \((5.2.22)\).

To prove the second identity in \((5.2.22)\) we let \( x^0, y, x_\lambda, y_\lambda \) be arbitrary vectors satisfying

\[
\begin{bmatrix} x^0 \\ y \end{bmatrix} \in \left( \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} - S \begin{bmatrix} 1_x & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_\lambda \\ y_\lambda \end{bmatrix}.
\]

In particular, this means that \( \begin{bmatrix} x^0 \\ 0 \end{bmatrix} \in \text{dom} \begin{bmatrix} S \end{bmatrix} \). The above condition is equivalent to the condition that

\[
\begin{bmatrix} -x^0 \\ y_\lambda - y \end{bmatrix} \in \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_\lambda \\ 0 \end{bmatrix}.
\]

By \((5.2.11c)\), this is equivalent to the condition

\[
\begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ \hat{\Sigma}(\lambda) & 1 \end{bmatrix} \begin{bmatrix} -x^0 \\ y_\lambda - y \end{bmatrix} + \begin{bmatrix} x_\lambda \\ 0 \end{bmatrix} = 0,
\]

which can be rewritten in the equivalent form

\[
\begin{bmatrix} \hat{\Phi}(\lambda) & 0 \\ \hat{\Sigma}(\lambda) & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ y \end{bmatrix} = \begin{bmatrix} x_\lambda \\ y_\lambda \end{bmatrix}.
\]

This gives the second identity in \((5.2.22)\). □

5.2.20. **Lemma.** Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a resolvable i/s/o node with s/s resolvent \( \tilde{\Phi} \).

(i) The following conditions are equivalent:
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(a) $S$ is single-valued;
(b) The main operator $A$ of $\Sigma$ is single-valued;
(c) $\mathbf{\mathcal{A}}(\lambda)$ is injective for some (and hence for all) $\lambda \in \rho(\Sigma)$.

(ii) The following conditions are equivalent:
(a) $\text{dom}(S)$ is dense in $[\chi X]$;
(b) The domain of the main operator $A$ of $\Sigma$ is dense in $\chi$;
(c) $\mathbf{\mathcal{A}}(\lambda)$ is has dense range for some (and hence for all) $\lambda \in \rho(\Sigma)$.

(iii) The following conditions are equivalent:
(a) $\Sigma$ is a regular i/s/o node with $\rho(\Sigma) \neq \emptyset$ (i.e., $S$ is single-valued and densely defined, and the i/s/o resolvent set of $S$ is nonempty);
(b) The main operator $A$ of $\Sigma$ is single-valued and densely defined.
(c) $\mathbf{\mathcal{A}}(\lambda)$ is injective and has dense range for some (and hence for all) $\lambda \in \rho(\Sigma)$.

Proof. (i) The function $\mathbf{\mathcal{A}}$ is the resolvent of the main (multi-valued) operator $A$ of $S$, and by Theorem 5.2.3(i), $A$ is single-valued if and only if $\mathbf{\mathcal{A}}(\lambda)$ is injective for some $\lambda \in \rho(A)$, or equivalently, for all $\lambda \in \rho(A)$. By (5.2.21f), $\text{mul}(S) = \{0\}$ if and only if $\text{ker}(\mathbf{\mathcal{A}}(\lambda)) = 0$, or equivalently, if and only if $A$ is single-valued.

(ii) The range of the operator $\left[ \begin{array}{c} \mathbf{\mathcal{S}}(\lambda) \\ 0 \end{array} \right]$ is dense in $[\chi X]$ for some $\lambda \in \rho(\Sigma) = \rho(A)$ if and only if the range of $\mathbf{\mathcal{A}}(\lambda)$ is dense in $\chi$, or equivalently, if and only if $\text{dom}(A)$ is dense in $\chi$. Thus by Lemma 5.2.19(i), $\text{dom}(S)$ is dense in $[\chi X]$ if and only if $\text{dom}(A)$ is dense in $\chi$.

(iii) Claim (iii) follows from (i) and (ii). \qed

5.2.21. Lemma. Let $\Sigma = (S; \chi, U, Y)$ be a resolvable i/s/o node. Denote the i/s/o resolvent matrix of $\Sigma$ by $\mathbf{\mathcal{A}} = \left[ \begin{array}{c} \mathbf{\mathcal{A}} \\ \mathbf{\mathcal{B}} \end{array} \right]$, and let $\lambda \in \rho(\Sigma)$. Then $\text{dom}(S)$ and $S$ itself can be recovered from $\mathbf{\mathcal{A}}(\lambda)$ by means of the following formulas:

(5.2.25a) $\text{dom}(S) = \text{rng} \left( \left[ \begin{array}{cc} \mathbf{\mathcal{A}}(\lambda) & \mathbf{\mathcal{B}}(\lambda) \\ 0 & 1_U \end{array} \right] \right)$,
(5.2.25b) $\text{dom}(S) = \text{dom} \left( \left[ \begin{array}{cc} \mathbf{\mathcal{A}}(\lambda) & 0 \\ \mathbf{\mathcal{B}}(\lambda) & 1_Y \end{array} \right]^{-1} \left[ \begin{array}{cc} -1_X & \mathbf{\mathcal{B}}(\lambda) \\ 0 & \mathbf{\mathcal{A}}(\lambda) \end{array} \right] \right)$,
(5.2.25c) $\text{dom}(S) = \text{dom} \left( \left[ \begin{array}{cc} \mathbf{\mathcal{A}}(\lambda) & \mathbf{\mathcal{B}}(\lambda) \\ 0 & 1_Y \end{array} \right]^{-1} \left[ \begin{array}{cc} -\mathbf{\mathcal{A}}(\lambda) & 0 \\ 0 & \mathbf{\mathcal{B}}(\lambda) \end{array} \right] \left[ \begin{array}{cc} \mathbf{\mathcal{A}}(\lambda) & 0 \\ 0 & 1_Y \end{array} \right]^{-1} \right)$,
(5.2.25d) $\text{dom}(S) = \left\{ \left\{ \begin{array}{c} x \\ u \end{array} \right\} \in \left[ \begin{array}{c} \chi \\ U \end{array} \right] \mid \mathbf{\mathcal{B}}(\lambda)u - x \in \text{dom}(A) \right\}$.

(5.2.26a) $S = \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \\ 0 \end{array} \right] + \left[ \begin{array}{cccc} -1_X & 0 \\ \mathbf{\mathcal{A}}(\lambda) & \mathbf{\mathcal{B}}(\lambda) \\ 0 & 1_U \end{array} \right] \left[ \begin{array}{cccc} \mathbf{\mathcal{A}}(\lambda) & \mathbf{\mathcal{B}}(\lambda) \\ 0 & 1_U \end{array} \right]^{-1}$,
(5.2.26b) $S = \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \\ 0 \end{array} \right] + \left[ \begin{array}{cccc} \mathbf{\mathcal{A}}(\lambda) \\ 0 \\ \mathbf{\mathcal{B}}(\lambda) \\ 0 \end{array} \right] \left[ \begin{array}{cccc} -1_X & \mathbf{\mathcal{A}}(\lambda) & \mathbf{\mathcal{B}}(\lambda) \\ 0 & 1_U \end{array} \right]^{-1}$,
(5.2.26c) $S = \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \\ 0 \end{array} \right] + \left[ \begin{array}{cccc} \mathbf{\mathcal{A}}(\lambda) \\ 0 \\ \mathbf{\mathcal{B}}(\lambda) \\ 0 \end{array} \right] \left[ \begin{array}{cccc} -\mathbf{\mathcal{A}}(\lambda) & 0 \\ 0 & \mathbf{\mathcal{B}}(\lambda) \\ 0 & 1_U \end{array} \right]^{-1}$.

Proof of Lemma 5.1.12. We begin by observing that (5.2.25a) and (5.2.25d) follow from parts (i) and (ii) of Lemma 5.2.19.
That (5.2.26a) holds can be seen in the following way. The operator 

\[
\begin{pmatrix}
\hat{X}(\lambda) & 0 \\
\hat{Y}(\lambda) & 1_Y
\end{pmatrix}
\]

maps \([\lambda]_U\) onto \(\text{dom}(S)\). Using this fact in (5.2.11b) we get

\[
gph \left( S - \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \right) = \text{rng} \left( \begin{pmatrix}
-1_X & 0 \\
\hat{X}(\lambda) & \hat{Y}(\lambda)
\end{pmatrix}
\right)
\]

\[
= \text{rng} \left( \begin{pmatrix}
-1_X & 0 \\
\hat{X}(\lambda) & \hat{Y}(\lambda)
\end{pmatrix}
\right)
\]

This gives (5.2.26a).

To prove (5.2.25b) and (5.2.26b) we observe from (5.2.11b) that 

\[
\begin{pmatrix}
x^0 \\
y
\end{pmatrix} \in \text{gph} \left( S - \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \right)
\]

if and only if

\[
\begin{pmatrix}
\hat{X}(\lambda) & 0 \\
\hat{Y}(\lambda) & 1_Y
\end{pmatrix}
\begin{pmatrix}
x^0 \\
y
\end{pmatrix} = \begin{pmatrix}
-1_X & \hat{Y}(\lambda) \\
\hat{X}(\lambda) & \hat{Y}(\lambda)
\end{pmatrix}
\begin{pmatrix}
x \\
u
\end{pmatrix}.
\]

Recall that for an arbitrary multi-valued operator \(T\) the condition \(y \in Tx\) is equivalent to the condition \(x \in T^{-1}y\). If we here replace \(T\) by \(\begin{pmatrix}
\hat{X}(\lambda) & 0 \\
\hat{Y}(\lambda) & 1_Y
\end{pmatrix}\), \(x\) by \(\begin{pmatrix}
x^0 \\
y
\end{pmatrix}\), and \(y\) by \(\begin{pmatrix}
-1_X & \hat{Y}(\lambda) \\
\hat{X}(\lambda) & \hat{Y}(\lambda)
\end{pmatrix}\), then we can rewrite the identity above into the equivalent condition

\[
\begin{pmatrix}
x^0 \\
y
\end{pmatrix} \in \text{gph} \left( S - \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \right)
\]

This gives (5.2.26b) and (5.2.25b).

The equality (5.2.25c) and the identity (5.2.26c) follow from (5.2.26b) since

\[
\begin{pmatrix}
\hat{X}(\lambda) & 0 \\
\hat{Y}(\lambda) & 1_Y
\end{pmatrix}
\begin{pmatrix}
S - \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \\
\hat{X}(\lambda) & \hat{Y}(\lambda)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1_X & \hat{Y}(\lambda) \\
\hat{X}(\lambda) & \hat{Y}(\lambda)
\end{pmatrix}
\begin{pmatrix}
\hat{X}(\lambda) & \hat{Y}(\lambda) \\
0 & \hat{Y}(\lambda)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & \hat{Y}(\lambda)
\end{pmatrix}
\]

Recall, in particular, that \(\begin{pmatrix}
\hat{X}(\lambda) & \hat{Y}(\lambda) \\
0 & \hat{Y}(\lambda)
\end{pmatrix}^{-1}\) maps \(\text{dom}(S)\) onto \(\begin{pmatrix} \lambda \\ \lambda \end{pmatrix}\), and that \(\begin{pmatrix}
\hat{X}(\lambda) & \hat{Y}(\lambda) \\
0 & \hat{Y}(\lambda)
\end{pmatrix}^{-1}\) maps \(\begin{pmatrix} \lambda \\ \lambda \end{pmatrix}\) onto \(\begin{pmatrix} \lambda \\ \lambda \end{pmatrix}\).

\[\square\]

5.2.22. Theorem. The multi-valued operator \(S: \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \supset \text{dom}(S) \rightarrow \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}\) is the system operator of a resolvable \(i/s/o\) node \(\Sigma = (S; \mathcal{X}, U, \mathcal{Y})\) if and only if the following five conditions hold:

(i) \(S\) is closed.

(ii) The operator \(\begin{pmatrix} 1_X & 0 \\ 0 & 0 \end{pmatrix}\) \(S\) is closed.

(iii) The main operator \(\hat{A}\) of \(S\) is closed and \(\rho(\hat{A}) \neq 0\).

(iv) \(\text{mul}(S) \cap \begin{pmatrix} 0 \\ \lambda \end{pmatrix} = \{0\} \)

(v) For every \(u \in \hat{U}\) there exists some \(x \in \mathcal{X}\) such that \(\begin{pmatrix} x \\ u \end{pmatrix} \in \text{dom}(S)\).
5.2.3. The i/s/o resolvent identity and i/s/o pseudo-resolvents. The i/s/o resolvent matrix $\mathcal{S}$ of an i/s/o node with $\rho(\Sigma) \neq \emptyset$ has some specific properties.

5.2.23. Theorem. Let $\Sigma = (S; X, U, Y)$ be a resolvable i/s/o node. Denote the i/s/o resolvent matrix of $\Sigma$ by $\mathcal{S}(\lambda) = \begin{bmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix}$.

(i) For every $\lambda, \mu \in \rho(\Sigma)$ we have

$$\mathcal{S}(\lambda) - \mathcal{S}(\mu) = \begin{bmatrix} \mu - \lambda & 0 \\ 0 & 0 \end{bmatrix} \mathcal{S}(\lambda) = \mathcal{S}(\lambda) \begin{bmatrix} \mu - \lambda & 0 \\ 0 & 0 \end{bmatrix} \mathcal{S}(\mu),$$

or equivalently,

$$\mathcal{A}(\lambda) - \mathcal{A}(\mu) = (\mu - \lambda)\mathcal{A}(\mu)\mathcal{A}(\lambda) = (\mu - \lambda)\mathcal{A}(\lambda)\mathcal{A}(\mu),$$

$$\mathcal{B}(\lambda) - \mathcal{B}(\mu) = (\mu - \lambda)\mathcal{A}(\mu)\mathcal{B}(\lambda) = (\mu - \lambda)\mathcal{A}(\lambda)\mathcal{B}(\mu),$$

$$\mathcal{C}(\lambda) - \mathcal{C}(\mu) = (\mu - \lambda)\mathcal{C}(\mu)\mathcal{A}(\lambda) = (\mu - \lambda)\mathcal{C}(\lambda)\mathcal{A}(\mu),$$

$$\mathcal{D}(\lambda) - \mathcal{D}(\mu) = (\mu - \lambda)\mathcal{C}(\mu)\mathcal{B}(\lambda) = (\mu - \lambda)\mathcal{C}(\lambda)\mathcal{B}(\mu).$$

(ii) For all $\lambda, \mu \in \rho(\Sigma)$,

$$\ker(\mathcal{A}(\lambda)) = \ker(\mathcal{A}(\mu)), \quad \ker(\mathcal{B}(\lambda)) = \ker(\mathcal{B}(\mu)), \quad \ker(\mathcal{C}(\lambda)) = \ker(\mathcal{C}(\mu)), \quad \ker(\mathcal{D}(\lambda)) = \ker(\mathcal{D}(\mu)).$$

$$\operatorname{rng}(\mathcal{A}(\lambda)) = \operatorname{rng}(\mathcal{A}(\mu)), \quad \operatorname{rng}(\mathcal{B}(\lambda)) = \operatorname{rng}(\mathcal{B}(\mu)), \quad \operatorname{rng}(\mathcal{C}(\lambda)) = \operatorname{rng}(\mathcal{C}(\mu)), \quad \operatorname{rng}(\mathcal{D}(\lambda)) = \operatorname{rng}(\mathcal{D}(\mu)).$$

$$\operatorname{rng}(\begin{bmatrix} \mathcal{B}(\lambda) - \mathcal{B}(\mu) \\ \mathcal{D}(\lambda) - \mathcal{D}(\mu) \end{bmatrix}) \subset \operatorname{rng}(\begin{bmatrix} \mathcal{A}(\lambda) \\ \mathcal{C}(\lambda) \end{bmatrix}),$$

$$\ker\left(\begin{bmatrix} \mathcal{A}(\lambda) \\ \mathcal{C}(\lambda) \end{bmatrix}\right) = \ker\left(\begin{bmatrix} \mathcal{A}(\mu) \\ \mathcal{C}(\mu) \end{bmatrix}\right),$$

$$\operatorname{rng}(\begin{bmatrix} \mathcal{A}(\lambda) \\ \mathcal{C}(\lambda) \end{bmatrix}) = \operatorname{rng}(\begin{bmatrix} \mathcal{A}(\mu) \\ \mathcal{C}(\mu) \end{bmatrix}).$$
(iii) For all \( \lambda, \mu \in \rho(\Sigma) \) the two operators
\[
\begin{bmatrix}
1_X + (\mu - \lambda) \mathcal{A}(\lambda) & (\mu - \lambda) \mathcal{B}(\lambda) \\
(\mu - \lambda) \mathcal{E}(\lambda) & 1_Y
\end{bmatrix}
\]
are boundedly invertible, and
\[
\begin{bmatrix}
1_X + (\mu - \lambda) \mathcal{A}(\lambda) & (\mu - \lambda) \mathcal{B}(\lambda) \\
0 & 1_Y
\end{bmatrix}^{-1}
= \begin{bmatrix}
1_X + (\lambda - \mu) \mathcal{A}(\mu) & (\lambda - \mu) \mathcal{B}(\mu) \\
0 & 1_Y
\end{bmatrix}, \quad \lambda, \mu \in \rho(\Sigma),
\]
(5.2.30)
\[
\begin{bmatrix}
1_X + (\mu - \lambda) \mathcal{A}(\lambda) & 0 \\
(\mu - \lambda) \mathcal{E}(\lambda) & 1_Y
\end{bmatrix}^{-1}
= \begin{bmatrix}
1_X + (\lambda - \mu) \mathcal{A}(\mu) & 0 \\
(\lambda - \mu) \mathcal{E}(\mu) & 1_Y
\end{bmatrix}, \quad \lambda, \mu \in \rho(\Sigma).
\]
Moreover, it is possible to solve \( \mathcal{S}(\lambda) \) in terms of \( \mathcal{S}(\mu) \) from 5.2.27 to get
\[
\mathcal{S}(\lambda) = \mathcal{S}(\mu) \begin{bmatrix}
1_X + (\lambda - \mu) \mathcal{A}(\mu) & (\lambda - \mu) \mathcal{B}(\mu) \\
0 & 1_Y
\end{bmatrix}^{-1}
\]
(5.2.31)
\[
= \begin{bmatrix}
1_X + (\lambda - \mu) \mathcal{A}(\mu) & 0 \\
(\lambda - \mu) \mathcal{E}(\mu) & 1_Y
\end{bmatrix}^{-1} \mathcal{S}(\mu), \quad \lambda, \mu \in \rho(\Sigma).
\]

(iv) \( \rho(\Sigma) \) is open, and \( \mathcal{S} \) is a \( \mathbb{B}([\lambda^\gamma]; [\gamma^\lambda]) \)-valued analytic function in \( \rho(\Sigma) \).

For each \( n \geq 1 \) the \( n \)-th order derivative \( \mathcal{S}^{(n)}(\lambda) \) of \( \mathcal{S} \) at the point \( \lambda \in \rho(\Sigma) \) are given by
\[
\mathcal{S}^{(n)}(\lambda) = (-1)^n n! \begin{bmatrix}
\mathcal{A}(\lambda) \\
\mathcal{E}(\lambda)
\end{bmatrix} \mathcal{A}(\lambda)^{n-1} \mathcal{B}(\lambda),
\]
(5.2.32)
or equivalently,
\[
\mathcal{S}^{(n)}(\lambda) = (-1)^n n! \mathcal{A}(\lambda)^{n+1},
\]
\[
\mathcal{S}^{(n)}(\lambda) = (-1)^n n! \mathcal{A}(\lambda)^{n} \mathcal{B}(\lambda),
\]
(5.2.33)
\[
\mathcal{S}^{(n)}(\lambda) = (-1)^n n! \mathcal{E}(\lambda) \mathcal{A}(\lambda)^n,
\]
\[
\mathcal{S}^{(n)}(\lambda) = (-1)^n n! \mathcal{E}(\lambda) \mathcal{A}(\lambda)^{n-1} \mathcal{B}(\lambda).
\]

**Proof.** (i) By 5.2.19, \[
\begin{bmatrix}
\mathcal{A}(\mu) & \mathcal{B}(\mu) \\
\mathcal{E}(\mu) & \mathcal{D}(\mu)
\end{bmatrix} \begin{bmatrix}
1_X + (\mu - \lambda) \mathcal{A}(\lambda) & (\mu - \lambda) \mathcal{B}(\lambda) \\
0 & 1_Y
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\
\mathcal{E}(\lambda) & \mathcal{D}(\lambda)
\end{bmatrix}.
\]
This is equivalent to the first set of identities in 5.2.28. To get the second set of identities in 5.2.28 it suffices to interchange \( \lambda \) and \( \mu \) with each other.

(ii) These identities and inclusions follow immediately from 5.2.27.

(iii) To show that the operators in 5.2.30 are inverses of each other in the case where both \( \lambda \in \rho(\Sigma) \) and \( \mu \in \rho(\Sigma) \) it suffices to multiply these operators (in either order) and to use 5.2.28 to show that the products are equal to \( \begin{bmatrix}
1_Y & 0 \\
0 & 1_Y
\end{bmatrix} \).

(iv) That \( \rho(\Sigma) \) is open follows from Theorem 5.2.16. The analyticity of \( \mathcal{S} \) follows from (iii) since the operators on the right-hand sides of 5.2.31 are analytic functions of \( \lambda \) (for fixed \( \mu \in \rho(\Sigma) \)). That 5.2.32 holds for \( n = 1 \) follows from 5.2.27 if we divide by \( \lambda - \mu \) and let \( \mu \rightarrow \lambda \). That 5.2.32 also holds for \( n > 1 \) can be proved by induction. \( \square \)
5.2.24. **Definition.** Let $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ be an $H$-spaces. A $\mathcal{B}(\mathcal{X}, \mathcal{Y})$-valued function $\hat{\Theta} := \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix}$ defined on some open set $\Omega$ is called an i/s/o pseudo-resolvent in $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ if it satisfies the equivalent i/s/o resolvent identities $(5.2.27)$ and $(5.2.28)$ for all $\lambda, \mu \in \Omega$.

5.2.25. **Corollary.** If $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is a resolvable i/s/o node, then the i/s/o node $\Sigma \hat{\Theta}$ is the restriction to $\Omega$ of the i/s/o resolvent matrix of a resolvable i/s/o node $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. Thus in particular, all the conclusions listed in Theorem 5.2.23 are valid for i/s/o pseudo-resolvent $\hat{\Theta}$ in $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ with $\rho(\Sigma)$ replaced by $\Omega$.

5.2.26. **Theorem.** Let $\hat{\Theta} := \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix}$ be an i/s/o pseudo-resolvent in $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ for some $H$-spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ and some open set $\Omega$ in $\mathbb{C}$. Then the following claims are true:

(i) $\hat{\Theta}$ is the restriction to $\Omega$ of the i/s/o resolvent matrix of a resolvable i/s/o node $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. Thus in particular, all the conclusions listed in Theorem 5.2.23 are valid for i/s/o pseudo-resolvent $\hat{\Theta}$ in $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ with $\rho(\Sigma)$ replaced by $\Omega$.

(ii) The i/s/o node $\Sigma$ in (i), and hence also the i/s/o pseudo-resolvent $\hat{\Theta}$ itself, is determined uniquely by the value of $\hat{\Theta}(\lambda)$ at some point $\lambda \in \Omega$.

(iii) The set $\rho(\Sigma)$, where $\Sigma$ is the i/s/o node in (i), is the maximal open set in $\mathbb{C}$ to which $\hat{\Theta}$ can be extended as an i/s/o pseudo-resolvent. More precisely, the i/s/o resolvent matrix of $\Sigma$ is an i/s/o pseudo-resolvent $\hat{\Theta}$ in $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \rho(A))$ whose restriction to $\Omega$ is equal to $\hat{\Theta}$, and there does not exist any i/s/o pseudo-resolvent in $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega')$ with $\Omega' \cap \sigma(\Sigma) \neq \emptyset$ whose restriction to $\Omega$ coincides with $\hat{\Theta}$.

**Proof.** (i) Fix $\lambda \in \Omega$, and let $S_\lambda$ be the closed multi-valued operator whose graph is equal to the right-hand side of $(5.2.12b)$. Then $\Sigma_\lambda = (S_\lambda; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o node with $\lambda \in \rho(\Sigma_\lambda)$, and $\hat{\Theta}(\lambda)$ is the value of the i/s/o resolvent matrix of $\Sigma_\lambda$ at the point $\lambda \in \Omega$.

If also $\mu \in \Omega$, then we can in the same way define an i/s/o node $\Sigma_\mu = (S_\mu; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ with $\mu \in \rho(\Sigma_\mu)$, such that $\hat{\Theta}(\mu)$ is the value of the i/s/o resolvent matrix of $\Sigma_\mu$ at the point $\mu \in \Omega$. In order to complete the proof it suffices to show that $\Sigma_\lambda = \Sigma_\mu$, or equivalently that $\text{gph} (S_\lambda) = \text{gph} (S_\mu)$.

The proof of $(5.2.30)$ which was given as a part of the proof of Theorem 5.2.23 was based entirely on the resolvent identity $(5.2.27)$, so by repeating the same procedure we find that $\begin{bmatrix} 1 + (\lambda - \mu)\hat{A}(\mu) \\ 0 \end{bmatrix} \begin{bmatrix} (\lambda - \mu)\hat{B}(\mu) \\ 1_{\mathcal{U}} \end{bmatrix}$ has a bounded inverse for every $\lambda, \mu \in \Omega$.

We can then solve $\hat{\Theta}(\lambda)$ in terms of $\hat{\Theta}(\mu)$ from $(5.2.27)$ to get

$$(5.2.34) \quad \hat{\Theta}(\lambda) = \hat{\Theta}(\mu) \begin{bmatrix} 1 + (\lambda - \mu)\hat{A}(\mu) \\ 0 \end{bmatrix} \begin{bmatrix} (\lambda - \mu)\hat{B}(\mu) \\ 1_{\mathcal{U}} \end{bmatrix}^{-1}.$$
Using this identity and the definitions of \( \text{gph} (S_\lambda) \) and \( \text{gph} (S_\mu) \) we get

\[
\text{gph} (S_\lambda) = \text{rng} \begin{pmatrix}
1_X & 0 & \lambda & 0 \\
0 & 1_Y & 0 & 0 \\
0 & 0 & 1_X & 0 \\
0 & 0 & 0 & 1_U
\end{pmatrix}
\begin{pmatrix}
-1_X & 0 \\
\mathcal{E}(\lambda) & \mathcal{D}(\lambda) \\
\mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\
0 & 1_U
\end{pmatrix}
= \text{rng} \begin{pmatrix}
1_X & 0 & \lambda & 0 \\
0 & 1_Y & 0 & 0 \\
0 & 0 & 1_X & 0 \\
0 & 0 & 0 & 1_U
\end{pmatrix}
\begin{pmatrix}
-1_X & 0 \\
\mathcal{E}(\lambda) & \mathcal{D}(\lambda) \\
\mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\
0 & 1_U
\end{pmatrix}
= \text{rng} \begin{pmatrix}
\mu \mathcal{A}(\mu) - 1_X & \mu \mathcal{B}(\mu) \\
\mathcal{E}(\mu) & \mathcal{D}(\mu) \\
\mathcal{A}(\mu) & \mathcal{B}(\mu) \\
0 & 1_U
\end{pmatrix}
= \text{gph} (S_\mu)
\]

Thus \( \text{gph} (S_\lambda) = \text{gph} (S_\mu) \) for all \( \lambda, \mu \in \Omega \).

(ii) That (ii) holds follows from the fact that an i/s/o node \( \Sigma \) is determined uniquely by the value of its i/s/o resolvent matrix evaluated at some point \( \lambda \in \rho(\Sigma) \).

(iii) By (i), the i/s/o resolvent of \( \Sigma \) is an i/s/o resolvent extension of \( \mathcal{S} \) to \( \rho(\Sigma) \). Conversely, if \( \mathcal{S} \) can be extended as a resolvent to some open set \( \Omega' \supset \Omega \), then it follows from (i) that \( \Omega' \subset \rho(\Sigma) \). Thus, \( \rho(\Sigma) \) is the maximal set to which \( \mathcal{S} \) can be extended as an i/s/o resolvent.

5.2.27. COROLLARY. If \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are two i/s/o pseudo-resolvents in \( (X, U, Y; \Omega_1) \) respectively \( (X, U, Y; \Omega_2) \) with \( \Omega_1 \cap \Omega_2 \neq \emptyset \), and if \( \mathcal{S}_1(\lambda) = \mathcal{S}_2(\lambda) \) at some point \( \lambda \in \Omega_1 \cap \Omega_2 \), then \( \mathcal{S}_1(\lambda) = \mathcal{S}_2(\lambda) \) for all \( \lambda \in \Omega_1 \cap \Omega_2 \). Moreover, if we define \( \mathcal{S} \) on \( \Omega := \Omega_1 \cup \Omega_2 \) by

\[
\mathcal{S}(\lambda) = \begin{cases} 
\mathcal{S}_1(\lambda), & \lambda \in \Omega_1, \\
\mathcal{S}_2(\lambda), & \lambda \in \Omega_2,
\end{cases}
\]

then \( \mathcal{S} \) is an i/s/o resolvent in \( (X, U, Y; \Omega) \).

PROOF. This follows from Theorem 5.2.26

5.2.4. The i/s/o resolvent matrices of transformed i/s/o nodes.

5.2.28. LEMMA. Let \( \Sigma = (S, X', U, Y) \) be a closed i/s/o node, and let \( \Sigma \overset{R}{=} (S, \overline{X}', U, Y) \) be the time reflected s/s node. Then \( \rho(\Sigma) \overset{R}{=} (\bar{\lambda} \mid \lambda \in \rho(\Sigma)) \). In particular, \( \rho(\Sigma) \overset{R}{=} (\emptyset) \) if and only if \( \rho(\Sigma) \overset{R}{=} (\emptyset) \), and when this is the case then the i/s/o resolvent matrices \( \mathcal{S} = \begin{pmatrix} \hat{S} & \hat{\mathcal{S}} \\ \hat{e} & \hat{Y} \end{pmatrix} \) and \( \mathcal{S} \overset{R}{=} \begin{pmatrix} \hat{\bar{\alpha}} & \hat{\mathcal{R}} \\ \hat{\bar{e}} & \hat{\bar{R}} \end{pmatrix} \) of \( \Sigma \) respectively \( \Sigma \overset{R}{=} \) satisfy

\[
(5.2.35) \quad \mathcal{S}(\bar{\lambda}) = \begin{pmatrix} \hat{\bar{S}} & \hat{\mathcal{S}} \\ \hat{e} & \hat{Y} \end{pmatrix} = \begin{pmatrix} -\hat{\alpha}(\bar{\lambda}) & \hat{\mathcal{S}}(\bar{\lambda}) \\ -\hat{e}(\bar{\lambda}) & \hat{Y}(\bar{\lambda}) \end{pmatrix}, \quad \bar{\lambda} \in \rho(\Sigma).
\]
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PROOF. If \( \lambda \in \rho(\Sigma) \), then by Definition 2.3.1 and Lemma 5.2.13

\[
\text{gph} \left( S \right) \left[ -\lambda \ 0 \ 0 \right] = \text{rng} \left( \begin{bmatrix}
1_{X} & 0 \\
\mathcal{C}(\lambda) & \mathcal{D}(\lambda) \\
\tilde{A}(\lambda) & \tilde{B}(\lambda) \\
0 & 1_{U}
\end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix}
-1_{X} & 0 \\
-\mathcal{C}(\lambda) & \mathcal{D}(\lambda) \\
-\tilde{A}(\lambda) & \tilde{B}(\lambda) \\
0 & 1_{U}
\end{bmatrix} \right).
\]

This together with Lemma 5.2.13 implies that \(-\lambda \in \rho(\Sigma) \). That also holds in this case follows from the same computation with \( \lambda \) replaced by \(-\lambda \).

By interchanging the roles of \( \Sigma \) and \( \Sigma \) in the above argument we find that if \( \lambda \in \rho(\Sigma) \), then \(-\lambda \in \rho(\Sigma) \). Thus, \( \lambda \in \rho(\Sigma) \) if and only if \(-\lambda \in \rho(\Sigma) \). \( \square \)

5.2.29. LEMMA. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o node, and let \( \Sigma_{P,Q,R} = (S_{P,Q,R}; \mathcal{X}_{1}, \mathcal{U}_{1}, \mathcal{Y}_{1}) \) be an i/s/o node which is \((P, Q, R)\)-similar to \( \Sigma \). Then \( \rho(\Sigma_{P,Q,R}) = \rho(\Sigma) \), and if these resolvent sets are nonempty, then the i/s/o resolvent matrices \( \mathcal{G} \) and \( \mathcal{H}_{P,Q,R} \) of \( \Sigma \) and \( \mathcal{H}_{P,Q,R} \) of \( \Sigma_{P,Q,R} \) satisfy

\[
\mathcal{H}_{P,Q,R}(\lambda) = \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \mathcal{G}(\lambda) \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}, \quad \lambda \in \rho(\Sigma).
\]

PROOF. By Definition 2.3.11 \( \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \mathcal{G}(\lambda) = \mathcal{H}_{P,Q,R}(\lambda) \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \). This together with Lemma 5.2.13 implies that if \( \lambda \in \rho(\Sigma) \), then

\[
\text{gph} \left( S_{P,Q,R} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q \end{bmatrix} \begin{bmatrix} -1_{X} & 0 \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \\ \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ 0 & 1_{U} \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} -1_{X} & 0 \\ R\mathcal{C}(\lambda) & R\mathcal{D}(\lambda) \\ P\tilde{A}(\lambda) & P\tilde{B}(\lambda) \\ 0 & Q \end{bmatrix} \begin{bmatrix} -1_{X} & 0 \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \\ \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ 0 & 1_{U} \end{bmatrix} \right).
\]

This together with Lemma 5.2.13 implies that \( \lambda \in \rho(\Sigma_{P,Q,R}) \) and that \( \rho(\Sigma_{P,Q,R}) = \rho(\Sigma) \).

5.2.30. LEMMA. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a resolvable i/s/o node, and let \( \Sigma_{1} = (S_{1}; \mathcal{X}_{1}, \mathcal{U}_{1}, \mathcal{Y}_{1}) \) be the static output feedback connection of \( \Sigma \) with feedback operator \( K \). Denote the i/s/o resolvent matrix of \( \Sigma \) by \( \mathcal{G} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \), and the i/s/o resolvent matrix of \( \Sigma_{1} \) by \( \mathcal{H}_{1} = \begin{bmatrix} \mathcal{A}_{1} & \mathcal{B}_{1} \\ \mathcal{C}_{1} & \mathcal{D}_{1} \end{bmatrix} \), and let \( \lambda \in \rho(\Sigma) \). Then \( \lambda \in \rho(\Sigma_{1}) \) if and only if \( 1_{U} - K\mathcal{D}(\lambda) \) has an inverse in \( \mathcal{B}(\mathcal{U}) \), or equivalently, if and only if \( 1_{Y} - \mathcal{D}(\lambda)K \) has an inverse in \( \mathcal{B}(\mathcal{Y}) \). If this is the case, then

\[
\mathcal{A}(\lambda) \begin{bmatrix} \mathcal{A}_{1}(\lambda) & \mathcal{B}_{1}(\lambda) \\ \mathcal{C}_{1}(\lambda) & \mathcal{D}_{1}(\lambda) \end{bmatrix}^{-1} = \mathcal{A}(\lambda) \begin{bmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} 1_{X} & 0 \\ -K\mathcal{C}(\lambda) & 1_{U} - K\mathcal{D}(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{bmatrix}^{-1}.
\]
or more explicitly,

\[
\hat{A}_1(\lambda) = \hat{A}(\lambda) + \hat{B}(\lambda)(1_{1U} - K\hat{D}(\lambda))^{-1}K\hat{E}(\lambda)
\]

\[
= \hat{A}(\lambda) + \hat{B}(\lambda)K(1_{1Y} - \hat{D}(\lambda)K)^{-1}\hat{E}(\lambda),
\]

(5.2.38) \[
\hat{B}_1(\lambda) = \hat{B}(\lambda)(1_{1U} - K\hat{D}(\lambda))^{-1} = \hat{B}(\lambda)[1_{1U} + K(1_{1Y} - \hat{D}(\lambda)K)^{-1}\hat{D}(\lambda)],
\]

\[
\hat{E}_1(\lambda) = [1_{1Y} + \hat{D}(\lambda)(1_{1U} - K\hat{D}(\lambda))^{-1}K]\hat{E}(\lambda) = (1_{1Y} - \hat{D}(\lambda)K)^{-1}\hat{E}(\lambda),
\]

\[
\hat{D}(\lambda) = \hat{D}(\lambda)(1_{1U} - K\hat{D}(\lambda))^{-1} = (1_{1Y} - \hat{D}(\lambda)K)^{-1}\hat{D}(\lambda).
\]

**Proof.** Let \(\lambda \in \rho(\Sigma)\). By using the representation (5.2.11b) of \(gph(S - [0 0])\) and the representation (2.3.10) of \(gph(S_1)\) we get

\[
gph\left(S_1 - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{rng} \left(\begin{bmatrix} -1_X & 0 \\ \hat{E}(\lambda) & \hat{D}(\lambda) \\ -K\hat{E}(\lambda) & 1_{1U} - K\hat{D}(\lambda) \end{bmatrix}\right),
\]

or equivalently,

\[
\begin{bmatrix} 0 & 0 & 1_X & 0 \\ -1_X & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{1U} \end{bmatrix} \text{gph}\left(S_1 - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{rng} \left(\begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{E}(\lambda) & \hat{D}(\lambda) \\ -K\hat{E}(\lambda) & 1_{1U} - K\hat{D}(\lambda) \end{bmatrix}\right).
\]

By Lemma 5.2.13 \(\lambda \in \rho(\Sigma_1)\) if and only if the right-hand side is the graph of a bounded linear operator, in which case this bounded linear operator is equal to \(\hat{S}_1(\lambda)\). We can now repeat the proof of Lemma 2.3.24 with \([\hat{A} \hat{B}]\) replaced by \(\hat{S}(\lambda)\) to show that is is true if and only \(1_{1U} - K\hat{D}(\lambda)\) has an inverse in \(B(\mathcal{U})\), or equivalently, if and only if \(1_{1Y} - \hat{D}(\lambda)K\) has an inverse in \(B(\mathcal{Y})\). Formulas (5.2.37) and (5.2.38) are obtained from formula (2.3.11) and (2.3.12) by simply replacing \([\hat{A} \hat{B}]\) by \([\hat{S}(\lambda) \hat{S}(\lambda)]\) and \([\epsilon \hat{S}_1]\).

5.2.31. **Lemma.** Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be an i/o node, and let \(\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1)\) be a bounded i/o extension of \(\Sigma\) with control operator \(B_1\), observation operator \(C_1\), and feedthrough operator \(D_1\). Then \(\rho(\Sigma_1) = \rho(\Sigma)\), and if these resolvent sets are nonempty, then the i/o resolvent matrices \(\hat{S} = [\hat{A} \hat{B}]\) and \(\hat{S}_1 = [\hat{A}_1 \hat{B}_1]\) of \(\Sigma\) respectively \(\Sigma_1\) satisfy

(5.2.39)

\[
\begin{bmatrix} \hat{A}_1(\lambda) & \hat{B}_1(\lambda) \\ \hat{E}_1(\lambda) & \hat{D}_1(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{E}(\lambda) & \hat{D}(\lambda) + D_{90} \\ C_1\hat{A}(\lambda) & C_1\hat{B}(\lambda) + D_{10} \end{bmatrix} \begin{bmatrix} \hat{A}(\lambda)B_1 \\ C_1\hat{A}(\lambda)B_1 + D_{11} \end{bmatrix}, \quad \lambda \in \rho(\Sigma) = \rho(\Sigma_1).
\]

**Proof.** By using the representation (5.2.12b) for \(gph(S)\) and the representation (2.3.17) for \(gph(S_1)\) we get for each \(\lambda \in \rho(\Sigma)\)

\[
gph(S_1) = \text{rng} \left(\begin{bmatrix} \lambda\hat{A}(\lambda) - 1_X & \lambda\hat{B}(\lambda) & B_1 \\ \hat{E}(\lambda) & \hat{D}(\lambda) + D_{90} & D_{01} \\ C_1\hat{A}(\lambda) & C_1\hat{B}(\lambda) + D_{10} & D_{11} \end{bmatrix}\right).
\]
The range of this block matrix operator does not change if it is multiplied by the invertible operator \[
\begin{bmatrix}
1_x & 0 & B_1 \\
0 & 1_u & 0 \\
0 & 0 & 1_{i_1}
\end{bmatrix}
\] to the right. Carrying out this multiplication we get
\[
\begin{pmatrix}
\lambda \hat{A}(\lambda) - 1_x & \lambda \hat{B}(\lambda) & \lambda \hat{A}(\lambda) B_1 \\
\hat{C}(\lambda) & \hat{D}(\lambda) + D_{00} & \hat{C}(\lambda) B_1 + D_{01} \\
C_1 \hat{A}(\lambda) & C_1 \hat{B}(\lambda) + D_{10} & C_1 \hat{A}(\lambda) B_1 + D_{11}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\hat{A}(\lambda) & \hat{B}(\lambda) & \hat{A}(\lambda) B_1 \\
0 & 1_u & 0 \\
0 & 0 & 1_{i_1}
\end{pmatrix}
\]
This representation is of the type (5.2.12b) with \(S\) replaced by \(S_1\) and \(\hat{\Sigma}(\lambda)\) replaced by the right-hand side of (5.2.39). By Lemma 5.2.13 \(\lambda \in \rho(\Sigma_1)\) and the fibers \(\hat{\Sigma}_1(\lambda)\) of the characteristic node bundle of \(\Sigma_1\) are given by (5.2.39) for all \(\lambda \in \rho(\Sigma)\). The converse inclusion \(\rho(\Sigma) \subset \rho(\Sigma_1)\) follows from the fact that \(\Sigma\) and \(\Sigma_1\) have the same s/s resolvent, and hence the same main operator (see Theorem 5.2.16).

5.2.5. The i/s/o resolvent matrices of interconnected i/s/o nodes.

5.2.32. Definition. By the cross product of two multi-valued operators \(A_1\) in \(X_1\) and \(A_2\) in \(X_2\) we mean the multivalued operator \(A_1 \times A_2\) in \(X := [X_1, X_2]\) whose graph is given by
\[(5.2.40)\]
\[
\text{gph}(A_1 \times A_2) = \left\{ \begin{bmatrix} [z_1, z_2] \\ x_1 \\ x_2 \end{bmatrix} \in [X_1, X_2, X] : \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \in \text{gph}(A_1), \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} \in \text{gph}(A_2), i = 1, 2 \right\}.
\]

5.2.33. Lemma. Let \(A := A_1 \times A_2\) be the cross product in \(X := [X_1, X_2]\) of the two multi-valued operators \(A_i\) in \(X_i, i = 1, 2\). Then \(\rho(A) = \rho(A_1) \cap \rho(A_2)\), and the resolvent \(\hat{A}\) of \(A\) is given in terms of the resolvents \(\hat{A}_i\) or \(\rho(A_1)\), \(i = 1, 2\), by
\[(5.2.41)\]
\[
\hat{A}(\lambda) = \begin{bmatrix}
\hat{A}_1(\lambda) & 0 \\
0 & \hat{A}_2(\lambda)
\end{bmatrix}, \quad \lambda \in \rho(\Sigma_1) \cap \rho(\Sigma_2).
\]

Proof. This follows from Lemma 5.2.2 and Definition 5.2.32.

5.2.34. Lemma. Let \(\Sigma_i = ([A \ B] ; X_i, U_i, Y_i), i = 1, 2\) be two resolvable i/s/o systems, and let \(\Sigma_x := \Sigma_1 \times \Sigma_2\) be the cross product of \(\Sigma_1\) and \(\Sigma_2\). Then the following claims are true.
(i) \(\rho(\Sigma_x) = \rho(\Sigma_1) \cap \rho(\Sigma_2)\). In particular, \(\Sigma_x\) is resolvable if and only if \(\rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset\).
(ii) If \(\rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset\) then the i/s/o resolvent matrix \(\hat{\Sigma}_x\) of \(\Sigma_x\) is given by
\[(5.2.42)\]
\[
\begin{pmatrix}
\hat{A}_1(\lambda) & 0 & \hat{B}_1(\lambda) & 0 \\
0 & \hat{A}_2(\lambda) & 0 & \hat{B}_2(\lambda) \\
\hat{C}_1(\lambda) & 0 & \hat{D}_1(\lambda) & 0 \\
0 & \hat{C}_2(\lambda) & 0 & \hat{D}_2(\lambda)
\end{pmatrix}, \quad \lambda \in \rho(\Sigma_1) \cap \rho(\Sigma_2).
\]

Proof. This follows from Definition 5.2.32 and Lemma 5.2.13.

5.2.35. Lemma. Let \(\Sigma_i = ([A \ B] ; X_i, U, Y), i = 1, 2\) be two resolvable i/s/o systems (with the same input and output spaces). Denote the parallel and difference connections of \(\Sigma_1\) and \(\Sigma_2\) by \(\Sigma_p := \Sigma_1 \mid \Sigma_2\) respectively \(\Sigma_d := \Sigma_1 \parallel \Sigma_2\). Then the following claims are true.
(i) $\Sigma_\parallel$ and $\Sigma_\perp$ have the same resolvent set, namely $\rho(\Sigma_\parallel) = \rho(\Sigma_\perp) = \rho(\Sigma_1) \cap \rho(\Sigma_2)$. In particular, $\Sigma_\parallel$ and $\Sigma_\perp$ are resolvable if and only if $\rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset$.

(ii) If $\rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset$ then the i/s/o resolvent matrices $\hat{S}_\parallel$ and $\hat{S}_\perp$ of $\Sigma_\parallel$ respectively $\Sigma_\perp$ are given by

$$\hat{S}_\parallel(\lambda) = \begin{bmatrix} \hat{A}_1(\lambda) & 0 & \hat{B}_1(\lambda) \\ \hat{C}_1(\lambda) & \hat{C}_2(\lambda) & \hat{D}_1(\lambda) + \hat{D}_2(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma_1) \cap \rho(\Sigma_2).$$

$$\hat{S}_\perp(\lambda) = \begin{bmatrix} \hat{A}_1(\lambda) & 0 & \hat{B}_1(\lambda) \\ \hat{C}_1(\lambda) & \hat{C}_2(\lambda) & \hat{D}_1(\lambda) - \hat{D}_2(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma_1) \cap \rho(\Sigma_2).$$

Proof. Below we prove only the claims about $\Sigma_\parallel$, and leave the analogous argument for $\Sigma_\perp$ to the reader.

It follows from Definitions 2.3.38 and 5.2.32 that the main operator of $\Sigma_\parallel$ is the cross product $A_1 \times A_2$ of the main operators $A_i$ of $\Sigma_i$, $i = 1, 2$. If $\rho(\Sigma_\parallel) \neq \emptyset$, then by Theorem 5.2.16 and Lemma 5.2.2, $\rho(\Sigma_\parallel) = \rho(A_1 \times A_2) = \rho(A_1) \cap \rho(A_2) = \rho(\Sigma_1) \cap \rho(\Sigma_2)$. Conversely, if $\lambda \in \rho(\Sigma_1) \cap \rho(\Sigma_2)$, then it follows from Definition 2.3.38 and Lemma 5.2.17 that $\lambda \in \rho(\Sigma_\parallel)$ and that (5.2.43) holds. \qed

5.2.36. Lemma. Let $\Sigma_1 = (S_1; X_1, U, Z)$ and $\Sigma_2 = (S_2; X_2, Z, Y)$ be two resolvable frequency domain i/s/o systems. Denote the cascade connection of $\Sigma_2$ and $\Sigma_1$ by $\Sigma = \Sigma_1 \circ \Sigma_2$. Then the following claims are true.

(i) $\rho(\Sigma) = \rho(\Sigma_1) \cap \rho(\Sigma_2)$. In particular, $\Sigma$ is resolvable if and only if $\rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset$.

(ii) If $\rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset$ then the i/s/o resolvent matrix $\hat{S}_\circ$ of $\Sigma$ is given by

$$\hat{S}_\circ(\lambda) = \begin{bmatrix} \hat{A}_2(\lambda) & \hat{B}_2(\lambda)\hat{C}_1(\lambda) & \hat{B}_2(\lambda)\hat{D}_1(\lambda) \\ 0 & \hat{A}_1(\lambda) & \hat{B}_1(\lambda) \\ \hat{C}_2(\lambda) & \hat{D}_2(\lambda)\hat{C}_1(\lambda) & \hat{D}_2(\lambda)\hat{D}_1(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma_1) \cap \rho(\Sigma_2).$$

Proof. The proof is analogous to the proof of Lemma 5.2.35. \qed

5.2.6. The resolvent family of bounded i/s/o nodes.

5.2.37. Definition. Let $\Sigma = (S; X, U, Y)$ be a resolvable i/s/o node with i/s/o resolvent matrix $\hat{S} = \begin{bmatrix} A & \tilde{B} \\ \tilde{C} & D \end{bmatrix}$. By the resolvent family of bounded i/s/o nodes induced by $\Sigma$ we mean the family of bounded i/s/o nodes $\Sigma^\lambda = (\hat{S}(\lambda); X, U, Y)$, $\lambda \in \rho(\Sigma)$.

5.2.38. Lemma. The system operators of the different members of the resolvent family $\Sigma^\lambda = (\hat{S}(\lambda); X, U, Y)$, $\lambda \in \rho(\Sigma)$, of a resolvable i/s/o node $\Sigma = (S; X, U, Y)$ are connected to each other via the identity (5.2.27).

Proof. See Theorem 5.2.23. \qed

5.2.39. Lemma. Let $\Sigma = (S; X, U, Y)$ and $\Sigma_i = (S_i; X_i, U_i, Y_i)$, $i = 1, 2$, be resolvable i/s/o systems, and let $\Sigma^\lambda = (\hat{S}(\lambda); X, U, Y)$ and $\Sigma_i^\lambda = (\hat{S}_i(\lambda); X_i, U_i, Y_i)$, $\lambda \in \rho(\Sigma_i)$, be the frequency domain families of bounded i/s/o nodes induced by $\Sigma$.
respectively \( \Sigma_i, i = 1, 2 \). Suppose, in addition that \( \rho(\Sigma) \cap \rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset \). Then the following claims are true:

(i) \( \Sigma_1 \) is \((P,Q,R)\)-similar to \( \Sigma \) if and only if \( \rho(\Sigma_1) = \rho(\Sigma) \) and \( \Sigma^\lambda_1 \) is \((P,Q,R)\)-similar to \( \Sigma^\lambda \) for some \( \lambda \in \rho(\Sigma) = \rho(\Sigma_1) \) (and hence for all \( \lambda \in \rho(\Sigma) = \rho(\Sigma_1) \)).

(ii) \( \Sigma_1 \) is the static output feedback connection of \( \Sigma \) if and only if \( \Sigma^\lambda_1 \) is the static output feedback connection of \( \Sigma^\lambda \) for some \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \) (and hence for all \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \)).

(iii) \( \Sigma_1 \) is the bounded i/o extension of \( \Sigma \) with control operator \( C_1 \), observation operator \( \hat{A}(\lambda) \), and feedthrough operator \( \hat{\Sigma}(\lambda) \) if and only if \( \rho(\Sigma_1) = \rho(\Sigma) \) and \( \Sigma^\lambda_1 \) is the bounded i/o extension of \( \Sigma^\lambda \) with control operator \( \hat{A}(\lambda) \), observation operator \( \hat{A}(\lambda) \), and feedthrough operator \( \hat{\Sigma}(\lambda) \) such that \( \rho(\Sigma) = \rho(\Sigma_1) \) (and hence for all \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \)).

(iv) \( \Sigma \) is the cross product of \( \Sigma_1 \) and \( \Sigma_2 \) if and only if \( \rho(\Sigma) = \rho(\Sigma_1) \cap \rho(\Sigma_2) \) and \( \Sigma^\lambda \) is the cross product of \( \Sigma^\lambda_1 \) and \( \Sigma^\lambda_2 \) for some \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \) (and hence for all \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \)).

(v) \( \Sigma \) is the parallel, difference, or cascade connection of \( \Sigma_1 \) and \( \Sigma_2 \) if and only if \( \rho(\Sigma) = \rho(\Sigma_1) \cap \rho(\Sigma_2) \) and \( \Sigma^\lambda \) is the parallel, difference, or cascade connection, respectively, of \( \Sigma^\lambda_1 \) and \( \Sigma^\lambda_2 \) for some \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \) (and hence for all \( \lambda \in \rho(\Sigma) \cap \rho(\Sigma_1) \)).

**Proof.** This follows from Lemmas \ref{lem:5.2.13} \ref{lem:5.2.29} \ref{lem:5.2.34} \ref{lem:5.2.35} \ref{lem:5.2.36} \ref{lem:5.2.30} and \ref{lem:5.2.31}.

**5.2.7. A finite-dimensional non-regular resolvable i/s/o system.** The notion of (non-regular) i/s/o nodes is useful already in the finite dimensional setting in the case where one wants to study nodes whose i/o resolvents have poles at infinity. For example, it is possible to construct a finite-dimensional resolvable i/s/o node \( \Sigma = (S; C; C; C) \) such that the i/o resolvent \( \hat{\Sigma} \) of \( \Sigma \) is given by \( \hat{\Sigma}(\lambda) = \lambda, \lambda \in \mathbb{C} \). This can be done as follows. First we construct an ordinary i/s/o node \( \Sigma_1 = (S_1; C; C; C) \) such that its i/s transfer function is \( \hat{\Sigma}_1(\lambda) = 1/\lambda, \lambda \neq 0 \). For example, we may take \( S_1 = [0 \ 1], \sigma(\Sigma_1) = \{0\} \), so that \ref{eq:2.1.12} becomes

\[
(5.2.45) \quad \dot{x}(t) = u(t), \quad y(t) = x(t), \quad t \in I.
\]

This is a bounded i/s/o node with main operator 0, so \( \rho(\Sigma_1) = \mathbb{C} \setminus \{0\} \) and \( \sigma(\Sigma_1) = \{0\} \). It follows from Corollary \ref{cor:5.1.8} that the i/s/o resolvent matrix of this node is \( \hat{\Sigma}_1(\lambda) = \begin{bmatrix} 1/\lambda & 1/\lambda \\ 1/\lambda & 1/\lambda \end{bmatrix} \). This can also be seen as follows: For all \( \lambda \neq 0 \) the graph of \( S_1 \) is given by

\[
\text{gph}(S_1) = \text{rng} \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1/\lambda & 1/\lambda \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1/\lambda & 1/\lambda \\ 1/\lambda & 1/\lambda \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} 0 & 1 \\ 1/\lambda & 1/\lambda \end{bmatrix} \right),
\]
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which is a representation of the type [5.2.12] of \( \text{gph}(S_1) \) with \( \hat{\Theta}_1(\lambda) = \begin{bmatrix} 1/\lambda & 1/\lambda \\ 1/\lambda & 1/\lambda \end{bmatrix} \).

To get an i/s/o node whose i/o resolvent matrix is \( \hat{\Theta}(\lambda) = \lambda \) we simply interchange the roles of the two variables \( u \) and \( y \), i.e., we reinterpret the graph of \( S_1 \) as a graph of the multi-valued operator \( S \) which maps \( [y] \) into \( [\dot{y}] \). Thus, the graph of \( S \) is given by \( \text{gph}(S) = \text{rng} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \). From here we see that \( \text{dom}(S) = \{ [x] \mid x \in \mathbb{C} \} \) and \( \text{rng}(S) = \text{mul}(S) = \{ [\dot{x}] \mid x \in \mathbb{C} \} \). In particular, \( S \) is not single-valued, and the domain of \( S \) is not dense in \( \mathbb{C}^2 \). The main operator \( A \) of \( S \) is purely multi-valued with \( \text{dom}(A) = \{ \} \) and \( \text{mul}(A) = \mathbb{C} \). However, we claim that \( \Sigma = (S; \mathbb{C}; \mathbb{C}; \mathbb{C}) \) is an i/s/o node with \( \rho(\Sigma) = \mathbb{C} \) and with i/s/o resolvent matrix equal to \( \hat{\Theta}(\lambda) = \begin{bmatrix} 0 & 1 \\ -\lambda & -\lambda \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \), \( \lambda \in \mathbb{C} \). An easy way to see that in the case of a classical trajectory the output \( u \) must be equal to the derivative of the input \( y \) is the following: The condition \( [x(t) \ y(t)] \in \text{dom}(S) = \{ [x] \mid x \in \mathbb{C} \} \) is equivalent to the condition \( x(t) = y(t) \), and the condition \( [\dot{x}(t) \ u(t)] \in \text{rng}(S) = \{ [\dot{x}] \mid x \in \mathbb{C} \} \) is equivalent to the condition \( \dot{x}(t) = u(t) \). Thus \( u(t) = \dot{y}(t) \).

It is possible to interpret the above example as a state/signal node \( \Sigma = (V; \mathbb{C}; \mathbb{C}^2) \) by taking the signal space to be \( \mathcal{W} = \mathbb{C}^2 \) and the generating subspace \( V \) to be given by \( [2.2.1] \). Note that \( V \) satisfies conditions (i)–(iv) in Definition [1.1.15] and hence \( \Sigma \) is a bounded s/s node. The above representation of the signal space is not i/s/o-admissible. However, it is frequency domain i/s/o-admissible for \( \Sigma \) in the sense of Definition [5.3.8] below.
5.3. Resolvable State/Signal Nodes (Jan 02, 2016)

In this section we introduce the notions of the resolvent set of a closed s/s node and define certain resolvents related to the node. We also investigate the relationships between the resolvent set and resolvents of a s/s node and the resolvent sets and i/s/o resolvent matrices of its i/s/o representations.

5.3.1. The resolvent set of a closed state/signal node. Before defining what we mean by the resolvent set of a closed s/s node, let us recall the corresponding definition for a regular i/s/o system. To motivate that definition we started by assuming that \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a Laplace transformable classical future trajectory of the i/s/o system \( \Sigma_{i/s/o} = (\mathcal{S}, \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) with initial state \( x(0) = x^0 \). By taking Laplace transforms in the equation (2.1.1) which describes the dynamics of \( \Sigma_{i/s/o} \) we arrived at the equation (5.1.7), which is valid for all those \( \lambda \in \mathbb{C} \) for which the Laplace transforms converge. In this equation we decided to interpret the initial state \( x^0 \) and the Laplace transform \( \hat{u}(\lambda) \) of the input as “given data” which can be chosen to be an arbitrary vector in \( [x, u] \), and required the remaining data, namely the Laplace transforms \( \hat{x}(\lambda) \) and \( \hat{y}(\lambda) \) of the state respectively output to be completely determined by \( \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix} \) and to depend continuously on \( \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix} \). The resolvent set \( \rho(\Sigma) \) was defined to be the set of point for which the above conditions hold.

In the case of a closed s/s system \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) we can proceed in the same way. As shown in Section 1.6, if \( \begin{bmatrix} x^0 \\ \hat{w}(\lambda) \end{bmatrix} \) is a Laplace transformable classical future trajectory of \( \Sigma \) with initial state \( x(0) = x^0 \), then by taking Laplace transforms in the equation (1.1.1) which describes the dynamics of \( \Sigma \) we arrived at the two equivalent equations

\begin{align}
\begin{bmatrix}
\lambda \hat{x}(\lambda) - x^0 \\
\hat{x}(\lambda) \\
\hat{w}(\lambda)
\end{bmatrix} \in V,
\end{align}

and

\begin{align}
\begin{bmatrix}
x^0 \\
\hat{x}(\lambda) \\
\hat{w}(\lambda)
\end{bmatrix} \in \tilde{\mathcal{E}}(\lambda),
\end{align}

where the characteristic node bundle \( \tilde{\mathcal{E}}(\lambda) \) is defined by

\begin{align}
\tilde{\mathcal{E}}(\lambda) = \left\{ \begin{bmatrix} x \\ z \\ w \end{bmatrix} \in \mathbb{R} \left| \begin{bmatrix} \lambda z - x \\ z \\ w \end{bmatrix} \in V \right. \right\}, \quad \lambda \in \mathbb{C}.
\end{align}

In equations (1.6.2) and (1.6.3) it is still possible to interpret the initial state \( x^0 \) as given data which can take any value in \( \mathcal{X} \), and to interpret the Laplace transform \( \hat{x}(\lambda) \) of the state as “dependent data” which is completely determined by \( x^0 \) and the Laplace transform \( \hat{w}(\lambda) \) of the signal. However, since \( \hat{w}(\lambda) \) is partly an input and partly an output, \( \hat{w}(\lambda) \) will be neither “free” or “dependent”. This leaves the following three natural conditions that should be satisfied at the point \( \lambda \in \mathbb{C} \) in order for this point to belong to the resolvent set of \( \Sigma \):

(i) To each vector \( \begin{bmatrix} x^0 \\ \hat{w}(\lambda) \end{bmatrix} \in [x, w] \) there is at most one vector \( \hat{x}(\lambda) \in \mathcal{X} \) such that conditions (1.6.2) and (1.6.3) hold. In other words, \( \hat{x}(\lambda) \) is determined uniquely by \( x^0 \) and \( \hat{w}(\lambda) \).
(ii) To each vector $x^0 \in \mathcal{X}$ there exists at least one vector $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in [\mathcal{X} \mathcal{W}]$ such that conditions (1.6.2) and (1.6.3) hold. In other words, $x^0$ is “free”.

(iii) The vector $\hat{x}(\lambda)$ in (1.6.2) and (1.6.3) depends continuously on $x^0$ and $\hat{w}(\lambda)$.

If conditions (i) above hold, then we can write $\hat{x}(\lambda)$ as a linear function of $\begin{bmatrix} x^0 \\ \hat{w}(\lambda) \end{bmatrix}$, i.e., there exists a linear operator $\hat{L}(\lambda)$ such that (1.6.2) and (1.6.3) hold if and only if $\hat{x}(\lambda) = \hat{L}(\lambda) \begin{bmatrix} x^0 \\ \hat{w}(\lambda) \end{bmatrix}$. Condition (ii) says that for every $x^0 \in \mathcal{X}$ there exists some $\hat{w}(\lambda) \in \mathcal{W}$ such that $\begin{bmatrix} x^0 \\ \hat{w}(\lambda) \end{bmatrix} \in \text{dom}(\hat{L}(\lambda))$. Finally, condition (iii) says that $\hat{L}(\lambda)$ is bounded. Since 

\[
5.3.1 \quad \text{gph}(\hat{L}(\lambda)) = \begin{bmatrix} 0 & 1_X & 0 \\ 1_X & 0 & 0 \\ 0 & 0 & 1_W \end{bmatrix} \hat{E}(\lambda) = \begin{bmatrix} 0 & 1_X & 0 \\ -1_X & \lambda & 0 \\ 0 & 0 & 1_W \end{bmatrix} V
\]

and $V$ is closed, the operator $\hat{L}(\lambda)$ is closed, and hence it is continuous if and only if $\text{dom}(\hat{L}(\lambda))$ is closed in $[\mathcal{X} \mathcal{W}]$.

By reformulating the above conditions directly in terms of the characteristic node bundle $\hat{E}$ we arrive at conditions (a)–(c) in the following definition:

5.3.1. Definition. Let $\Sigma = (V; \mathcal{W}, \mathcal{Y})$ be a closed s/s node.

(i) A point $\lambda \in \mathbb{C}$ belongs to the resolvent set of $\Sigma$ if the fiber $\hat{E}(\lambda)$ of the characteristic node bundle of $\Sigma$ at the point $\lambda$ satisfies the following three conditions:

(a) $\hat{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} = \{0\}$.

(b) For every $x^0 \in \mathcal{X}$ there exist some $x_\lambda \in \mathcal{X}$ and $w_\lambda \in \mathcal{W}$ such that $\begin{bmatrix} x^0 \\ w_\lambda \end{bmatrix} \in \hat{E}(\lambda)$.

(c) The set $\left\{ \begin{bmatrix} x^0 \\ w_\lambda \end{bmatrix} \in [\mathcal{X} \mathcal{W}] \mid \begin{bmatrix} x^0 \\ w_\lambda \end{bmatrix} \in \hat{E}(\lambda) \text{ for some } x_\lambda \in \mathcal{X} \right\}$ is closed.

This set of points is denoted by $\rho(\Sigma)$.

(ii) The spectrum of $\Sigma$ is the complement of the resolvent set of $\Sigma$. This set is denoted by $\sigma(\Sigma)$.

(iii) By a resolvable s/s node we mean a closed s/s node with a nonempty resolvent set.

The resolvent set of a s/s node can also be characterized in the following way:

5.3.2. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a closed s/s node with characteristic node and signal bundles $\hat{E}$ respectively $\hat{F}$.

(i) For each $\lambda \in \mathbb{C}$ the set

\[
5.3.2 \quad V^\lambda := \begin{bmatrix} 0 & 1_X & 0 \\ 1_X & 0 & 0 \\ 0 & 0 & 1_W \end{bmatrix} \hat{E}(\lambda) = \begin{bmatrix} 0 & 1_X & 0 \\ -1_X & \lambda & 0 \\ 0 & 0 & 1_W \end{bmatrix} V
\]

is a closed subspace of $[\mathcal{X} \mathcal{W}]$. Thus, $\Sigma_\lambda = (V^\lambda; \mathcal{X}, \mathcal{W})$ is a closed s/s node.

(ii) For each $\lambda \in \mathbb{C}$ the following conditions are equivalent:

(a) $\lambda \in \rho(\Sigma)$

(b) The s/s node $\Sigma_\lambda$ defined in (i) is bounded.
Conditions (a) and (b) in Definition 5.3.1(i) hold and \( \tilde{F}(\lambda) \) is closed.

The right-hand side of (5.3.1) is the graph of a continuous linear operator \( \tilde{L}(\lambda) \) with closed domain satisfying

\[
(5.3.3) \quad \text{for every } x \in X \text{ there is some } w \in W \text{ such that } \begin{bmatrix} x \\ w \end{bmatrix} \in \text{dom} \left( \tilde{L}(\lambda) \right).
\]

In particular, \( \Sigma \) is resolvable if and only if the equivalent conditions (a)–(d) above hold for at least one point \( \lambda \in \mathbb{C} \) (in which case they hold for all \( \lambda \in \rho(\Sigma) \)).

**Proof.** (i) That \( V^\lambda \) is closed follows from (5.3.2) and the fact that \( V \) is closed.

(a) \( \iff \) (b): This follows from Definitions 1.1.15 and 5.3.1.

(b) \( \iff \) (c): This follows from part (i) of Theorem 2.2.27 applied to the s/s node \( \Sigma_\lambda \).

(b) \( \iff \) (d): This follows from Lemma 1.1.17 applied to the s/s node \( \Sigma_\lambda \). \( \square \)

In practice, if one wants to check whether a given s/s node \( \Sigma \) has a nonempty resolvent set, and to compute this resolvent set, then the most useful part of Lemma 5.3.2 seems to be the equivalence of (a) and (c). For easy reference, let us therefore restate this particular result as follows:

5.3.3. **Corollary.** Let \( \Sigma = (V; W, Y) \) be a s/s node or node with characteristic node bundle \( \tilde{E} \) and characteristic signal bundle \( \tilde{F} \). Then a point \( \lambda \in \mathbb{C} \) belongs to the resolvent set of \( \Sigma \) if and only if the following three conditions hold:

(i) \( \tilde{E}(\lambda) \cap \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \{ 0 \} \).

(ii) For every \( x^0 \in X \) there exist some \( x_\lambda \in X \) and \( w_\lambda \in W \) such that \( \begin{bmatrix} x^0 \\ w_\lambda \end{bmatrix} \in \tilde{E}(\lambda) \).

(iii) \( \tilde{F}(\lambda) \) is closed.

In particular, \( \Sigma \) is resolvable if and only if conditions (i)–(iii) above hold for at least one point \( \lambda \in \mathbb{C} \) (in which case they hold for all \( \lambda \in \rho(\Sigma) \)).

**Proof.** This is the equivalence of (a) and (c) in part (ii) of Lemma 5.3.2. \( \square \)

5.3.4. **Lemma.** Let \( \Sigma = (V; X, W) \) be a closed s/s node.

(i) Let \( \Sigma_{\text{R}} = (V_{\text{R}}; X_{\text{R}}, W) \) be the time reflection of \( \Sigma \). Then \( \rho(\Sigma_{\text{R}}) = \{-\lambda \mid \lambda \in \rho(\Sigma)\} \).

In particular, \( \rho(\Sigma_{\text{R}}) \neq \emptyset \) if and only if \( \rho(\Sigma) \neq \emptyset \).

(ii) Let \( \Sigma_{P,Q,R} = (S_{P,Q}; X_{1}, W_{1}) \) be a s/s node which is \( (P, Q) \)-similar to \( \Sigma \).

Then \( \rho(\Sigma_{P,Q}) = \rho(\Sigma) \). In particular, \( \rho(\Sigma_{P,Q}) \neq \emptyset \) if and only if \( \rho(\Sigma) \neq \emptyset \).

**Proof.** This follows from Lemmas 1.6.8 and 1.6.9 and Corollary 5.3.3. \( \square \)

Although there do exist s/s nodes which have an empty resolvent set (see Example 5.3.5 below), in this book we are mainly interested in s/s nodes which have a nonempty resolvent set. As we shall prove in Theorem 5.3.9 below, also the resolvent set of an s/s node is open (like the resolvent sets of single-valued or multi-valued operators, and the resolvent sets of i/s/o nodes).

5.3.5. **Example.** Let \( \Sigma_i = (V_i, X_i, W_i) \), \( i = 1, 2, \ldots, 16 \) be the s/s node induced by the i/s/o node \( \Sigma^3_i \) in Example 5.1.35. Then

(i) \( \Sigma_j \) is bounded (and hence \( \rho(\Sigma_j) \neq \emptyset \)) for \( j = 5, 6, 8, \) and 9;

(ii) \( \Sigma_j \) is not bounded but \( \rho(\Sigma_j) \neq \emptyset \) for \( j = 1, 3, 7, 10, 12, 13, 14, \) and 15;
(iii) $\rho(T) \subset \rho(\Sigma_j)$ for $j = 1, 3, 7, 9, 10, 12, 13, 14, \text{ and } 15$, and $\mathbb{C} \setminus \{0\} \subset \rho(\Sigma_j)$ for $j = 5, 6, 8$;
(iv) $\Sigma_j$ is unbounded and $\rho(\Sigma_j) = \emptyset$ for $j = 2$ and 4;
(v) $\Sigma_{11}$ is not bounded, and $\rho(\Sigma_{11}) \neq \emptyset$ if and only if $T^2$ is closed (in which case $\mathbb{C} \setminus \{0\} \subset \rho(\Sigma_{11})$);
(vi) $\Sigma_{16}$ is not bounded, and $\rho(\Sigma_{16}) \neq \emptyset$ if and only if $\frac{1}{\lambda}T^2 + T$ is closed for some $\lambda \neq 0$ (in which case all $\lambda \neq 0$ for which $\frac{1}{\lambda}T^2 + T$ is closed belongs to $\rho(\Sigma_{16})$).

The analysis below of this example is based on Theorem 5.2.3 and Corollary 5.3.3. This means that we need the characteristic node bundles of each of the given examples. These are listed below for $\lambda \in \mathbb{C}$:

\[
\begin{align*}
\hat{\mathcal{E}}_1(\lambda) &= \begin{bmatrix} \lambda - T & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} [\text{dom}(T)]^T X, \\
\hat{\mathcal{E}}_2(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} [\text{dom}(T)]^T X, \\
\hat{\mathcal{E}}_3(\lambda) &= \begin{bmatrix} \lambda - T & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} [\text{dom}(T)]^T X, \\
\hat{\mathcal{E}}_4(\lambda) &= \begin{bmatrix} \lambda - T & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} [\text{dom}(T)]^T X, \\
\hat{\mathcal{E}}_5(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} [\text{dom}(T)]^T X, \\
\hat{\mathcal{E}}_6(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} [\text{dom}(T)]^T X, \\
\hat{\mathcal{E}}_7(\lambda) &= \begin{bmatrix} \lambda - T & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(S_7), \\
\hat{\mathcal{E}}_8(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(S_8), \\
\hat{\mathcal{E}}_9(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(S_9), \\
\hat{\mathcal{E}}_{10}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T), \\
\hat{\mathcal{E}}_{11}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T), \\
\hat{\mathcal{E}}_{12}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T), \\
\hat{\mathcal{E}}_{13}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T), \\
\hat{\mathcal{E}}_{14}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T), \\
\hat{\mathcal{E}}_{15}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T), \\
\hat{\mathcal{E}}_{16}(\lambda) &= \begin{bmatrix} \lambda & 0 \\ 1_x & 0 \\ 0 & 1_x \end{bmatrix} \text{dom}(T).
\end{align*}
\]

We shall also need the signal bundles of these s/s nodes at some points $\lambda \in C$. To get these we set the first component in the characteristic signal bundle to be equal to zero, and project the result onto the signal space $\mathcal{W} = [X]$. The values of the different characteristic signal bundles are listed below under the additional assumption that $\lambda \in \rho(T)$ if $i = 1, 3, 7, 9, 10, 12, 13, 14, 15,$ and $\lambda \neq 0$ if $i = 2, 4, 5, 6, 8, 11,$ and $16$ (in other words, $\lambda$ belong to the resolvent set of the formal main
operator (which is either $T$ or 0) of the i/o node $\Sigma_i^{i/o}$:

\[
\begin{align*}
\hat{\mathcal{X}}_1(\lambda) &= \hat{\mathcal{X}}_2(\lambda) = \hat{\mathcal{X}}_3(\lambda) = \hat{\mathcal{X}}_7(\lambda) = [\{0\} X], \\
\hat{\mathcal{X}}_4(\lambda) &= \mathrm{gph}(\lambda T - T^{-1}) , \\
\hat{\mathcal{X}}_5(\lambda) &= \hat{\mathcal{X}}_6(\lambda) = \hat{\mathcal{X}}_8(\lambda) = \hat{\mathcal{X}}_{10}(\lambda) = \hat{\mathcal{X}}_{14}(\lambda) = \hat{\mathcal{X}}_{15}(\lambda) = \hat{\mathcal{X}}_{16}(\lambda) = \mathrm{gph}(T) , \\
\hat{\mathcal{X}}_9(\lambda) &= \mathrm{gph}(\lambda T - T^{-1}) , \\
\hat{\mathcal{X}}_{11}(\lambda) &= \mathrm{gph}(\frac{1}{2}T^2) , \\
\hat{\mathcal{X}}_{12}(\lambda) &= \mathrm{gph}(\lambda T - T^{-1} - 2T) , \\
\hat{\mathcal{X}}_{13}(\lambda) &= \mathrm{gph}(\lambda T - T^{-1} - T) , \\
\hat{\mathcal{X}}_{16}(\lambda) &= \mathrm{gph}(\frac{1}{2}T^2 + T) .
\end{align*}
\]

(5.3.5)

Finally, if we want to check which of these s/s nodes are bounded we also need the different sets $\mathcal{X}_0$ in (1.1.2) and $\mathcal{W}_0$ in (1.1.7), which are listed below:

(5.3.6)

\[
\begin{align*}
\mathcal{X}_0^1 &= \mathcal{X}_0^2 = \mathcal{X}_0^3 = \mathcal{X}_0^{10} = \mathcal{X}_0^{11} = \mathcal{X}_0^{12} = \mathcal{X}_0^{13} = \mathcal{X}_0^{14} = \mathcal{X}_0^{15} = \mathcal{X}_0^{16} = \mathrm{dom}(T) , \\
\mathcal{X}_0^4 &= \mathcal{X}_0^5 = \mathcal{X}_0^6 = \mathcal{X}_0^7 = \mathcal{X}_0^8 = \mathcal{X}_0^9 = \mathcal{X} , \\
\mathcal{W}_0^0 &= \mathcal{W}_0^3 = \mathcal{W}_0^4 = \mathcal{W}_0^5 = \mathcal{W}_0^6 = \mathcal{W}_0^7 = \mathcal{W}_0^{11} = \mathcal{W}_0^{13} = \{0\}_{\mathrm{dom}(T)} , \\
\mathcal{W}_0^5 &= \mathcal{W}_0^6 = \mathcal{W}_0^8 = \mathcal{W}_0^9 = \mathcal{W}_0^{10} = \mathcal{W}_0^{14} = \mathcal{W}_0^{15} = \mathcal{W}_0^{16} = \mathrm{gph}(T) , \\
\mathcal{W}_0^{12} &= \mathrm{gph}(-T) .
\end{align*}
\]

From the above lists combined with Theorem 2.2.27 and Corollary 5.3.3 we can draw the following conclusions.

(i) $\Sigma_i$ is bounded (and hence $\rho(\Sigma_i) \neq \emptyset$) if and only if $i = 5, 6, 8,$ or $9$, because these are the values of $i$ for which $\mathcal{X}_0^i = \mathcal{X}$ and $\mathcal{W}_0^i$ is closed.

(ii) All the examples satisfy condition (ii) in Corollary 5.3.3 for either all $\lambda \in \rho(T)$ or for all $\lambda \neq 0$ (depending on whether the formal main operator of $\Sigma_i^{i/o}$ is $T$ or 0).

(iii) $\rho(\Sigma_2) = \emptyset$ because condition (i) in Lemma 5.3.3 is not satisfied when $i = 2$. However, for all other values of $i$ condition (i) in Lemma 5.3.3 is satisfied.

(iv) $\rho(\Sigma_4) = \emptyset$ because $\hat{\mathcal{X}}_4(\lambda)$ is not closed for any $\lambda \neq 0$ (and the spectrum of $\Sigma_4$ is closed, so if every $\lambda \neq 0$ belongs to the spectrum, then also 0 belongs to the spectrum).

(v) All the remaining examples satisfy condition (iii) in Corollary (under the additional assumptions on $\lambda$ when $i = 11$ or 16 listed above). Thus, they have a nonempty resolvent set.

5.3.2. Frequency domain i/o admissible i/o representations. Above we introduced the notion of the resolvent set of a closed s/s node $\Sigma = (V; X, W)$. We proceed to investigate the connection between the resolvent set of a closed s/s system and the resolvent sets of its i/o representations, and begin with the following theorem.

5.3.6. Theorem. Let $\Sigma = (V; X, W)$ be a closed s/s node with characteristic node bundle $\hat{\mathcal{E}}$ and characteristic signal bundle $\hat{\mathcal{X}}$, let $(U, Y)$ be an i/o representation of $W$, and denote the corresponding i/o representation of $\Sigma$ by $\Sigma_{i/o} = (S; X, U, Y)$. Then for each $\lambda \in \mathbb{C}$ the following conditions are equivalent:
When the equivalent conditions (i)–(vi) above hold, then the following additional claims are true:

(i) \( \lambda \in \rho(\Sigma_{i/s/o}) \).

(ii) The s/s node \( \Sigma_\lambda \) defined in Lemma 5.3.2 is bounded and the i/o representation \((\mathcal{U}, \mathcal{Y})\) is boundedly i/s/o-admissible for \( \Sigma_\lambda \) (cf. Definition 2.2.14).

(iii) \( V \) has the equivalent representations

\[
V = \text{rng} \left( \begin{bmatrix} \lambda \hat{A}_{i/s/o}(\lambda) - 1 \chi & \lambda \hat{B}_{i/s/o}(\lambda) \\ \hat{I}_{y} \hat{C}_{i/s/o}(\lambda) & \hat{I}_{y} \hat{D}_{i/s/o}(\lambda) + \hat{I}_{d} \end{bmatrix} \right),
\]

\[
V = \ker \left( \begin{bmatrix} -\hat{A}_{i/s/o}(\lambda) & \lambda \hat{A}_{i/s/o}(\lambda) - 1 \chi & \hat{B}_{i/s/o}(\lambda) P_{U}^{y} \\ -\hat{C}_{i/s/o}(\lambda) & \lambda \hat{C}_{i/s/o}(\lambda) & \hat{D}_{i/s/o}(\lambda) P_{U}^{y} - P_{d}^{y} \end{bmatrix} \right),
\]

for some bounded linear operators \( \hat{A}_{i/s/o}(\lambda), \hat{B}_{i/s/o}(\lambda), \hat{C}_{i/s/o}(\lambda), \) and \( \hat{D}_{i/s/o}(\lambda) \). Here the block matrix operator in (5.3.7a) maps \([\mathcal{Y}]\) into \( \mathcal{X} \) and the block matrix operator in (5.3.7b) maps \( \mathcal{X} \) into \([\mathcal{Y}]\).

(iv) \( \hat{\mathcal{E}}(\lambda) \) has the equivalent representations

\[
\hat{\mathcal{E}}(\lambda) = \left\{ x^0 \atop x_\lambda \atop w_\lambda \right\} \in \mathcal{X} \times \mathcal{X} \times \mathcal{W} \text{ s.t. } P_{d}^{y} w_\lambda = \hat{\mathcal{E}}_{i/s/o}(\lambda) x^0 + \hat{\mathcal{D}}_{i/s/o}(\lambda) P_{U}^{y} w_\lambda,
\]

\[
\hat{\mathcal{E}}(\lambda) = \text{rng} \left( \begin{bmatrix} 1x \\ \hat{A}_{i/s/o}(\lambda) & \lambda \hat{A}_{i/s/o}(\lambda) - 1 \chi & \hat{B}_{i/s/o}(\lambda) P_{U}^{y} \\ \hat{I}_{y} \hat{C}_{i/s/o}(\lambda) & \hat{I}_{y} \hat{D}_{i/s/o}(\lambda) + \hat{I}_{d} \end{bmatrix} \right),
\]

\[
\hat{\mathcal{E}}(\lambda) = \ker \left( \begin{bmatrix} \hat{A}_{i/s/o}(\lambda) & \lambda \hat{A}_{i/s/o}(\lambda) - 1 \chi & 0 \\ \hat{C}_{i/s/o}(\lambda) & \lambda \hat{C}_{i/s/o}(\lambda) & \hat{D}_{i/s/o}(\lambda) P_{U}^{y} - P_{d}^{y} \end{bmatrix} \right),
\]

for some bounded linear operators \( \hat{A}_{i/s/o}(\lambda), \hat{B}_{i/s/o}(\lambda), \hat{C}_{i/s/o}(\lambda), \) and \( \hat{D}_{i/s/o}(\lambda) \). Here the block matrix operator in (5.3.8a) maps \([\mathcal{X}]\) into \( \mathcal{Y} \) and the block matrix operator in (5.3.8b) maps \( \mathcal{Y} \) into \([\mathcal{X}]\).

(v) \( \lambda \in \rho(\Sigma) \) and \( \hat{\mathcal{S}}(\lambda) \) has the equivalent representations

\[
\hat{\mathcal{S}}(\lambda) = \left\{ w \in \mathcal{W} \text{ s.t. } P_{d}^{y} w = \hat{\mathcal{D}}_{i/s/o}(\lambda) P_{U}^{y} w \right\},
\]

\[
\hat{\mathcal{S}}(\lambda) = \text{rng} (\hat{I}_{y} \hat{D}_{i/s/o}(\lambda) + \hat{I}_{d}),
\]

\[
\hat{\mathcal{S}}(\lambda) = \ker (\hat{D}_{i/s/o}(\lambda) P_{U}^{y} - P_{d}^{y}),
\]

for some \( \hat{D}_{i/s/o}(\lambda) \in \mathcal{B}(\mathcal{U}; \mathcal{Y}) \).

(vi) \( \lambda \in \rho(\Sigma) \) and \( \mathcal{Y} \) is a direct complement to \( \hat{\mathcal{S}}(\lambda) \) (and \( \mathcal{U} \) is a direct complement to \( \mathcal{Y} \)).

When the equivalent conditions (i)–(vi) above hold, then the following additional claims are true:

(vii) both \( \Sigma \) and \( \Sigma_{i/s/o} \) are resolvable,

(viii) the operators \( \hat{A}_{i/s/o}(\lambda), \hat{B}_{i/s/o}(\lambda), \hat{C}_{i/s/o}(\lambda), \) and \( \hat{D}_{i/s/o}(\lambda) \) in (iii)–(v) are the state/state, the input/state, the state/output, and the input/output resolvents of \( \Sigma_{i/s/o} \) evaluated at \( \lambda \),

(ix) the i/s/o representation of \( \Sigma_\lambda \) corresponding to the i/o representation \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) is the bounded i/s/o node \( \Sigma^{(i/s/o)}_{i/s/o} := (\hat{\mathcal{E}}_{i/s/o}(\lambda); \mathcal{X}, \mathcal{U}, \mathcal{Y}) \),

where \( \hat{\mathcal{S}}_{i/s/o}(\lambda) = \begin{bmatrix} \hat{A}_{i/s/o}(\lambda) & \hat{B}_{i/s/o}(\lambda) \\ \hat{C}_{i/s/o}(\lambda) & \hat{D}_{i/s/o}(\lambda) \end{bmatrix} \).
for some bounded linear operators $A$ to $\hat{B}$ is bounded and the i/o representation $(U, \lambda)$ is of the type (2.2.33b), and according to Theorem 2.2.27 $\Sigma$ is bounded and the i/o representation $(\hat{W}, \hat{Y})$ is boundedly i/s/o-admissible for $\Sigma$.

(ii) $\Rightarrow$ (i): By Theorem 2.2.27 $V^\lambda$ has a representation of the type

$$V^\lambda = \text{rng} \left( \begin{bmatrix} A_\lambda & B_\lambda \\ 1_\chi & 0 \end{bmatrix} \right).$$

for some bounded linear operators $A_\lambda$, $B_\lambda$, $C_\lambda$, and $D_\lambda$. This together with (5.3.2) implies that $V$ has the representation

$$V = \text{rng} \left( \begin{bmatrix} -1_\chi + \lambda A_\lambda & \lambda B_\lambda \\ A_\lambda & B_\lambda \\ I_\chi C_\chi & I_\chi D_\chi + I_\chi \end{bmatrix} \right).$$

Denote i/s/o representation of $\Sigma$ corresponding to the i/o representation $(\hat{\mathcal{F}}(\lambda), \hat{\mathcal{Y}})$ of $W$ by $\Sigma_{i/s/o} = (S, X, \hat{\mathcal{F}}(\lambda), \hat{\mathcal{Y}})$. Then from the above representation of $V$ and (2.2.16) we find that $\text{gph} (S)$ has the representation

$$\text{gph} (S) = \text{rng} \left( \begin{bmatrix} -1_\chi + \lambda A_\lambda & \lambda B_\lambda \\ A_\lambda & B_\lambda \\ C_\lambda & V^\lambda \\ 0 & 1_\chi \end{bmatrix} \right).$$

Therefore, by Lemma 5.2.13 $\lambda \in (\Sigma_{i/s/o})$.

(iii) $\Rightarrow$ (v): We can apply Theorem 2.2.27 to the bounded s/s node $\Sigma_\lambda$. The subspace $W_0$ in this case is isomorphic to $\hat{\mathcal{F}}(\lambda)$, and (v) follows from condition (v) in Theorem 2.2.29.

(v) $\Rightarrow$ (vi): If (v) holds, then every vector $w \in \hat{\mathcal{F}}(\lambda)$ can be written in the form $w = u + y$, where $u = P_{U}^{\lambda}w \in U$ and $y = P_{U}^{\lambda}w = \hat{\mathcal{F}}_{i/s/o}(\lambda)u$. This means that every $w \in W$ can be written in the form

$$w = P_{U}^{\lambda}w + P_{U}^{\hat{\mathcal{F}}(\lambda)}w = (P_{U}^{\lambda}w + \hat{\mathcal{F}}_{i/s/o}(\lambda)P_{U}^{\hat{\mathcal{F}}(\lambda)}w) + (P_{U}^{\hat{\mathcal{F}}(\lambda)}w - \hat{\mathcal{F}}_{i/s/o}(\lambda)P_{U}^{\hat{\mathcal{F}}(\lambda)}w),$$

where $P_{U}^{\lambda}w + \hat{\mathcal{F}}_{i/s/o}(\lambda)P_{U}^{\hat{\mathcal{F}}(\lambda)}w \in \hat{\mathcal{F}}(\lambda)$ and $P_{U}^{\hat{\mathcal{F}}(\lambda)}w - \hat{\mathcal{F}}_{i/s/o}(\lambda)P_{U}^{\hat{\mathcal{F}}(\lambda)}w \in \mathcal{Y}$. Thus $W = \hat{\mathcal{F}}(\lambda) + \mathcal{Y}$. On the other hand, if $w \in \mathcal{Y} \cap \hat{\mathcal{F}}(\lambda)$, then $P_{U}^{\lambda}w = 0$ and hence also $P_{U}^{\hat{\mathcal{F}}(\lambda)}w = \hat{\mathcal{F}}_{i/s/o}(\lambda)P_{U}^{\hat{\mathcal{F}}(\lambda)}w = 0$. Thus $\mathcal{Y} \cap \hat{\mathcal{F}}(\lambda) = \{0\}$. This shows that $W = \hat{\mathcal{F}}(\lambda) + \mathcal{Y}$.

(vi) $\Rightarrow$ (ii): By Lemma 5.3.2 $\Sigma_\lambda$ is a bounded s/s node, and hence we may apply Theorem 2.2.29 to $\Sigma_\lambda$. The subspace $W_0$ in Theorem 2.2.29 is again equal to $\hat{\mathcal{F}}(\lambda)$. By Theorem 2.2.29 the i/o representation $(U, \mathcal{Y})$ of $W$ is boundedly i/s/o-admissible for $\Sigma_\lambda$.

We have now proved that conditions (i)–(vi) are equivalent. The additional claims at the end follow from Definitions 5.2.8 and 5.3.1, Theorem 2.2.27 and Lemmas 5.2.13 and 5.3.2. 

Proof.
5.3.7. Corollary. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a closed s/s node with characteristic node bundle \( \mathcal{E} \) and characteristic signal bundle \( \mathfrak{F} \), let \((\mathcal{U}, \mathcal{Y})\) be an i/o representation of \( \mathcal{W} \), and denote the corresponding i/s/o representation of \( \Sigma \) by \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \). Then the following conditions are equivalent:

(i) \( \rho(\Sigma_{i/s/o}) \neq \emptyset \).

(ii) The six equivalent conditions in Theorem 5.3.6 hold for some \( \lambda \in \mathbb{C} \).

If these equivalent conditions hold, then \( \Sigma \) is resolvable.

Proof. This follows immediately from Theorem 5.3.6. \( \square \)

5.3.8. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a s/s node, and let \( \mathcal{W} = (\mathcal{U}, \mathcal{Y}) \) be an i/o representation of \( \mathcal{W} \), and let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be the corresponding i/s/o representation of \( \Sigma \).

(i) \((\mathcal{U}, \mathcal{Y})\) is called frequency domain i/s/o-admissible for \( \Sigma \) at the point \( \lambda \in \mathbb{C} \) if the equivalent conditions (i)–(vi) in Theorem 5.3.6 hold.

(ii) \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) is called frequency domain i/s/o-admissible for \( \Sigma \) if \( \Sigma_{i/s/o} \) is resolvable, i.e., if the equivalent conditions (i) and (ii) in Corollary 5.3.7 hold.

5.3.9. Theorem. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a closed s/s node. Then \( \rho(\Sigma) \) is the union of the resolvent sets \( \rho(\Sigma_{i/s/o}) \) over all i/s/o representations \( \Sigma_{i/s/o} \) of \( \Sigma \). Thus in particular, \( \rho(\Sigma) \) is open, and \( \Sigma \) is resolvable if and only if \( \Sigma \) has at least one resolvable i/s/o representation.

Proof. That \( \rho(\Sigma_{i/s/o}) \subset \rho(\Sigma) \) for every i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \) follows from Theorem 5.3.6.

Conversely, suppose that \( \lambda \in \rho(\Sigma) \). Then the s/s node \( \Sigma_{\lambda} \) in Lemma 5.3.2 is bounded. By Theorem 2.2.27, \( \Sigma_{\lambda} \) has a bounded i/s/o representation \( \Sigma_{i/s/o}^{bnd} = (S_{\lambda}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \). Let \( \Sigma_{i/s/o} \) be the i/s/o representation of \( \Sigma \) corresponding to the i/o representation \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \). Then by Theorem 5.3.6, \( \lambda \in \rho(\Sigma_{i/s/o}) \). \( \square \)

5.3.10. Remark. In Definition 4.4.2 we introduced the notion of the i/s/o-bounded resolvent set \( \rho^{bnd}(\Sigma) \) of a bounded s/s node. Of course, it is also possible to apply Definition 5.3.1 in this case to get the resolvent set \( \rho(\Sigma) \) of \( \Sigma \). By comparing Definition 4.4.2 and Theorem 5.3.9 to each other we find that it is always true that \( \rho^{bnd}(\Sigma) \subset \rho(\Sigma) \), but at the moment is not clear to what extent the converse inclusion holds. Suppose that \( \lambda_0 \in \rho(\Sigma) \), and let \((\mathcal{U}, \mathcal{Y})\) be an i/o representation of \( \Sigma \). Then by Theorem 5.3.9 the corresponding i/s/o representation \( \Sigma_{i/s/o}^{bnd} = (\mathcal{U}^{bnd}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) of \( \Sigma \) satisfies \( \lambda_0 \in \rho(\Sigma_{i/s/o}^{bnd}) \) if and only if \( \mathcal{Y} \) is a direct complement to \( \mathfrak{F}(\lambda_0) \) (and \( \mathcal{U} \) is a direct complement to \( \mathcal{Y} \)). However, these conditions do not yet imply that \( \Sigma_{i/s/o} \) is bounded. By Theorem 2.2.27, \( \Sigma_{i/s/o} \) is bounded if and only if, in addition, \( \mathcal{Y} \) is a direct complement to the canonical input space \( \mathcal{W}_0 \) of \( \Sigma \). Thus the following question arises: Is it true for all \( \lambda_0 \in \rho(\Sigma) \) that there exists a closed subspace \( \mathcal{Y} \) of \( \mathcal{W} \) such that \( \mathcal{Y} \) is a direct complement to both \( \mathfrak{F}(\lambda_0) \) and \( \mathcal{W}_0 \)? If, and only if, the answer to this question is “yes”, then \( \rho(\Sigma) = \rho^{bnd}(\Sigma) \). The same remark also applies to the notion of the i/s/o-semi-bounded resolvent set of a semi-bounded i/s/o system introduced in Definition 4.3.7.

5.3.11. Remark. It follows from Theorem 5.3.9 that if \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) is a resolvable s/s node, then \( \Sigma \) has at least one resolvable i/s/o representation \( \Sigma_{i/s/o} \).
It is an interesting open question if $\Sigma_{i/s/o}$ always has a regular resolvable i/s/o representation $\Sigma_{i/s/o}$ whenever $\Sigma$ is regular and resolvable.

5.3.12. **Theorem.** Let $\Sigma = (V; X, W)$ be a resolvable s/s node with characteristic signal bundle $\mathring{\Sigma}$, let $(U_1, Y_1)$ be an frequency domain i/s/o-admissible i/o representation of $W$, and denote the corresponding i/s/o resolvent matrix of $\Sigma$ by $\mathring{\Sigma}_1 := \begin{bmatrix} \mathring{a}_1 & \mathring{b}_1 \\ \mathring{c}_1 & \mathring{d}_1 \end{bmatrix}$. Let $(U_2, Y_2)$ be another i/o representation of $W$, and let $\Theta$ and $\mathring{\Theta}$ be the transition matrices from $(U_1, Y_1)$ to $(U_2, Y_2)$ respectively from $(U_2, Y_2)$ to $(U_1, Y_1)$ (these are given by $(2.2.8)$ and $(2.2.10)$).

(i) For each $\lambda \in \text{dom}(\mathring{\Sigma}_1)$ the following conditions are equivalent:

(a) The i/o representation $(U_2, Y_2)$ is frequency domain i/s/o-admissible for $\Sigma$ at the point $\lambda$;
(b) $Y_2$ is a direct complement to $\mathring{\Sigma}(\lambda)$;
(c) The operator $\Theta_{11} + \Theta_{12} \mathring{\Sigma}(\lambda)$ maps $U_1$ one-to-one onto $U_2$ (and hence it has a bounded inverse);
(d) The operator $\Theta_{22} - \mathring{\Sigma}(\lambda) \Theta_{12}$ maps $Y_2$ one-to-one onto $Y_1$ (and hence it has a bounded inverse).

(ii) Suppose that the i/o representation $(U_2, Y_2)$ is frequency domain i/s/o-admissible for $\Sigma$ and that $\text{dom}(\mathring{\Sigma}_1) \cap \text{dom}(\mathring{\Sigma}_2) \neq \emptyset$, where $\mathring{\Sigma}_2 := \begin{bmatrix} \mathring{a}_2 & \mathring{b}_2 \\ \mathring{c}_2 & \mathring{d}_2 \end{bmatrix}$ is the i/s/o resolvent matrix of $\Sigma$ corresponding to the i/o decomposition $(U_2, Y_2)$ of $W$. Then for $\lambda \in \text{dom}(\mathring{\Sigma}) \cap \text{dom}(\mathring{\Sigma}_2)$ the following claims are true:

(a) The operators listed below are boundedly invertible, with the following inverses:

\begin{align*}
(5.3.10a) \quad & (\Theta_{11} + \Theta_{12} \mathring{\Sigma}_1(\lambda))^{-1} = \Theta_{11} + \mathring{\Theta}_{12} \mathring{\Sigma}_2(\lambda), \\
(5.3.10b) \quad & (\Theta_{22} - \mathring{\Sigma}_1(\lambda) \Theta_{12})^{-1} = \Theta_{22} - \mathring{\Sigma}_2(\lambda) \Theta_{12}, \\
(5.3.10c) \quad & \begin{bmatrix} 1_X & 0 \\ \Theta_{12} \mathring{C}_1(\lambda) & \Theta_{11} + \Theta_{12} \mathring{D}_1(\lambda) \end{bmatrix}^{-1} = \begin{bmatrix} 1_X & 0 \\ \Theta_{12} \mathring{C}_2(\lambda) & \Theta_{11} + \Theta_{12} \mathring{D}_2(\lambda) \end{bmatrix}, \\
(5.3.10d) \quad & \begin{bmatrix} 1_X & -\mathring{B}_1(\lambda) \Theta_{12} \\ 0 & \Theta_{22} - \mathring{D}_1(\lambda) \Theta_{12} \end{bmatrix}^{-1} = \begin{bmatrix} 1_X & -\mathring{B}_2(\lambda) \Theta_{12} \\ 0 & \Theta_{22} - \mathring{D}_2(\lambda) \Theta_{12} \end{bmatrix}.
\end{align*}

(b) $\mathring{\Sigma}_2(\lambda)$ can be obtained from the $\mathring{\Sigma}_1(\lambda)$ in the following way:

\begin{align*}
(5.3.11) \quad & \begin{bmatrix} \mathring{a}_2(\lambda) \\ \mathring{c}_2(\lambda) \\ \mathring{b}_2(\lambda) \\ \mathring{d}_2(\lambda) \end{bmatrix} = \begin{bmatrix} \mathring{a}_1(\lambda) \\ \Theta_{22} \mathring{C}_1(\lambda) & \Theta_{21} + \Theta_{22} \mathring{D}_1(\lambda) \end{bmatrix} \begin{bmatrix} 1_X & 0 \\ \Theta_{12} \mathring{C}_1(\lambda) & \Theta_{11} + \Theta_{12} \mathring{D}_1(\lambda) \end{bmatrix}^{-1},
\end{align*}

or equivalently,

\begin{align*}
(5.3.12) \quad & \mathring{a}_2(\lambda) = \mathring{a}_1(\lambda) - \mathring{B}_1(\lambda)(\Theta_{11} + \Theta_{12} \mathring{D}_1(\lambda))^{-1} \Theta_{12} \mathring{C}_1(\lambda), \\
& \mathring{b}_2(\lambda) = \mathring{B}_1(\lambda)(\Theta_{11} + \Theta_{12} \mathring{D}_1(\lambda))^{-1}, \\
& \mathring{c}_2(\lambda) = \Theta_{22} \mathring{C}_1(\lambda) - (\Theta_{21} + \Theta_{22} \mathring{D}_1(\lambda))(\Theta_{11} + \Theta_{12} \mathring{D}_1(\lambda))^{-1} \Theta_{12} \mathring{C}_1(\lambda), \\
& \mathring{d}_2(\lambda) = (\Theta_{21} + \Theta_{22} \mathring{D}_1(\lambda))(\Theta_{11} + \Theta_{12} \mathring{D}_1(\lambda))^{-1}.
\end{align*}

(c) $\mathring{\Sigma}_2(\lambda)$ can also be obtained from the $\mathring{\Sigma}_1(\lambda)$ in the following way:

\begin{align*}
(5.3.13) \quad & \begin{bmatrix} \mathring{a}_2(\lambda) \\ \mathring{b}_2(\lambda) \\ \mathring{c}_2(\lambda) \\ \mathring{d}_2(\lambda) \end{bmatrix} = \begin{bmatrix} 1_X & -\mathring{B}_1(\lambda) \Theta_{12} \\ 0 & \Theta_{22} - \mathring{D}_1(\lambda) \Theta_{12} \end{bmatrix}^{-1} \begin{bmatrix} \mathring{a}_1(\lambda) \\ \mathring{b}_1(\lambda) \Theta_{11} \\ \mathring{c}_1(\lambda) \\ \mathring{d}_1(\lambda) \Theta_{11} \end{bmatrix}.
\end{align*}
or equivalently,

\begin{align}
\hat{\mathbf{A}}_2(\lambda) &= \hat{\mathbf{A}}_1(\lambda) + \hat{\mathbf{B}}_1(\lambda)\hat{\Theta}_{12}(\Theta_{22} - \hat{\Theta}_1(\Theta_{12})^{-1}\hat{\Theta}_1(\lambda),
\hat{\mathbf{B}}_2(\lambda) &= \hat{\mathbf{B}}_1(\lambda)\hat{\Theta}_{11} + \hat{\mathbf{B}}_1(\lambda)\hat{\Theta}_{12}(\Theta_{22} - \hat{\Theta}_1(\Theta_{12})^{-1}(-\hat{\Theta}_{21} + \hat{\Theta}_1(\Theta_{11})),
\hat{\mathbf{C}}_2(\lambda) &= (\Theta_{22} - \hat{\Theta}_1(\Theta_{12})^{-1}\hat{\Theta}_1(\lambda),
\hat{\mathbf{D}}_2(\lambda) &= (\Theta_{22} - \hat{\Theta}_1(\Theta_{12})^{-1}(-\hat{\Theta}_{21} + \hat{\Theta}_1(\Theta_{11})).
\end{align}

**Proof.** This follows from Lemma 5.3.2 and Theorems 2.2.31 and 5.3.6 \(\square\)

In order to apply Theorem 5.3.12 we must first have a frequency domain i/o admissible i/o representation \((\mathcal{U}_1, \mathcal{Y}_1)\) of the signal space \(\mathcal{W}\). Our following theorem can be used even in the case where the i/o representation \((\mathcal{U}_1, \mathcal{Y}_1)\) is not frequency domain i/s/o admissible.

5.3.13. THEOREM. Let \(\Sigma_{i/o} = (S; X, \mathcal{U}_1, \mathcal{Y}_1)\) be an i/o representation of the closed s/s node \(\Sigma = (V; X, \mathcal{W})\) with characteristic signal bundle \(\hat{g}\), and let \((\mathcal{U}_2, \mathcal{Y}_2)\) be an i/o representation of \(\mathcal{W}\). Let \(\Theta\) be the transition matrix from \((\mathcal{U}_1, \mathcal{Y}_1)\) to \((\mathcal{U}_2, \mathcal{Y}_2)\) (see (2.2.8)). Then the following conditions are equivalent:

(i) The i/o representation \((\mathcal{U}_2, \mathcal{Y}_2)\) is frequency domain i/s/o-admissible for \(\Sigma\) at the point \(\lambda\).

(ii) \(\mathcal{Y}_2\) is a direct complement to \(\hat{g}(\lambda)\).

(iii) There right-hand side of the equation

\begin{equation}
\text{gph} (\hat{\mathbf{S}}_2(\lambda)) = \begin{bmatrix} 0 & 0 & 1_X & 0 \\
0 & \Theta_{22} & 0 & \Theta_{21} \\
-1_X & 0 & \lambda & 0 \\
0 & \Theta_{12} & 0 & \Theta_{11} \end{bmatrix}
\text{gph} (S)
\end{equation}

is the graph of a bounded linear operator \(\hat{\mathbf{S}}_2(\lambda) = \begin{bmatrix} \hat{\mathbf{S}}_2(\lambda) & \hat{\mathbf{S}}_2(\lambda) \\
\hat{\mathbf{C}}_2(\lambda) & \hat{\mathbf{D}}_2(\lambda) \end{bmatrix} \in \mathcal{B}([X], [Y]).

Suppose that these equivalent conditions hold, and denote the i/o representation of \(\Sigma\) corresponding to the i/o representation \((\mathcal{U}_2, \mathcal{Y}_2)\) of \(\mathcal{W}\) by \(\Sigma_{i/o}^2 = (S_2; X, \mathcal{U}_2, \mathcal{Y}_2)\).

Then both \(\Sigma_{i/o}^2\) and \(\Sigma\) are resolvable, and \(\hat{\mathbf{S}}_2(\lambda)\) coincides with the i/s/o resolvent matrix of \(\Sigma_{i/o}^2\) at the point \(\lambda\).

If \(S\) is semi-regular, then conditions (i)–(iii) above are further equivalent to the condition

(iv) The operators \(K(\lambda): [X] \to [Y]\) and \(M(\lambda): [X] \to [\lambda T]\) (with \(\text{dom} (K(\lambda)) = \text{dom} (M(\lambda)) = \text{dom} (S)\)) defined by

\begin{align}
K(\lambda) &= \begin{bmatrix} 1_X & 0 \\
0 & \Theta_{21} \end{bmatrix} S, \\
M(\lambda) &= \begin{bmatrix} \lambda & 0 \\
0 & \Theta_{11} \end{bmatrix} + \begin{bmatrix} -1_X & 0 \\
0 & \Theta_{12} \end{bmatrix} S,
\end{align}

satisfy \(\text{rng} (M(\lambda)) = [X], \text{ker} (M(\lambda)) \subset \text{ker} (K(\lambda)).\)

In this case

\begin{equation}
\hat{\mathbf{S}}_2(\lambda) = K(\lambda)M(\lambda)^{-1}.
\end{equation}

Note that (5.1.11) can be interpreted as the special case of (5.3.18) where \(\mathcal{U}_2 = \mathcal{U}_1\) and \(\mathcal{Y}_2 = \mathcal{Y}_1\) and hence \(\Theta = \begin{bmatrix} 1_X & 0 \\
0 & 1_{\mathcal{U}_1} \end{bmatrix}\).
Proof of Theorem 5.3.13. (i) ⇔ (ii): See Theorem 5.3.12.

(ii) ⇔ (iii): This equivalence follows from Definition 5.3.8 and the representation (2.2.47) of \( \mathcal{C}(\lambda) \) in terms of \( \text{gph}(S_2) \) and the representation (2.2.48) of \( \text{gph}(S) \) in terms of \( \text{gph}(S) \).

(iii) ⇔ (iv): If \( S \) is single-valued, then both \( K(\lambda) \) and \( M(\lambda) \) are single-valued, and the right-hand side of (5.3.15) is equal to \( \text{rng} \left( \left[ \lambda(\lambda) M(\lambda) \right] \right) \). The right-hand side of (5.3.15) is closed since \( S \) is closed, and it defines a closed multi-valued operator \( \left[ \lambda^X \right] \rightarrow \left[ \lambda^X \right] \), which we may denote by \( \mathcal{S}_2(\lambda) \). By the closed graph theorem, \( \mathcal{S}(\lambda) \in \mathcal{B}(\left[ \lambda^X \right] ; \left[ \lambda^X \right] ) \) if and only if \( \mathcal{S}(\lambda) \) is single-valued and \( \text{dom}(\mathcal{S}(\lambda)) = \left[ \lambda^X \right] \).

5.3.14. Proposition. A regular \( s/s \) node \( \Sigma = (V; X, W) \) is bounded if and only if the characteristic node bundle \( \mathcal{E} \) of \( \Sigma \) is analytic at infinity and

\[
\mathcal{E}(\infty) = \begin{bmatrix} X^I \\ \{0\} \end{bmatrix} \quad \text{for some closed subspace } W_0 \text{ of } W.
\]

Proof. The necessity of the above condition for the boundedness of \( \Sigma \) follows from (5.4.3). Conversely, suppose that \( \mathcal{E} \) is analytic at infinity and that (5.3.19) holds. Define \( \mathcal{U} = \{0 \ 0 \ 1_{W_0}\} \mathcal{E}(\infty) \), and let \( \mathcal{Y} \) be a direct complement to \( \mathcal{U} \) in \( W \).

Then by Lemma A.3.3 there exists an analytic \( \mathcal{B}(\left[ \lambda^X \right] ; \left[ \lambda^X \right] ) \)-valued function defined in some neighborhood \( \mathcal{O}(\infty) \) such that \( \mathcal{E} \) has the graph representations (5.3.8) with \( \mathcal{U} = W_0 \) and \( \mathcal{Y} \) equal to an arbitrary direct complement to \( \mathcal{U} \) for all \( \lambda \in \mathcal{O}(\infty) \) for some functions \( \mathcal{A}_{i/s/o}, \mathcal{B}_{i/s/o}, \mathcal{C}_{i/s/o}, \mathcal{D}_{i/s/o} \) which are analytic at infinity and vanish at infinity. From this follows that \( \mathcal{C} \) has the representation (5.3.7a) for all \( \lambda \in \mathcal{O}(\infty) \). In particular, for any \( x \in X \) it is true that the vector \( \begin{bmatrix} \lambda x \\ w_\lambda \end{bmatrix} := \begin{bmatrix} \lambda \mathcal{A}_{i/s/o}(x) x \\ \mathcal{B}_{i/s/o}(\lambda) \end{bmatrix} \). Since \( V \) is closed, the analyticity of \( \mathcal{A}_{i/s/o} \) and \( \mathcal{C}_{i/s/o} \) and the conditions \( \mathcal{A}_{i/s/o}(\infty) = 0 \) and \( \mathcal{C}_{i/s/o}(\infty) = 0 \) imply that the limit \( \begin{bmatrix} \lambda x \\ w_\lambda \end{bmatrix} := \lim_{\lambda \rightarrow \infty} \begin{bmatrix} \lambda x \\ w_\lambda \end{bmatrix} \) exists and belongs to \( V \). The regularity of \( V \) implies that \( z_\infty = 0 \). Thus, \( \lim_{\lambda \rightarrow \infty} \lambda \mathcal{A}_{i/s/o}(\lambda) = 1_X \).

Denote

\[
A := \lim_{\lambda \rightarrow \infty} \lambda(\lambda \mathcal{A}_{i/s/o}(\lambda) - 1_X), \quad B := \lim_{\lambda \rightarrow \infty} \lambda \mathcal{B}_{i/s/o}(\lambda),
\]

\[
C := \lim_{\lambda \rightarrow \infty} \lambda \mathcal{C}_{i/s/o}(\lambda), \quad D := 0.
\]

For \( 0 \neq \lambda \in \mathcal{O}(\infty) \) the representation (5.3.7a) is equivalent to the representation

\[
V = \text{rng} \left( \begin{bmatrix} \lambda \mathcal{A}_{i/s/o}(\lambda) - 1_X & \lambda \mathcal{B}_{i/s/o}(\lambda) \\ \lambda \mathcal{A}_{i/s/o}(\lambda) & \mathcal{B}_{i/s/o}(\lambda) \end{bmatrix} \right).
\]
The operator on the right-hand side is analytic at infinity, and the value at infinity of this operator is
\[
\lim_{\lambda \to \infty} \begin{bmatrix}
\lambda(\mathfrak{A}_{i/o}(\lambda) - 1_X) & \lambda\mathfrak{B}_{i/o}(\lambda) \\
\lambda\mathfrak{A}_{i/o}(\lambda) & \mathfrak{B}_{i/o}(\lambda) \\
I_Y \lambda & 0 \\
I_Y \lambda & I_{\mathfrak{A}_{i/o}(\lambda)} + I_d)
\end{bmatrix} = \begin{bmatrix} A & B \\ 1_X & 0 \end{bmatrix}.
\]

Using this fact it is not difficult to show that (2.2.33) holds with \( D = 0 \). This shows that \( \Sigma \) has the bounded i/o representation \( \Sigma_{i/o} = ([A B] : X, U, Y) \), and consequently \( \Sigma \) is bounded.

5.3.15. Example (cf. Example 1.4.1). The s/s node \( \Sigma = (V; X, W) \) in Example 1.4.1 with \( X = L^2(\mathbb{R}^+) \) and \( W = \{0\} \) has only one possible i/o decomposition of the signal space \( \{0\} \), namely the one where both the input space \( U = \{0\} \) and the output space \( Y = 0 \). The corresponding i/o node is the one given in Example 2.4.12. Thus by Theorem 5.3.9, \( \Sigma \) has the same resolvent set as the i/o system in Example 2.4.12, i.e., \( \rho(\Sigma) = \mathbb{C}^+ \).

5.3.16. Example (cf. Example 1.4.2). The s/s node \( \Sigma = (V; X, W) \) in Example 1.4.2 with \( X = L^2(\mathbb{R}^+) \) and \( W = \{0\} \) has only one possible i/o decomposition of the signal space \( \{0\} \), namely the one where both the input space \( U = \{0\} \) and the output space \( Y = 0 \). The corresponding i/o node is the one given in Example 2.4.13. Thus by Theorem 5.3.9, \( \Sigma \) has the same resolvent set as the i/o system in Example 2.4.13, i.e., \( \rho(\Sigma) = \mathbb{C}^+ \).

5.3.17. Example (cf. Example 1.4.3). The s/s node \( \Sigma = (V; X, W) \) in Example 1.4.3 with \( X = L^2(\mathbb{R}^+) \) and \( W = \{0\} \) has only one possible i/o decomposition of the signal space \( \{0\} \), namely the one where both the input space \( U = \{0\} \) and the output space \( Y = 0 \). The corresponding i/o node is the one given in Example 2.4.14. Thus by Theorem 5.3.9, \( \Sigma \) has the same resolvent set as the i/o system in Example 2.4.14, i.e., \( \rho(\Sigma) = \mathbb{C}^- \).

5.3.18. Example (cf. Example 1.4.4). The s/s node \( \Sigma = (V; X, W) \) in Example 1.4.4 with \( X = L^2(\mathbb{R}^+) \) and \( W = \{0\} \) has only one possible i/o decomposition of the signal space \( \{0\} \), namely the one where both the input space \( U = \{0\} \) and the output space \( Y = 0 \). The corresponding i/o node is the one given in Example 2.4.15. Thus by Theorem 5.3.9, \( \Sigma \) has the same resolvent set as the i/o system in Example 2.4.15, i.e., \( \rho(\Sigma) = \mathbb{C}^- \).

5.3.19. Example (cf. Example 1.4.5). The s/s node \( \Sigma = (V; X, W) \) in Example 1.4.5 with \( X = L^2(\mathbb{R}^+) \) and \( W = \mathbb{C} \) has two possible i/o decompositions of the signal space, namely \( \{0\}, \mathbb{C} \) and \( \{0\}, \mathbb{C} \). As we observed in Example 2.2.20, both of these are i/o-admissible, i.e., the corresponding i/o representations are regular. The i/o representation corresponding to the decomposition \( \{0\}, \mathbb{C} \) is the one described in Example 2.4.16. Recall that the s/s node \( \Sigma \) in Example 1.4.7 is the time reflection of the s/s node in Example 1.4.6, and therefore the i/o representation of \( \Sigma \) corresponding to the decomposition \( \{0\}, \mathbb{C} \) of the signal space is the time reflection of the i/o node described in Example 2.4.17. The resolvent set of the former i/o representation is \( \mathbb{C}^+ \), and the resolvent set of the latter i/o representation is \( \mathbb{C}^- \). Therefore, by Theorem 5.3.9, \( \rho(\Sigma) = \mathbb{C}^+ \cup \mathbb{C}^- = \mathbb{C} \setminus i\mathbb{R} \). From Lemma 1.6.3, Examples 5.1.33 and 5.1.34, and Theorem 5.3.6, and we get the following formulas for the characteristic signal bundle \( \hat{\mathfrak{E}} \) and the characteristic
5.3. RESOLVABLE STATE/SIGNAL NODES (Jan 02, 2016)

signal bundle $\hat{F}$ of $\Sigma$:

\[(5.3.20)\]

\[
\hat{E}(\lambda) = \begin{cases} \\
\xi \mapsto \int_{\xi}^{\infty} e^{\lambda(\xi-\zeta)} \varphi(\zeta) \, d\zeta \mid \varphi \in L^2(\mathbb{R}^+) \end{cases}, \quad \lambda \in \mathbb{C}^+, \\
\hat{F}(\lambda) = \begin{cases} 0 \end{cases}, \quad \lambda \in \mathbb{C}^+, \\
\hat{F}(\lambda) = \mathbb{C}, \quad \lambda \in \mathbb{C}^{-}.
\]

5.3.20. Example (cf. Example 1.4.6). The s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ in Example 1.4.6 with $\mathcal{X} = L^2(\mathbb{R}^+)$ and $\mathcal{W} = \mathbb{C}$ is the time reflection of Example 5.3.19. The resolvent set of $\Sigma$ is therefore consists of all $\lambda \in \mathbb{C}$ for which $-\lambda$ belongs to the resolvent set of Example 5.3.19. Thus, $\rho(\Sigma) = \mathbb{C}^+ \cup \mathbb{C}^- = \mathbb{C} \setminus j\mathbb{R}$ and the characteristic signal bundle $\hat{E}$ and the characteristic signal bundle $\hat{F}$ of $\Sigma$ have the following representations:

\[(5.3.21)\]

\[
\hat{E}(\lambda) = \begin{cases} \\
\xi \mapsto \left( \int_{0}^{\xi} e^{-\lambda(\xi-\zeta)} \varphi(\zeta) \, d\zeta + e^{-\lambda \xi} \right) \mid \varphi \in L^2(\mathbb{R}^+), \, u \in \mathbb{C} \end{cases}, \quad \lambda \in \mathbb{C}^+, \\
\hat{E}(\lambda) = \begin{cases} \varphi \mid \varphi \in L^2(\mathbb{R}^+), \, u \in \mathbb{C} \end{cases}, \quad \lambda \in \mathbb{C}^-,
\]

\[
\hat{F}(\lambda) = \{0\}, \quad \lambda \in \mathbb{C}^+, \quad \hat{F}(\lambda) = \mathbb{C}, \quad \lambda \in \mathbb{C}^-.
\]

5.3.3. The state/signal resolvents.

5.3.21. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a resolvable s/s node with characteristic node bundle $\hat{E}$, characteristic signal/state bundle $\hat{G}$, and characteristic signal bundle $\hat{F}$, let $\lambda \in \rho(\Sigma)$, and let $\Sigma^\lambda = (V^\lambda; \mathcal{X}, \mathcal{W})$ be the bounded s/s node defined in Lemma 5.3.2.
(i) The operators $\hat{L}(\lambda): V \rightarrow X$, $\hat{A}(\lambda): X \rightarrow X$, and $\hat{B}(\lambda): W \rightarrow X$ whose graphs are given by

$$gph(\hat{L}(\lambda)) := \left\{ \begin{bmatrix} x_\lambda \\ x^0 \\ w_\lambda \end{bmatrix} \mid \begin{bmatrix} x^0 \\ x_\lambda \\ w_\lambda \end{bmatrix} \in \mathcal{E}(\lambda) \right\},$$

(5.3.22)

$$gph(\hat{A}(\lambda)) := \left\{ \begin{bmatrix} 0 & 1_x & 0 \\ 1_x & 0 & 0 \\ 0 & 0 & 1_w \end{bmatrix} \begin{bmatrix} x_\lambda \\ x^0 \\ w_\lambda \end{bmatrix} \right\},$$

(5.3.23)

$$gph(\hat{B}(\lambda)) := \mathcal{V}_{1_w},$$

(5.3.24)

are continuous operators with closed domains (so that they are bounded operators from their respective domains to their range spaces).

(ii) The multi-valued operator $\hat{C}(\lambda): X \rightarrow W$ whose graph is given by

$$gph(\hat{C}(\lambda)) := \left\{ \begin{bmatrix} w_\lambda \\ x_\lambda \\ x^0 \end{bmatrix} \mid \begin{bmatrix} x_\lambda \\ x^0 \\ w_\lambda \end{bmatrix} \in \mathcal{E}(\lambda) \right\},$$

(5.3.25)

is closed and $\text{dom}(\hat{C}(\lambda)) = X$.

(iii) The operators $\hat{A}(\lambda)$, $\hat{B}(\lambda)$, and $\hat{C}(\lambda)$ can be recovered from $\hat{L}(\lambda)$ through the formulas

$$\hat{A}(\lambda) = \hat{L}(\lambda) \left[ \begin{bmatrix} 1_x \\ 0 \end{bmatrix} \right],$$

(5.3.26)

$$\hat{B}(\lambda) = \hat{L}(\lambda) \left[ \begin{bmatrix} 0 \\ 1_w \end{bmatrix} \right],$$

(5.3.27)

$$gph(\hat{C}(\lambda)^{-1}) = \text{dom}(\hat{L}(\lambda)),$$

(5.3.28)

and the domains of $\hat{A}(\lambda)$ and $\hat{B}(\lambda)$ and the multi-valued part of $\hat{C}(\lambda)$ are given by

$$\text{dom}(\hat{A}(\lambda)) = \ker(\hat{C}(\lambda)), \quad \text{dom}(\hat{B}(\lambda)) = \text{null}(\hat{C}(\lambda)) = \hat{S}(\lambda).$$

Proof. This follows from Definition 5.3.1 and 1.6.7

5.3.22. Definition. Let $\Sigma = (V, X, W)$ be a resolvable s/s node. The (single-valued and multi-valued) functions in Lemma 5.3.21 defined on $\rho(\Sigma)$ are called as follows:

(i) $\hat{L}$ is the state-signal/state resolvent of $\Sigma$.

(ii) $\hat{A}$ is the unobservable state/state resolvent of $\Sigma$.

(iii) $\hat{B}$ is the signal/state resolvent of $\Sigma$.

(iv) $\hat{C}$ is the (multi-valued) state/signal resolvent of $\Sigma$.

It will be shown in Corollary 5.3.26 below that $\hat{L}$, $\hat{A}$, and $\hat{C}$ are analytic on $\rho(\Sigma)$ in a certain sense.
5.3.23. Remark. It is, of course, possible to define \( \hat{L}(\lambda), \hat{A}(\lambda), \hat{B}(\lambda), \) and \( \hat{D}(\lambda) \) by (5.3.22)–(5.3.25) also for \( \lambda \notin \rho(\Sigma) \), but for \( \lambda \notin \rho(\Sigma) \) these operators need no longer be bounded, and also \( \hat{L}(\lambda), \hat{A}(\lambda) \) and \( \hat{B}(\lambda) \) may then be multi-valued. By Lemma 5.3.2, \( \lambda \in \rho(\Sigma) \) if and only if \( \hat{L}(\lambda) \) is a continuous operator with closed domain satisfying (5.3.3). All of these four (possibly multi-valued) operators are still closed even when \( \lambda \notin \rho(\Sigma) \). Condition (iii) in Definition 1.1.9 holds with \( \Sigma \) replaced by \( \Sigma_\lambda \) if and only if condition (a) in Definition 5.3.1(i) holds, and in this case also \( \hat{A}(\lambda) \) and \( \hat{B}(\lambda) \) are single-valued. The node \( \Sigma_\lambda \) is regular if and only if, in addition, condition (iii) in 1.1.9 holds with \( \Sigma \) replaced by \( \Sigma_\lambda \), i.e., the subspace
\[
X_0^\lambda = \{ x^0 \in X \mid [x^w] \in \text{dom}(\hat{L}(\lambda)) \text{ for some } w \in W \}
\]
is dense in \( [X/W] \).

5.3.24. Lemma. Let \( \Sigma = (V; X, W) \) be a resolvable s/s node. Then, for each \( \lambda \in \rho(\Sigma) \) the generating subspace \( V \) of \( \Sigma \) can be recovered from the value at the point \( \lambda \in \rho(\Sigma) \) of the state-signal/state resolvent \( \hat{L} \) by the formula
\[
(5.3.30) \quad V = \begin{bmatrix}
\lambda & -1_X & 0 \\
1_X & 0 & 0 \\
0 & 0 & 1_W \\
\end{bmatrix} \text{gph}(\hat{L}(\lambda)).
\]

Proof. This follows from (5.3.2) and (5.3.22). \( \square \)

We saw in Lemma 1.6.4 that the characteristic node bundle \( \hat{E} \) of a s/s node \( \Sigma \) is analytic in the full complex plane. As the following lemma shows, the graph of \( \hat{L} \) in (5.3.22) is also analytic in the full complex plane, and the graphs of \( \hat{B} \) in (5.3.24) and \( \hat{C} \) in (5.3.25) as well as the characteristic signal bundle \( \hat{F} \) of \( \Sigma \) are analytic in \( \rho(\Sigma) \).

5.3.25. Lemma. Let \( \Sigma = (V, X, W) \) be a resolvable s/s node with characteristic node bundle \( \hat{E} \), let \( X_1 \) be a closed subspace of \( X \), and let \( E \in B(X; Z) \) for some \( H \)-space \( Z \).

(i) The graph of the signal-signal/state resolvent \( \hat{L} \) in (5.3.22) is an analytic vector bundle in the full complex plane.

(ii) The vector bundle \( \mathcal{G}_E \) in \([X_1/W]\) whose fibers are given by
\[
(5.3.31) \quad \mathcal{G}_E(\lambda) := \begin{bmatrix}
1_X & 0 & 0 \\
0 & E & 0 \\
0 & 0 & 1_W \\
\end{bmatrix} \left( \hat{E}(\lambda) \cap \begin{bmatrix} X_1 \\ X \end{bmatrix} \right)
\]
is analytic in \( \rho(\Sigma) \).

(iii) The graph of the signal/state resolvent \( \hat{B} \) in (5.3.24) and the graph of the state/signal resolvent \( \hat{B} \) in (5.3.24) are analytic in \( \rho(\Sigma) \).

(iv) The characteristic signal bundle \( \hat{F} \) of \( \Sigma \) (cf. Definition 1.6.7) is analytic in \( \rho(\Sigma) \).

Proof. The analyticity of \( \hat{L} \) in the full complex plane follows from (5.3.22) and the analyticity of the characteristic node bundle \( \hat{E} \) in the full complex plane. The proof of claim (ii) is analogous to the proof of claim (ii) in Lemma 3.4.3 with (3.4.1) replaced by (5.3.8). Finally, Claims (iii) and (iv) are essentially special cases of (i) (take \( X_1 = \{0\} \) or \( E = 0 \) in (ii), and change the order of the components in the case of \( \mathcal{E} \)). \( \square \)
5.3.26. Corollary. Let \( \Sigma = (V; X, W) \) be a resolvable s/s node. Then the (single-valued and multi-valued) functions \( \hat{S}, \hat{B}, \) and \( \hat{C} \) defined in Lemma 5.3.21 are all analytic in \( \rho(\Sigma) \) in the sense that both the domains and the graphs of these (single-valued or multi-valued) operators are analytic vector bundles in the respective domain and graph spaces.

Proof. This follows from Lemmas 5.3.21 and 5.3.25.

Above we do not claim that the unobservable state/state resolvents \( \hat{\mathcal{A}} \) is analytic (its domain need not be an analytic vector bundle). However, the restriction of \( \hat{\mathcal{A}} \) to \( \cap_{\lambda \in \Omega} \text{ker}(\hat{C}(\lambda)) \) is analytic. (As we see in lemma 6.1.14 below, this is the \( \Omega \)-unobservable subspace of \( \Sigma \), and this is the motivation for the name “unobservable state/state resolvent.”)

It is possible to express the operators in Definition 5.3.22 in terms of the input/state/output resolvent matrix of an i/s/o representation \( \Sigma(i/s/o) \) of \( \Sigma \) whenever the parameter \( \lambda \) belongs to \( \rho(\Sigma(i/s/o)) \).

5.3.27. Lemma. Let \( \Sigma = (V; X, W) \) be a resolvable s/s node with characteristic node bundle \( \mathcal{C} \) and characteristic signal bundle \( \mathcal{F} \). Define \( \hat{\mathcal{S}}, \hat{\mathcal{A}}, \hat{\mathcal{B}}, \) and \( \hat{\mathcal{C}} \) as above. Let \( \Sigma(i/s/o) = (S; \tilde{X}, \tilde{Y}, \tilde{Z}) \) be an i/s/o representation of \( \Sigma \) with i/s/o resolvent matrix \( \hat{\mathcal{E}}(i/s/o) = \left[ \begin{array}{c} \hat{\mathcal{E}}_{i/s/o}(\lambda) \\ \hat{\mathcal{E}}_{i/s/o}(\lambda) \end{array} \right] \) and let \( \lambda \in \rho(\Sigma(i/s/o)) \). Then

(i) \( W = \hat{\mathcal{S}}(\lambda) + \mathcal{Y} \),
(ii) \( w_\lambda \in \hat{\mathcal{S}}(\lambda) \) if and only if \( P_{\mathcal{Y}} w_\lambda = \hat{\mathcal{D}}_{i/s/o}(\lambda) P_{\mathcal{U}} w_\lambda \),
(iii) \( x_\lambda = \hat{\mathcal{A}}(\lambda) x_0 \) if and only if \( \hat{\mathcal{C}}_{i/s/o}(\lambda) x_0 = 0 \) and \( x_\lambda = \hat{\mathcal{A}}_{i/s/o}(\lambda) x_0 \),
(iv) \( w_\lambda \in \hat{\mathcal{C}}(\lambda) x_0 \) if and only if \( \hat{\mathcal{F}} w_\lambda = \hat{\mathcal{E}}_{i/s/o}(\lambda) x_0 + \hat{\mathcal{E}}_{i/s/o}(\lambda) P_{\mathcal{Y}} w_\lambda \),
(v) \( x_\lambda = \hat{\mathcal{B}}(\lambda) w_\lambda \) if and only if \( \hat{\mathcal{E}} w_\lambda = \hat{\mathcal{C}}_{i/s/o}(\lambda) x_0 \) and \( x_\lambda = \hat{\mathcal{C}}_{i/s/o}(\lambda) P_{\mathcal{U}} w_\lambda \).
(vi) The following conditions are equivalent:

(a) \( \left[ \begin{array}{c} x_0 \\ w_\lambda \end{array} \right] \in \hat{\mathcal{C}}(\lambda) \),
(b) \( x_\lambda = \hat{\mathcal{A}}(\lambda) \left[ \begin{array}{c} x_0 \\ w_\lambda \end{array} \right] \),
(c) \( w_\lambda \in \hat{\mathcal{C}}(\lambda) x_0 \) and \( x_\lambda = \hat{\mathcal{A}}_{i/s/o}(\lambda) x_0 + \hat{\mathcal{C}}_{i/s/o}(\lambda) P_{\mathcal{U}} w_\lambda \).

Proof. This follows from Theorem 5.3.6 and Lemma 5.3.21.

The same conditions can also be written in the following alternative way:

5.3.28. Lemma. Under the same assumptions and with the same notations as in Lemma 5.3.27 the following claims are true:

(i) \( \hat{\mathcal{S}}(\lambda) = \left[ \hat{\mathcal{A}}_{i/s/o}(\lambda) \hat{\mathcal{B}}_{i/s/o}(\lambda) P_{\mathcal{U}} \right] |_{\text{dom}(\hat{\mathcal{C}}(\lambda))} \),
(ii) \( \hat{\mathcal{A}}(\lambda) = \hat{\mathcal{A}}_{i/s/o}(\lambda) |_{\text{dom}(\hat{\mathcal{C}}(\lambda))} \),
(iii) \( \hat{\mathcal{B}}(\lambda) = \hat{\mathcal{B}}_{i/s/o}(\lambda) P_{\mathcal{U}} |_{\hat{\mathcal{C}}(\lambda)} \) and \( \hat{\mathcal{B}}_{i/s/o}(\lambda) = \hat{\mathcal{B}}(\lambda) (I_{\mathcal{Y}} \hat{\mathcal{D}}_{i/s/o}(\lambda) + I_{\mathcal{U}}) \),
(iv) \( \text{gph}(\hat{\mathcal{C}}(\lambda)) = \text{gph}(\hat{\mathcal{E}}_{i/s/o}(\lambda)) + \hat{\mathcal{S}}(\lambda) \) and \( \hat{\mathcal{E}}_{i/s/o}(\lambda) = P_{\mathcal{Y}}(\hat{\mathcal{S}}(\lambda)) \).

Proof. This follows from Lemma 5.3.27.

5.3.29. Lemma. Let \( \Sigma = (V; X, W) \) be a resolvable s/s node with signal/state, state/signal, and unobservable state/state resolvents \( \hat{\mathcal{B}}, \hat{\mathcal{C}}, \) and \( \hat{\mathcal{A}} \), respectively.
Then the following formulas are valid for all \( \lambda \in \rho(\Sigma) \):

\[
\text{rng} (\hat{B}(\lambda)) = [0 \ 1_X \ 0] V, \quad (5.3.32a)
\]

\[
= \left\{ x \in X \mid \begin{bmatrix} \frac{z}{u} \end{bmatrix} \in V \text{ for some } \begin{bmatrix} \frac{x}{w} \end{bmatrix} \in [\frac{X}{W}] \right\},
\]

\[
\text{ker} \left( \begin{bmatrix} \hat{A}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} (0) \right) = \begin{bmatrix} 1_X \ 0 \ 0 \end{bmatrix} (V \cap [\begin{bmatrix} 0 \ 0 \end{bmatrix}]), \quad (5.3.32b)
\]

\[
= \left\{ z \in X \mid \begin{bmatrix} \frac{z}{0} \end{bmatrix} \in V \right\}.
\]

Thus in particular, the left-hand sides of the formulas in \(5.3.32\) are independent of \( \lambda \), as long as \( \lambda \in \rho(\Sigma) \).

**Proof.** This follows from Theorems 5.3.6 and 5.3.9 and Lemma 5.3.28.

\[\square\]

5.3.30. **Lemma.** Let \( \Sigma = (V; X, W) \) be a resolvable s/s node with signal/state, state/signal, and unobservable state/state resolvents \( \hat{B}, \hat{C}, \) and \( \hat{A} \), respectively.

(i) The following conditions are equivalent:
(a) \( \Sigma \) satisfies condition (ii) in Definition 1.1.9.
(b) \( \begin{bmatrix} \hat{A}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} \) is injective for some (and hence for all) \( \lambda \in \rho(\Sigma) \).

(ii) The following conditions are equivalent:
(a) \( \Sigma \) satisfies condition (iii) in Definition 1.1.9.
(b) \( \text{rng} (\hat{B}(\lambda)) \) is dense in \( X \) for some (and hence for all) \( \lambda \in \rho(\Sigma) \).

(iii) The following conditions are equivalent:
(a) \( \Sigma \) is a regular i/s/o, i.e., conditions (i)–(iii) in Definition 1.1.9 hold,
(b) \( \begin{bmatrix} \hat{A}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} \) is injective and \( \text{rng} (\hat{B}(\lambda)) \) is dense in \( X \) for some (and hence for all) \( \lambda \in \rho(\Sigma) \).

**Proof.** This follows immediately from Lemma 5.3.29.

\[\square\]

5.3.34. **The resolvent family of bounded s/s nodes.**

5.3.31. **Definition.** Let \( \Sigma = (V; X, W) \) be a resolvable s/s node. By the resolvent family of bounded s/s nodes induced by \( \Sigma \) we mean the family of bounded s/s nodes \( \Sigma^\lambda = (V^\lambda; X, W) \), \( \lambda \in \rho(\Sigma) \), where \( V^\lambda \) is defined by (5.3.2).

5.3.32. **Lemma.** Let \( \Sigma = (V; X, W) \) be a resolvable s/s node with characteristic signal bundle \( \hat{F} \), and let \( \Sigma^\lambda = (V^\lambda; X, W) \), \( \lambda \in \rho(\Sigma) \), be the family of bounded s/s nodes induced by \( \Sigma \). Then the following claims are true for all \( \lambda \in \rho(\Sigma) \):

(i) The canonical input space of \( \Sigma^\lambda \) is equal to \( \hat{F}(\lambda) \) (see Definition 2.2.30).

(ii) The operator \( F \) in (1.1.3) with \( \Sigma \) replaced by \( \Sigma^\lambda \) is equal to \( \hat{L}(\lambda) \), where \( \hat{L} \) is the state-signal/state resolvent of \( \Sigma \) (see Definition 5.3.22).

**Proof.** This follows from Definitions 1.6.7, 2.2.30, 5.3.22, and 5.3.31.

\[\square\]
5.4. Notes and Comments (Jan 02, 2016)

All results in this chapter remain valid if we allow the state spaces of the i/s/o and s/s systems to be $B$-spaces instead of $H$-spaces.

All results in this chapter about i/s/o systems remain valid if we allow the input and output spaces of the i/s/o systems to be $B$-spaces instead of $H$-spaces. However, since Theorem 2.2.27 is not valid in a $B$-spaces setting, and since we make heavy use of bounded i/s/o representations in the study of the resolvent set of a s/s system, many of the proofs of the results about the resolvent set of a s/s system in Chapter 5.3 are not valid if the signal space is a $B$-space.

As Theorem 5.1.13 shows, our class of resolvable i/s/o nodes coincides with the class of i/s/o nodes described in [Staffans 2005, Section 4.7], and hence it is a slight extension of the class of i/s/o nodes described in [Salamon 1987, Section 2.1], [Weiss 1994, Definition 2.1], and [Arov and Nudelman 1996, Section 2], etc. In Staffans [2005] these i/s/o nodes were called “operator nodes”, in Salamon [1987] they were called “semigroup control systems”, in Weiss [1994] they were called “abstract linear systems”, and in Arov and Nudelman [1996] they were called “linear stationary dynamical systems in continuous time”. In Salamon [1987], Weiss [1994], and Arov and Nudelman [1996] it is assumed, in addition, that the main operator generates a $C_0$ semigroup. Another still slightly smaller class of i/s/o nodes is the class of “regular linear systems” in the sense of Weiss [1994, Definition 2.2].
CHAPTER 6

Input/State/Output Systems in the Frequency Domain (Feb 02, 2016)

In Chapters 1 and 2 we defined various time domain notions for i/s/o systems, such as the reachable and unobservable subspaces, controllability and observability, strong and unobservable invariance, external equivalence, intertwinements, compressions, dilations, restrictions, extensions, and projections. For i/s/o nodes with nonempty resolvent sets all these notions have natural frequency domain counterparts, which will be described in this chapter. The frequency domain definitions are analogous to their time domain counterparts, with time domain trajectories throughout replaced by (frequency domain) Ω-trajectories, where Ω is a nonempty open subset of the resolvent set of the i/s/o node.
6.1. Frequency Domain Input/State/Output Systems (Feb 02, 2016)

In this section we first introduce the notion of a frequency domain trajectory of an i/s/o node, and then we use this notion to define and study various frequency domain notions and properties of frequency domain i/s/o systems.

6.1.1. Introduction to frequency domain i/s/o systems. Equation (2.1.1) describes the time domain evolution of a regular i/s/o system Σ. From the time domain equation (2.1.1) we get the frequency domain equation (5.1.8a) by taking (formal) Laplace transforms as explained in the connection with (5.1.8a). The corresponding result for a general i/s/o system is given in (5.2.10a).

It is possible to introduce the notion of a frequency domain trajectory induced by an i/s/o node by replacing (2.1.10) by (5.2.10a), and at the same time replacing the time domain interval \( I \) by some open set \( Ω \) in the complex (frequency domain) plane \( \mathbb{C} \).

6.1.1. Definition. Let \( Σ = (S; X, U, Y) \) be an i/s/o node.

(i) By a (frequency domain) \( Ω \)-trajectory generated by \( Σ \), where \( Ω \) is some nonempty open set in \( \mathbb{C} \), we mean a quadruple \((\hat{x}, \hat{y}; x^0, \hat{u})\) where \( \hat{x}, \hat{y}, \hat{u} \) are analytic functions defined on \( Ω \) with values in \( X, Y, U \) and \( x^0 \) is a constant in \( X \), which satisfy the equivalent equations

\[
\begin{align*}
\lambda \hat{x}(\lambda) - x^0 & \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \\
-\hat{y}(\lambda) & \in \left(S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \\
\hat{x}(\lambda) & \in \text{dom}(S),
\end{align*}
\]

for all \( λ \in Ω \) (if \( Σ \) is a regular i/s/o node then we may replace the second “\( \in \)” in the above formulas by “\( = \)”). The different components of such a trajectory are called as follows: \( \hat{x} \) is the state component, \( \hat{y} \) is the output component, \( \hat{u} \) is the input component, and \( x^0 \) is the initial state of the trajectory \((\hat{x}, \hat{y}; x^0, \hat{u})\).

(ii) By the i/s/o (input/state/output) frequency domain system induced by the i/s/o node \( Σ \) we mean the node \( Σ \) itself together with sets of all \( Ω \)-trajectories generated by \( Σ \). We use the same notation \( Σ = (S; X, U, Y) \) for the frequency domain i/s/o system as for the i/s/o node, and alternatively write “\( Ω \)-trajectories of the i/s/o frequency domain system \( Σ \)” instead of “\( Ω \)-trajectories generated by the i/s/o node \( Σ \)”.

(iii) When the i/s/o node \( Σ \) is closed, or regular, or resolvable, then we also call the frequency domain i/s/o system \( Σ \) closed, or regular, or resolvable, respectively. (See Definitions 2.1.1, 2.1.5, and 5.3.1.)

6.1.2. Remark. Throughout this and the next chapter, when we assume that some set \( Ω \) in \( \mathbb{C} \) is open we at the same time tacitly assume that \( Ω \neq \emptyset \).

Without any further assumptions on \( Σ \) and \( Ω \) there may not exist any nonzero \( Ω \)-trajectories. However, as will be shown in Lemma 6.1.7 below, if \( Σ \) is resolvable and \( Ω \subset ρ(Σ) \), then to each analytic \( U \)-valued function \( \hat{u} \) on \( Ω \) and each \( x^0 \in X \) there exists a unique \( Ω \)-trajectory \((\hat{x}, \hat{y}; x^0, \hat{u})\) of \( Σ \) (with initial state \( x^0 \) and input function \( \hat{u} \)).
6.1.3. REMARK. By the argument at the beginning Section 5.1 that was used to motivate the theory presented in that section, in the case where \( \Omega \) is a right-half plane the set of frequency domain \( \Omega \)-trajectories generated by a regular i/s/o node \( \Sigma \) can be interpreted as formal Laplace transforms of time domain future trajectories of \( \Sigma \). The same argument can also be extended to the case where \( \Sigma \) is a general i/s/o node.

6.1.4. LEMMA. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a frequency domain i/s/o system, and let \( \Omega_i, i = 1, 2 \) be open sets in \( \mathbb{C} \).

(i) If \( \Omega_1 \subset \Omega_2 \), then the restriction of an \( \Omega_2 \)-trajectory of \( \Sigma \) to \( \Omega_1 \) is an \( \Omega_1 \)-trajectory of \( \Sigma \).

(ii) Let \( (\hat{x}_i, \hat{y}_i; x^0_i, \hat{u}_i) \) be two \( \Omega_i \)-trajectories of \( \Sigma, i = 1, 2 \), which coincide on \( \Omega_1 \cap \Omega_2 \) (if \( \Omega_1 \cap \Omega_2 \neq \emptyset \)). Define \( (\hat{x}, \hat{y}; x^0, \hat{u}) \) by

\[
(\hat{x}(\lambda), \hat{y}(\lambda); x^0, \hat{u}(\lambda)) = (\hat{x}_i(\lambda), \hat{y}_i(\lambda); x^0_i, \hat{u}_i(\lambda)), \quad \lambda \in \Omega_i, \quad i = 1, 2.
\]

Then \( (\hat{x}, \hat{y}; x^0, \hat{u}) \) is an \( \Omega \)-trajectory of \( \Sigma \), where \( \Omega := \Omega_1 \cup \Omega_2 \).

Proof. This follows immediately from Definition 6.1.1. \( \square \)

6.1.5. LEMMA. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a frequency domain i/s/o system, and let \( \Omega \) be a open set in \( \mathbb{C} \). Then the following conditions are equivalent:

(i) the initial state \( x^0 \) of every \( \Omega \)-trajectory \( (\hat{x}, \hat{y}; x^0, \hat{u}) \) of \( \Sigma \) is determined uniquely by \( \hat{x}, \hat{y}, \) and \( \hat{u} \).

(ii) \( \text{mul}(S) \cap \left[ \frac{x^0}{(\hat{0})} \right] = \{0\} \).

In particular, these conditions are true if \( \Sigma \) is a regular i/s/o system.

Proof. This follows immediately from Definition 6.1.1. \( \square \)

For a resolvable frequency domain i/s/o system \( \Sigma \) satisfying \( \Omega \subset \rho(\Sigma) \) the question of existence of \( \Omega \)-trajectories has the following natural answer. Since this class of systems will play a central role in this chapter we introduce the following terminology.

6.1.6. DEFINITION (cf. Definitions 5.1.2 and 5.2.8). Let \( \Omega \) be a (nonempty) open set in \( \mathbb{C} \). By an \( \Omega \)-resolvable i/s/o node or frequency domain i/s/o system we mean a resolvable i/s/o node respectively frequency domain i/s/o system \( \Sigma \) satisfying \( \Omega \subset \rho(\Sigma) \).

6.1.7. LEMMA. Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an \( \Omega \)-resolvable frequency domain i/s/o system with i/s/o resolvent matrix \( \hat{\Theta} = \begin{bmatrix} \hat{\Theta} & \hat{\Theta} \\ \hat{\Theta} & \hat{\Theta} \end{bmatrix} \).

(i) For every \( x^0 \in \mathcal{X} \) and for every analytic \( \mathcal{U} \)-valued function \( \hat{u} \) in \( \Omega \) the frequency domain i/s/o system \( \Sigma \) has a unique \( \Omega \)-trajectory \( (\hat{x}, \hat{y}; x^0, \hat{u}) \) with initial state \( x^0 \) and input function \( \hat{u} \). The state and output components of this trajectory are given by

\[
(6.1.2) \quad \begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \hat{\Theta}(\lambda) \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{\Theta}(\lambda)x^0 + \hat{\Theta}(\lambda)\hat{u}(\lambda) \\ \hat{\Theta}(\lambda)x^0 + \hat{\Theta}(\lambda)\hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \Omega.
\]

(ii) \( \Sigma \) is determined uniquely by the set of all \( \Omega \)-trajectories of \( \Sigma \) evaluated at some point \( \lambda \in \Omega \).
Proof. (i) Claim (i) follows from Definition 6.1.1 and Lemma 5.2.13.

(ii) Claim (ii) follows from the fact that the set of Ω-trajectories evaluated at \( \lambda \in \Omega \) determines \( \hat{S}(\lambda) \) uniquely, and the system operator \( S \) of \( \Sigma \) is uniquely determined by \( \hat{S}(\lambda) \) through formula (5.2.12a).

\[ \Box \]

6.1.2. The frequency domain behavior and external equivalence.

6.1.8. Definition. Let \( \Sigma = (S; X, U, Y) \) be a frequency domain i/s/o system, and let \( \Omega \) be a open set in \( \mathbb{C} \). By the \( \Omega \)-behavior \( W_\Omega \Sigma \) of \( \Sigma \) we mean the set of all analytic \([U; Y]\)-valued functions \([\hat{u}; \hat{y}]\) in \( \Omega \) for which there exists some analytic \( X \)-valued function \( \hat{x} \) in \( \Omega \) such that \((\hat{x}, \hat{y}; 0, \hat{u})\) is an \( \Omega \)-trajectory of \( \Sigma \) (with initial state \( x^0 = 0 \)).

6.1.9. Lemma. Let \( \Sigma \) be an open set in \( \mathbb{C} \), and let \( \Sigma = (S; X, U, Y) \) be an \( \Omega \)-resolvable frequency domain i/s/o system with with i/o resolvent \( \hat{D} \) and \( \Omega \)-behavior \( W_\Omega \Sigma \). Then

\[ W_\Omega \Sigma = \left\{ \left[ \frac{\hat{u}}{\hat{y}} \right] \bigg| \hat{u} \text{ is an analytic } U \text{-valued function in } \Omega \right\}. \]

Proof. This follows from Lemma 6.1.7 and Definition 6.1.10.

\[ \Box \]

6.1.10. Definition. Let \( \Sigma_i = (S_i; X_i, U, Y) \) be two frequency domain i/s/o systems (with the same input and output spaces), and let \( \Omega \) be a open set in \( \mathbb{C} \). We say that \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent if they have the same \( \Omega \)-behavior.

6.1.11. Lemma. Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma_i = (S_i; X_i, U, Y) \) be two \( \Omega \)-resolvable frequency domain i/s/o systems with i/o resolvent matrices \( \hat{D}_i \), \( i = 1, 2 \). Then \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent if and only if \( \hat{D}_1(\lambda) = \hat{D}_2(\lambda) \) for all \( \lambda \in \Omega \).

Proof. This follows from Definitions 6.1.8 and 6.1.10.

\[ \Box \]

6.1.3. Frequency domain controllability and observability.

6.1.12. Definition. Let \( \Sigma = (S; X, U, Y) \) be a frequency domain i/s/o system, and let \( \Omega \) be an open set in \( \mathbb{C} \).

(i) A state vector \( x_0 \in X \) is called exactly \( \Omega \)-reachable for \( \Sigma \) if there exists an \( \Omega \)-trajectory \((\hat{x}, \hat{y}; 0, \hat{u})\) of \( \Sigma \) (with zero initial state) such that \( x_0 = \hat{x}(\lambda) \) for some \( \lambda \in \Omega \).

(ii) An \( \Omega \)-trajectory \((\hat{x}, \hat{y}; x^0, \hat{u})\) of \( \Sigma \) is called \( \Omega \)-unobservable if \( \hat{u} = 0 \) and \( \hat{y} = 0 \).

(iii) A state vector \( x^0 \in X \) is called \( \Omega \)-unobservable for \( \Sigma \) if there exists an \( \Omega \)-unobservable \( \Omega \)-trajectory \((\hat{x}, 0; x^0, 0)\) of \( \Sigma \) with this initial state.

It is easy to see that the sets of all \( \Omega \)-unobservable states is a subspace of \( X \).

6.1.13. Definition. Let \( \Sigma = (S; X, U, Y) \) be a frequency domain i/s/o system, and let \( \Omega \) be an open set in \( \mathbb{C} \).

(i) The linear span of all exactly \( \Omega \)-reachable states of \( \Sigma \) is called the exactly \( \Omega \)-reachable subspace of \( \Sigma \) and it is denoted by \( R_{\Omega,\text{exact}}^\Omega \Sigma \).

(ii) The closure of \( R_{\Omega,\text{exact}}^\Omega \Sigma \) is called the (approximately) \( \Omega \)-reachable subspace of \( \Sigma \) and it is denoted by \( R_{\text{approx}}^\Omega \Sigma \).
(iii) The subspace of all $\Omega$-unobservable states of $\Sigma$ is called the $\Omega$-unobservable subspace of $\Sigma$, and it is denoted by $\Omega^1$.
(iv) $\Sigma$ is $\Omega$-controllable if $\mathcal{R}_\Sigma^\Omega = \mathcal{X}$, and $\Sigma$ is $\Omega$-observable if $\Omega^1 = \{0\}$.

6.1.14. Lemma. Let $\Omega$ be an open set in $\mathbb{C}$, and let $\Sigma = (S, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an $\Omega$-resolvable frequency domain $i/s/o$ system with $i/s$ and $s/o$ resolvents $\mathcal{B}$ respectively $\mathcal{E}$. Then, with the notation introduced in Definition 6.1.13
\[(6.1.4) \quad \mathcal{R}_\Sigma^{\Omega, \text{exact}} = \text{span}_{\lambda \in \Omega} \text{rng} (\mathcal{B}(\lambda)), \]
\[(6.1.5) \quad \mathcal{R}_\Sigma^{\Omega} = \bigvee_{\lambda \in \Omega} \text{rng} (\mathcal{B}(\lambda)), \]
\[(6.1.6) \quad \Omega^1 = \bigcap_{\lambda \in \Omega} \ker (\mathcal{E}(\lambda)). \]

In particular, $\Omega^1$ is closed.

Proof. It follows from Lemma 6.1.7 and Definition 6.1.13. \qed

6.1.15. Definition. Let $\Sigma = (S, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a frequency domain $i/s/o$ system, let $\mathcal{Z}$ be a subspace of $\mathcal{X}$, and let $\Omega$ be an open set in $\mathbb{C}$.

(i) $\mathcal{Z}$ is an $\Omega$-invariant subspace for $\Sigma$ if it is true for every $\Omega$-trajectory $(\dot{x}, \dot{y}, x^0, 0)$ with initial state $x^0 \in \mathcal{Z}$ (and zero input) that $\dot{x}(\lambda) \in \mathcal{Z}$ for all $\lambda \in \Omega$.
(ii) $\mathcal{Z}$ of $\mathcal{X}$ is strongly $\Omega$-invariant for $\Sigma$ if it is true for every $\Omega$-trajectory $(\dot{x}, \dot{y}, x^0, \dot{u})$ of $\Sigma$ with initial state $x^0 \in \mathcal{Z}$ that $\dot{x}(\lambda) \in \mathcal{Z}$ for all $\lambda \in \Omega$.
(iii) A subspace $\mathcal{Z}$ of $\mathcal{X}$ is unobservably $\Omega$-invariant for $\Sigma$ if it is true that for every $x^0 \in \mathcal{Z}$ there exists an $\Omega$-unobservable $\Omega$-trajectory $(\dot{x}, 0; x^0, 0)$ of $\Sigma$ with initial state $x^0$ satisfying $\dot{x}(\lambda) \in \mathcal{Z}$ for all $\lambda \in \Omega$.

6.1.16. Lemma. Let $\Omega$ be an open set in $\mathbb{C}$, and let $\Sigma = (S, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an $\Omega$-resolvable frequency domain $i/s/o$ system with $i/s/o$ resolvent matrix $\hat{\mathcal{S}} = \begin{bmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{bmatrix}$, and let $\mathcal{Z}$ be a subspace of $\mathcal{X}$. Then the following claims are true.

(i) $\mathcal{Z}$ is an $\Omega$-invariant subspace for $\Sigma$ if and only if $\hat{\mathcal{A}}(\lambda) \mathcal{Z} \subset \mathcal{Z}$ for all $\lambda \in \Omega$.
(ii) $\mathcal{Z}$ is a strongly $\Omega$-invariant subspace for $\Sigma$ if and only if $\hat{\mathcal{A}}(\lambda) \mathcal{Z} \subset \mathcal{Z}$ and $\text{rng} (\hat{\mathcal{B}}(\lambda)) \subset \mathcal{Z}$ for all $\lambda \in \Omega$.
(iii) $\mathcal{Z}$ is an unobservably $\Omega$-invariant subspace for $\Sigma$ if and only if $\hat{\mathcal{A}}(\lambda) \mathcal{Z} \subset \mathcal{Z}$ and $\mathcal{Z} \subset \ker (\hat{\mathcal{E}}(\lambda))$ for all $\lambda \in \Omega$.

Proof. This follows from Lemma 6.1.7 and Definition 6.1.15. \qed

6.1.17. Lemma. Let $\Omega$ be an open set in $\mathbb{C}$, and let $\Sigma = (S, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an $\Omega$-resolvable frequency domain $i/s/o$ system.

(i) If $\mathcal{Z}$ is a strongly $\Omega$-invariant or unobservable $\Omega$-invariant subspace for $\Sigma$, then the closure of $\mathcal{Z}$ is also strongly $\Omega$-invariant respectively $\Omega$-unobservably invariant for $\Sigma$.
(ii) If both $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are strongly $\Omega$-invariant for $\Sigma$, then $\mathcal{Z}_1 + \mathcal{Z}_2$ and $\mathcal{Z}_1 \lor \mathcal{Z}_2$ are strongly $\Omega$-invariant for $\Sigma$.
(iii) If both $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are unobservably $\Omega$-invariant for $\Sigma$, then $\mathcal{Z}_1 \cap \mathcal{Z}_1$ is unobservably $\Omega$-invariant for $\Sigma$. 

PROOF. If the invariance conditions (i)–(iii) listed in Lemmas 6.1.16 hold for some subspace $Z$ of $X$, then they also hold if we replace $Z$ by $\bar{Z}$. From this (i) follows. Also the assertions (ii) and (iii) follow from Lemma 6.1.16. □

6.1.18. Lemma. Let $\Omega$ be an open set in $\mathbb{C}$, and let $\Sigma = (S; X, U, Y)$ be an $\Omega$-resolvable frequency domain input/output system.

(i) The subspace
\[ \mathcal{R}_\Omega^\Sigma := \text{span}_{\lambda \in \Omega, n \in \mathbb{Z}} \text{rng} (\hat{A}(\lambda)^n \hat{B}(\lambda)), \]

is the minimal strongly $\Omega$-invariant subspace for $\Sigma$, i.e., $\mathcal{R}_\Omega^\Sigma$ is strongly $\Omega$-invariant for $\Sigma$, and $\mathcal{R}_\Omega^\Sigma$ is contained in every other strongly $\Omega$-invariant subspace for $\Sigma$. This subspace satisfies $\mathcal{R}_\Omega^{\Sigma,\text{exact}} \subset \mathcal{R}_\Omega^\Sigma \subset \mathcal{R}_\Omega^{\Sigma,\text{in}},$ and hence $\mathcal{R}_\Omega^\Sigma = \mathcal{R}_\Omega^{\Sigma,\text{in}}$.

(ii) The $\Omega$-reachable subspace $\mathcal{R}_\Omega^\Sigma$ is the minimal closed strongly $\Omega$-invariant subspace for $\Sigma$, i.e., $\mathcal{R}_\Omega^\Sigma$ is strongly $\Omega$-invariant for $\Sigma$, and $\mathcal{R}_\Omega^\Sigma$ is contained in every other closed strongly $\Omega$-invariant subspace for $\Sigma$.

(iii) The $\Omega$-unobservable subspace $\mathcal{U}_\Omega^\Sigma$ is the maximal unobservable $\Omega$-invariant subspace for $\Sigma$, i.e., $\mathcal{U}_\Omega^\Sigma$ is unobservable $\Omega$-invariant, and $\mathcal{U}_\Omega^\Sigma$ contains every other unobservable $\Omega$-invariant subspace for $\Sigma$.

PROOF. (i) If $Z$ is strongly $\Omega$-invariant for $\Sigma$, then it follows from Lemma 6.1.16 that $\hat{A}(\lambda) Z \subset Z$ and $\text{rng}(\hat{B}(\lambda)) \subset Z$ for all $\lambda \in \Omega$. This implies that $\text{rng}(\hat{A}(\lambda)^n \hat{B}(\lambda)) \subset Z$ for all $n \in \mathbb{Z}^+$, and hence $\mathcal{R}_\Omega^\Sigma \subset Z$.

We next show that $\mathcal{R}_\Omega^\Sigma$ is strongly $\Omega$-invariant. One half of condition (ii) in Lemma 6.1.16 is clearly satisfied since $\text{rng}(\hat{B}(\lambda)) \subset \mathcal{R}_\Omega^\Sigma$ for all $\lambda \in \Omega$, but before appealing to Lemma 6.1.16 we still have to check that $\hat{A}(\lambda) \mathcal{R}_\Omega^\Sigma \subset \mathcal{R}_\Omega^\Sigma$. Every vector in $\mathcal{R}_\Omega^\Sigma$ is a finite sum of terms of the form $\hat{A}(\lambda)^n \hat{B}(\lambda) u$ for some $n \in \mathbb{Z}^+$, $\mu \in \Omega$, and some $u \in U$, and we must show that also $\hat{A}(\lambda) \hat{A}(\mu)^n \hat{B}(\mu) u$ is of the same form. This is clear if $\lambda = \mu$ (simply replace $n$ by $n + 1$). If $\lambda \neq \mu$, then we can use the second equation in (5.2.28) to get

\[ \hat{A}(\lambda) \hat{A}(\mu)^n \hat{B}(\mu) u = \hat{A}(\mu)^n \hat{A}(\lambda) \hat{B}(\mu) u = \frac{1}{\mu - \lambda} \hat{A}(\mu)^n (\hat{B}(\lambda)) - \hat{B}(\mu) u. \]

Here $\hat{A}(\mu)^n \hat{B}(\mu) u \in \mathcal{R}_\Omega^\Sigma$, and

\[ \hat{A}(\mu)^n \hat{B}(\lambda) u = \hat{A}(\mu)^{n-1} \hat{A}(\mu) \hat{B}(\lambda) u = \hat{A}(\mu)^{n-1} \hat{A}(\lambda) \hat{B}(\mu) u = \hat{A}(\lambda) \hat{A}(\mu)^{n-1} \hat{B}(\mu) u, \]

which is otherwise the same as the original term, but $n$ has been replaced by $n - 1$. The same argument can be repeated until $n$ has been replaced by zero, and we are left with a term of the form $\hat{A}(\lambda) \hat{B}(\mu) u$. This term can be rewritten as

\[ \hat{A}(\lambda) \hat{B}(\mu) u = \frac{1}{\mu - \lambda} (\hat{B}(\lambda)) - \hat{B}(\mu) u \in \mathcal{R}_\Omega^\Sigma. \]

Thus $\hat{A}(\lambda) \hat{A}(\mu)^n \hat{B}(\mu) u \in \mathcal{R}_\Omega^\Sigma$ for all $n \in \mathbb{Z}^+$, $\mu \in \Omega$, and some $u \in U$, and we conclude that $\mathcal{R}_\Omega^\Sigma$ is strongly $\Omega$-invariant.

That $\mathcal{R}_\Omega^{\Sigma,\text{exact}} \subset \mathcal{R}_\Omega^\Sigma$ follows from Lemma 6.1.14 and the inclusion $\mathcal{R}_\Omega^\Sigma \subset \mathcal{R}_\Omega^{\Sigma,\text{in}}$ follows from the second identity in (5.2.33) and the fact that $\text{rng}(\hat{B}(\lambda)^n) \subset \mathcal{R}_\Omega^\Sigma$. 
for all $n \in \mathbb{Z}^+$ (see Lemmas 6.1.14 and A.3.6). This implies that $\mathcal{R}_\Omega^{\Sigma_2} = \mathcal{R}_\Sigma^{\Sigma_2}$ since $\mathcal{R}_\Sigma^{\Sigma_2} = \mathcal{R}_\Sigma^{\Sigma_2,\text{exact}}$.

(ii) That (ii) holds follows from (i) and Lemma 6.1.17.

(iii) Trivially every unobservably $\Omega$-invariant subspace for $\Sigma$ must be contained in $\mathcal{U}_\Sigma^{\Omega}$, and by Lemma 6.1.14 $\mathcal{U}_\Sigma^{\Omega} \subset \ker (\mathcal{C}(\lambda))$ for every $\lambda \in \Omega$. Thus, to complete the proof of (iii) we must still show that $\mathcal{U}_\Sigma^{\Omega}$ is $\Omega$-invariant for $\Sigma$, and to do this it suffices to show that $\mathcal{C}(\lambda) \hat{A}(\mu)x = 0$ for all $\lambda, \mu \in \Omega$ whenever $\mathcal{C}(\lambda)x = 0$ for all $\lambda \in \Omega$. That this is true follows from the third identity in (5.2.28) in the case where $\lambda \neq \mu$ and from the third identity in (5.2.33) in the case where $\lambda = \mu$. □

6.1.5. Frequency domain intertwinements.

6.1.19. Definition. Let $\Sigma_i = (S_i, \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two frequency domain i/s/o systems (with the same input and output spaces), and let $\Omega$ be an open set in $\mathbb{C}$. We say that $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ if the following two conditions hold:

(i) If $(\hat{x}_1, \hat{y}_1; \hat{x}_1^0, \hat{u})$ is an $\Omega$-trajectory of $\Sigma_1$ with $x_1^0 \in \text{dom}(P)$, then for every $\hat{x}_2^0 \in P x_1^0$ there exists an $\Omega$-trajectory $(\hat{x}_2, \hat{y}_2; \hat{x}_2^0, \hat{u})$ of $\Sigma_2$ satisfying $\hat{x}_2(\lambda) \in P \hat{x}_1(\lambda)$ for all $\lambda \in \Omega$.

(ii) Condition (i) above also holds if we interchange $\Sigma_1$ and $\Sigma_2$ and replace $P$ by $P^{-1}$. In other words, if $(\hat{x}_2, \hat{y}_2; \hat{x}_2^0, \hat{u})$ is an $\Omega$-trajectory of $\Sigma_2$ with $x_2^0 \in \text{rng}(P)$, then for every $\hat{x}_1^0 \in P^{-1} x_2^0$ there exists an $\Omega$-trajectory $(\hat{x}_1, \hat{y}_1; \hat{x}_1^0, \hat{u})$ of $\Sigma_1$ satisfying $\hat{x}_1(\lambda) \in P^{-1} \hat{x}_2(\lambda)$ for all $\lambda \in \Omega$.

6.1.20. Definition. Let $\Sigma_i = (S_i, \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two frequency domain i/s/o systems (with the same input and output spaces), and let $\Omega$ be an open set in $\mathbb{C}$.

(i) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-pseudo-similar if $\Sigma_1$ and $\Sigma_2$ are intertwinied by a closed single-valued injective linear operator $P: \mathcal{X} \to \mathcal{X}_1$ with dense domain and dense range, called the $\Omega$-pseudo-similarity operator.

(ii) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-similar if $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by a bounded linear operator $P \in \mathcal{B}(\mathcal{X}_1; \mathcal{X}_2)$ with a bounded inverse $P^{-1} \in \mathcal{B}(\mathcal{X}_2; \mathcal{X}_1)$, called the $\Omega$-similarity operator.

6.1.21. Lemma. Let $\Sigma_i = (S_i, \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two frequency domain i/s/o systems, let $\Omega$ be an open set in $\mathbb{C}$, and denote the exactly $\Omega$-reachable subspaces and the $\Omega$-unobservable subspaces of $\Sigma_i$ by $\mathcal{R}_\Sigma^{\Omega,\text{exact}}$, respectively $\mathcal{U}_\Sigma^{\Omega}$, $i = 1, 2$. If $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by some $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$, then the following claims hold:

(i) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent.

(ii) $\text{dom}(P)$ is strongly $\Omega$-invariant for $\Sigma_1$. In particular, $\mathcal{R}_\Sigma^{\Omega,\text{exact}} \subset \text{dom}(P)$.

(iii) $\text{rng}(P)$ is strongly $\Omega$-invariant for $\Sigma_2$. In particular, $\mathcal{R}_\Sigma^{\Omega,\text{exact}} \subset \text{rng}(P)$.

(iv) $\ker(P)$ is unobservably $\Omega$-invariant for $\Sigma_1$. In particular, $\ker(P) \subset \mathcal{U}_\Sigma^{\Omega}$.

(v) $\text{mul}(P)$ is unobservably $\Omega$-invariant for $\Sigma_2$. In particular, $\text{mul}(P) \subset \mathcal{U}_\Sigma^{\Omega}$.

Proof. The proof is analogous to the proof of Lemma 1.5.27. □

6.1.22. Lemma. Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma_i = (S_i, \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two $\Omega$-resolveable frequency domain i/s/o systems (with the same input and output spaces), and let $P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$. Then the following claims hold:


of \( \Sigma \) has a trajectory \((\hat{x}, \hat{y}_i; x^0_i, \hat{u}_i)\) are \( \Omega \) trajectories of \( \Sigma_i \), and if \( x^0_i \in Px^0_1 \) and \( \hat{u}_i(\lambda) = \hat{u}_2(\lambda) \) for all \( \lambda \in \Omega \), then \( \hat{x}_2(\lambda) \in Px_1(\lambda) \) and \( \hat{y}_1(\lambda) = \hat{y}_2(\lambda) \) for all \( \lambda \in \Omega \).

In (ii) we have used the convention that the conditions \( x^0_i \in Px^0_1 \) and \( \hat{x}_2(\lambda) \in Px_1(\lambda) \) imply that \( x^0_i \in \text{dom}(P) \) and \( x_1(\lambda) \in \text{dom}(P) \).

**Proof of Lemma 6.1.21** (i) Suppose that condition (i) in Definition 6.1.19 holds. Let \( x^0_2 \in \text{rng}(P) \) and \( x^0_1 \in P^{-1}x^0_2 \), and let \((\hat{x}_2, \hat{y}_2; x^0_2, \hat{u}_2)\) be an \( \Omega \)-trajectory of \( \Sigma_2 \), where \( x^0_2 \in Px^0_1 \). By Lemma 6.1.7, there exists an \( \Omega \)-trajectory \((\hat{x}_1, \hat{y}_1; x^0_1, \hat{u}_2)\) of \( \Sigma_1 \). It follows from condition (i) in Definition 6.1.19 that \( \Sigma_2 \) has an \( \Omega \)-trajectory \((\hat{x}_3, \hat{y}_3; x^0_1, \hat{u}_2)\) satisfying \( \hat{x}_3(\lambda) \in P\hat{x}_1(\lambda) \) for all \( \lambda \in \Omega \). However, since such a trajectory is determined uniquely by \( x^0_2 \) and \( \hat{u}_2 \), it follows that \( \hat{x}_3 = \hat{x}_2 \) and \( \hat{y}_3 = \hat{y}_2 \). In other words, \((\hat{x}_1, \hat{y}_2; x^0_1, \hat{u}_2)\) is an \( \Omega \)-trajectory of \( \Sigma_1 \) satisfying \( \hat{x}_1(\lambda) \in P^{-1}x^0_2 \) for all \( \lambda \in \Omega \). This shows that condition (i) in Definition 6.1.19 implies condition (ii). The converse implication is proved in the same way by interchanging the two systems \( \Sigma_1 \) and \( \Sigma_2 \).

(ii) Suppose first that \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \). Let \( \hat{u} \) be an analytic \( \mathcal{U} \)-valued function in \( \Omega \), let \( x^0_2 \in Px^0_1 \), and let \((\hat{x}_1, \hat{y}_1; x^0_1, \hat{u})\) be the \( \Omega \)-trajectories of \( \Sigma_i \), \( i = 1, 2 \) given by Lemma 6.1.7. By condition (i) in Definition 6.1.19 \( \Sigma_2 \) also has a trajectory \((\hat{x}_3, \hat{y}_1; x^0_2, \hat{u})\) satisfying \( \hat{x}_3(\lambda) \in P\hat{x}_1(\lambda) \), \( \lambda \in \Omega \). By the uniqueness part of Lemma 6.1.7, \( \hat{x}_3 = \hat{x}_2 \). This shows that condition (b) holds.

Conversely, suppose that condition (b) holds. Let \((\hat{x}_1, \hat{y}_1; x^0_1, \hat{u})\) be an \( \Omega \)-trajectory of \( \Sigma_1 \) with \( x^0_1 \subset \text{dom}(P) \). Let \( x^0_2 \in Px^0_1 \), and let \((\hat{x}_2, \hat{y}_2; x^0_2, \hat{u})\) be the \( \Omega \)-trajectories of \( \Sigma_2 \) given by Lemma 6.1.7. By the condition in (ii), we have \( \hat{y}_2 = \hat{y} \). This shows that for every \( \Omega \)-trajectory \((\hat{x}_1, \hat{y}; x^0_1, \hat{u})\) of \( \Sigma_1 \) with \( x^0_1 \subset \text{dom}(P) \) and for every \( x^0_2 \in Px_1(0) \) there exists an \( \Omega \)-trajectory \((\hat{x}_2, \hat{y}; x^0_2, \hat{u})\) of \( \Sigma_1 \). Thus, condition (b) implies that condition (i) in Definition 6.1.19 and according part (i) that we proved above, conditions (i) and (ii) in Definition 6.1.19 are equivalent to each other. Thus, condition (iii) implies that \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \).

**6.1.23. Lemma.** Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma_i = (\mathcal{S}_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}), i = 1, 2 \), be two \( \Omega \)-resolvable frequency domain \( \text{i/s/o} \) systems (with the same input and output spaces). Denote the \( \text{i/s/o} \) resolvent matrices of \( \Sigma_i \) by \( \mathcal{G}_i = \left[ \frac{\mathcal{A}_i \mathcal{G}_i}{\mathcal{B}_i \mathcal{G}_i} \right], \quad i = 1, 2 \). Then \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) if and only if the following four conditions hold for all \( \lambda \in \Omega \):

(i) \( \mathcal{B}_2(\lambda)x_2 \in \text{P}\mathcal{A}_2(\lambda)x_1 \) for all \( x_2 \in P\mathcal{X}_1 \).
(ii) \( \mathcal{B}_2(\lambda)u_0 \in \text{P}\mathcal{B}_2(\lambda)u_0 \) for all \( u_0 \in \mathcal{U} \).
(iii) \( \mathcal{B}_2(\lambda)x_2 = \mathcal{B}_1(\lambda)x_1 \) for all \( x_2 \in P\mathcal{X}_1 \).
(iv) \( \mathcal{B}_2(\lambda) = \mathcal{B}_1(\lambda) \).

Above we have used the convention that the condition \( x_2 \in P\mathcal{X}_1 \) implies that \( x_1 \in \text{dom}(P) \), and the same convention is also used in the inclusions in (i) and (ii).

**Proof of Lemma 6.1.23** This follows immediately from Lemmas 6.1.7 and 6.1.22. \( \square \)
6.1.24. **Lemma.** Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma_i = (S_i; X_i, U, Y), i = 1, 2,$ be two $\Omega$-resolvable frequency domain i/o systems (with the same input and output spaces). If $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by some $P \in \mathcal{ML}(X_1; X_2)$, then they are also $\Omega$-intertwined by the closure of $P$.

**Proof.** This follows directly from Lemma 6.1.23.

6.1.25. **Lemma.** Let $\Sigma_i = (S_i; X_i, U, Y), i = 1, 2, 3,$ be three frequency domain i/o systems, and let $\Omega$ be an open set in $\mathbb{C}$.

(i) If $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P_1 \in \mathcal{ML}(X_1; X_2)$ and $\Sigma_2$ and $\Sigma_3$ are $\Omega$-intertwined by $P_2 \in \mathcal{ML}(X_2; X_3)$, then $\Sigma_1$ and $\Sigma_3$ are $\Omega$-intertwined by $P_3 := P_2 P_1 \in \mathcal{ML}(X_1; X_3)$.

(ii) If, in addition, $\rho(\Sigma_1) \cap \rho(\Sigma_3) \neq \emptyset$, and $\Omega \subset \rho(\Sigma_1) \cap \rho(\Sigma_3)$, then $\Sigma_1$ and $\Sigma_3$ are also $\Omega$-intertwined by the closure of $P_3$.

**Proof.** Claim (i) follows immediately from Definition 6.1.19 and the definition of the composition of two multi-valued operators. Claim (ii) follows from Lemma 6.1.24.

6.1.26. **Lemma.** Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma_i = (S_i; X_i, U, Y), i = 1, 2,$ be two $\Omega$-resolvable frequency domain i/o systems (with the same input and output spaces), and let $\Sigma = \Sigma_2 \dashv \Sigma_1$ be the difference connection of $\Sigma_2$ and $\Sigma_1$ (see Definition 2.3.38). Then $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P \in \mathcal{ML}(X_1; X_2)$ if and only if the i/o resolvent $\mathfrak{D}$ of $\Sigma$ satisfies $\mathfrak{D}(\lambda) = 0$ for all $\lambda \in \Omega$, and in addition, $\text{gph}(P)$ is both a strongly $\Omega$-invariant and an unobservably $\Omega$-invariant subspace for $\Sigma$.

**Proof.** This follows from Definitions 6.1.15 and 6.1.19 and Lemma 5.2.35.

6.1.27. **Lemma.** Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma_i = (S_i; X_i, U, Y), i = 1, 2,$ be two $\Omega$-resolvable frequency domain i/o systems (with the same input and output spaces), let $P \in \mathcal{ML}(X_1; X_2)$ be closed, and let $\Sigma = (S; \text{gph}(P); U, Y)$ be the gph($P$)-short circuit connection of $\Sigma_2$ and $\Sigma_1$ (see Definition 2.3.37). Then $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P$ if an only if $\Sigma$ is $\Omega$-resolvable.

**Proof.** The proof is analogous to the proof of Lemma 3.2.17.

6.1.28. **Theorem.** Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma_i = (S_i; X_i, U, Y), i = 1, 2,$ be two $\Omega$-resolvable frequency domain i/o systems (with the same input and output spaces), and let $\Sigma = \Sigma_2 \dashv \Sigma_1$ be the difference connection of $\Sigma_2$ and $\Sigma_1$ (see Definition 2.3.38). Denote the i/o resolvent matrices of $\Sigma_i$ by $\mathfrak{G}_i = \left[ \begin{array}{c|c} \mathfrak{A}_i & \mathfrak{B}_i \\ \mathfrak{C}_i & \mathfrak{D}_i \end{array} \right]$, $i = 1, 2,$ and the i/o resolvent matrix of $\Sigma_1$ by $\left[ \begin{array}{c|c} \mathfrak{A}_1 & \mathfrak{B}_1 \\ \mathfrak{C}_1 & \mathfrak{D}_1 \end{array} \right]$. Then the following claims are true.

(i) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by some $P \in \mathcal{ML}(X_1; X_2)$ if and only if $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent.

(ii) If $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent, then the following claims are true.

(a) There exists a unique minimal $P_{\Omega}^{\text{min}} \in \mathcal{ML}(X_1; X_2)$ which $\Omega$-intertwines $\Sigma_1$ and $\Sigma_2$, i.e., there exists a unique $P_{\Omega}^{\text{min}} \in \mathcal{ML}(X_1; X_2)$ which $\Omega$-intertwines $\Sigma_1$ and $\Sigma_2$ such that $\text{gph}(P_{\Omega}^{\text{min}}) \subset \text{gph}(P)$ for any other
Among all multi-valued respectively closed multi-valued operators that \( \Omega \)-intertwine the i/o resolvent matrices of \( \Sigma_1 \) and \( \Sigma_2 \), The graph of \( P_{\min}^\Omega \) is given by

\[
gph (P_{\min}^\Omega) = \text{span}_{\lambda \in \Omega, n \in \mathbb{Z}^+} \text{rng} \left( \left[ \begin{array}{c} \hat{B}_2(\lambda)^n \hat{B}_2(\lambda) \\ \hat{B}_1(\lambda)^n \hat{B}_1(\lambda) \end{array} \right] \right) = \text{span}_{\lambda \in \Omega, n \in \mathbb{Z}^+} \text{rng} \left( \hat{A}(\lambda)^n \hat{B}(\lambda) \right).
\]

(b) The closure \( \overline{P_{\min}^\Omega} \) of \( P_{\min}^\Omega \) is the minimal closed multi-valued operator which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( \overline{P_{\min}^\Omega} \) is equal to the reachable subspace of \( \Sigma_{ir} \), and it is given by

\[
gph (\overline{P_{\min}^\Omega}) = \bigvee_{\lambda \in \Omega} \text{rng} \left( \left[ \begin{array}{c} \hat{B}_2(\lambda) \\ \hat{B}_1(\lambda) \end{array} \right] \right) = \bigvee_{\lambda \in \Omega} \text{rng} \left( \hat{B}(\lambda) \right).
\]

(c) There also exists a unique maximal \( P_{\max}^\Omega \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \), i.e., there exists a unique \( P_{\max}^\Omega \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \) such that \( \text{gph}(P) \subseteq \text{gph}(P_{\max}^\Omega) \) for any other \( P \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( P_{\max}^\Omega \) is equal to the unobservable subspace of \( \Sigma_{ir} \), and it is given by

\[
gph (P_{\max}^\Omega) = \bigcap_{\lambda \in \Omega} \ker \left( \left[ \begin{array}{c} \hat{C}_2(\lambda) \\ -\hat{C}_1(\lambda) \end{array} \right] \right) = \bigcap_{\lambda \in \Omega} \ker \left( \hat{C}(\lambda) \right).
\]

In particular, \( P_{\max}^\Omega \) is closed.

Thus, if \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) intertwines \( \Sigma_1 \) and \( \Sigma_2 \), then

\[
\text{gph}(P_{\min}^\Omega) \subset \text{gph}(P) \subset \text{gph}(P_{\max}^\Omega).
\]

**Proof.** Proof of (i): By Lemma 6.1.21, if \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by some \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \), then they are externally \( \Omega \)-equivalent. The converse part of (i) follows from (ii). (The proof of (ii) does not use (i).)

Proof of (ii)(a)–(b): Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent. Let \( P_{\min}^\Omega \) be the operator whose graph is equal to the right-hand side of (6.1.7). By Lemma 6.1.18, \( \text{gph}(P_{\min}^\Omega) \) is the minimal strongly \( \Omega \)-invariant subspace for \( \Sigma_{ir} \).

By the same lemma, the closure of \( \text{gph}(P_{\min}^\Omega) \) is given by (6.1.8), and \( \text{gph}(\overline{P_{\min}^\Omega}) \) is the minimal closed strongly \( \Omega \)-invariant subspace for \( \Sigma_{ir} \). By Lemma 6.1.11, the i/o resolvent matrices of \( \Sigma_1 \) and \( \Sigma_2 \) coincide in \( \Omega \), and therefore \( \hat{D}_1(\lambda) = \hat{D}_2(\lambda) \), \( \hat{D}_1(\lambda) - \hat{D}_2(\lambda) = 0 \) for all \( \lambda \in \Omega \).

We claim that both \( \text{gph}(P_{\min}^\Omega) \) and \( \text{gph}(\overline{P_{\min}^\Omega}) \) are unobservably \( \Omega \)-invariant for \( \Sigma_{ir} \). We know that they are both \( \Omega \)-invariant since they are strongly \( \Omega \)-invariant, so by Lemma 6.1.16 we still have to show that they are both contained in \( \text{gph}(P_{\max}^\Omega) \), where \( P_{\max}^\Omega \) be the operator whose graph is equal to the right-hand side of (6.1.9). Since \( \text{gph}(P_{\min}^\Omega) \subset \text{gph}(\overline{P_{\min}^\Omega}) \), it suffices to show that

\[
\hat{C}_\nu(\lambda)\hat{D}_\nu(\mu) = \left[ \begin{array}{c} \hat{C}_2(\lambda) \\ -\hat{C}_1(\lambda) \end{array} \right] \left[ \begin{array}{c} \hat{B}_2(\mu) \\ \hat{B}_1(\mu) \end{array} \right] = 0, \quad \lambda, \mu \in \Omega.
\]

For \( \lambda \neq \mu \) this follows from (5.2.27) and the fact that \( \hat{D}_1(\lambda) = \hat{D}_2(\lambda) \) for all \( \lambda \in \Omega \), and by continuity, (6.1.10) then holds for \( \lambda = \mu \) as well. By Lemma 6.1.26 \( \Sigma_1 \) and \( \Sigma_2 \) are intertwine by both \( P_{\min}^\Omega \) and \( P_{\max}^\Omega \) and \( P_{\min}^\Omega \) and \( P_{\max}^\Omega \) are minimal among all multi-valued respectively closed multi-valued operators that \( \Omega \)-intertwine
Σ₁ and Σ₂. This proves the converse part of claim (i), as well as claims (ii)(a) and (ii)(b).

Proof of (ii)(c): Suppose again that Σ₁ and Σ₂ are externally Ω-equivalent. By Lemma 6.1.18, \( \text{gph} (P_{\max}^Ω) \) is the maximal unobservably Ω-invariant subspace for \( \Sigma = \Sigma_1 \). We again have \( \Omega_1(\lambda) = \Omega_2(\lambda) \) for all \( \lambda \in \Omega \), and we know that \( \text{gph} (P_{\max}^Ω) \) is Ω-invariant for \( \Sigma = \Sigma_1 \), so it only remains to show that \( \Omega(\lambda)\Omega_2(\mu) = 0 \) for all \( \mu, \lambda \in \Omega \). But this was already done in (6.1.10) above.

6.1.29. Corollary. Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma_i = (\Sigma_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \), \( i = 1, 2 \), be two \( \Omega \)-resolvable frequency domain i/s/o systems (with the same input and output spaces). Moreover, suppose that both \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-controllable and \( \Omega \)-observable. Then \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-pseudo-similar if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent. Among all the \( \Omega \)-pseudo-similarities between \( \Sigma_1 \) and \( \Sigma_2 \), there is a (unique) minimal one \( P_{\min}^Ω \) and a (unique) maximal one \( P_{\max}^Ω \), namely those defined in Theorem 6.1.28 (both of which in this case are single-valued densely defined injective operators with dense range).

Proof. If \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-pseudo-similar, then it follows from Theorem 6.1.28(i) that \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent.

Conversely, suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent. By Theorem 6.1.28 \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some \( P \in \mathcal{M}(\Sigma_1; \Sigma_2) \). By Lemma 6.1.21, \( \text{rg}(B_1(\lambda)) \subset \text{dom} (P) \) and \( \text{rg}(B_2(\lambda)) \subset \text{rg} (P) \), and hence the controllability of \( \Sigma_1 \) and \( \Sigma_2 \) implies that \( \text{dom} (P) \) is dense in \( \mathcal{X}_1 \) and \( \text{rg} (P) \) is dense in \( \mathcal{X}_2 \). By the same lemma, \( \text{ker} (P) \subset \mathcal{U}_1 \) and \( \text{mul} (P) \subset \mathcal{U}_2 \), where \( \mathcal{U}_i \) is the \( \Omega \)-unobservable subspace of \( \Sigma_i \), \( i = 1, 2 \), and hence the \( \Omega \)-observability of \( \Sigma_1 \) and \( \Sigma_2 \) implies that \( P \) is injective and single-valued.

6.1.6. Frequency domain compressions, restrictions, and projections.

6.1.30. Definition. Let \( \Sigma_i = (\Sigma_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \) be two frequency domain i/s/o systems (with the same input and output spaces), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \), and let \( \Omega \) be an open set in \( \mathbb{C} \). We call \( \Sigma_1 \) a \( \Omega \)-compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \), and we call \( \Sigma_2 \) a \( \Omega \)-dilation of \( \Sigma_1 \) along \( \mathcal{Z}_1 \), if the following two condition holds for all \( \lambda \in \Omega \):

(i) If \( (\hat{x}_2, \hat{y}; x^0, \hat{u}) \) is an \( \Omega \)-trajectory of \( \Sigma_2 \) with \( x^0 \in \mathcal{X}_1 \), then \( (P_{\mathcal{Z}_1}\hat{x}_2, \hat{y}; x^0, \hat{u}) \) is an \( \Omega \)-trajectory of \( \Sigma_1 \).

(ii) For each \( \Omega \)-trajectory \( (\hat{x}_1, \hat{y}; x^0, \hat{u}) \) of \( \Sigma_1 \) there exists some \( \Omega \)-trajectory \( (\hat{x}_2, \hat{y}; x^0, \hat{u}) \) of \( \Sigma_2 \) such that \( \hat{x}_1 = P_{\mathcal{Z}_1}\hat{x}_2 \).

6.1.31. Remark. In the above definition we do not require \( \Omega \)-compressions to preserve regularity in the sense that even in the case where the i/s/o node \( \Sigma_2 \) in Definition 6.1.30 is regular we do not require the node \( \Sigma_1 \) in Definition 6.1.30 to be regular. This is important in the study of \( \Omega \)-minimality of frequency domain i/s/o systems. See also Remarks 6.1.45 and 6.2.6 below.

6.1.32. Remark. In the definitions of an \( \Omega \)-compression \( \Sigma_1 = (\Sigma_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) of a frequency domain i/s/o system \( \Sigma_2 = (\Sigma_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}) \) we first fix the open set \( \Omega \), and require that \( \Omega \)-trajectories of \( \Sigma_1 \) and \( \Sigma_2 \) have certain properties. Of course, \( \Sigma_1 \) and \( \Sigma_2 \) also have \( \Omega \)-trajectories for \( \Omega' \neq \Omega \), but these \( \Omega \)-trajectories are not required to satisfy the same conditions that we impose on the \( \Omega \)-trajectories. The
question of when an $\Omega$-compression is also an $\Omega'$-compression will be discussed in Lemma 6.1.34. Analogous comments apply to the notions of $\Omega$-restrictions and $\Omega$-projections that are introduced in Definitions 6.1.36 and 6.1.38 below.

6.1.33. Lemma. If the frequency domain i/s/o system $\Sigma_1$ is the $\Omega$-compression of the frequency domain i/s/o system $\Sigma_2$, then $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent.

Proof. This follows immediately from Definitions 6.1.10 and 6.1.30. □

6.1.34. Lemma. Let $\Sigma_i = (S_i; X_i, U, Y)$ be three frequency domain i/s/o systems, and let $\Omega$ be an open set in $\mathbb{C}$. If $\Sigma_2$ is the $\Omega$-compression of $\Sigma_3$ onto $X_2$ along $Z_2$, and if $\Sigma_1$ is the $\Omega$-compression of $\Sigma_2$ onto $X_1$ along $Z_1$, then $\Sigma_1$ is the $\Omega$-compression of $\Sigma_3$ onto $X_1$ along $Z_1 + Z_2$.

Proof. This follows from Definition 6.1.30 since $P_{X_1}^{Z_1} + Z_2 = P_{X_1}^{Z_1} P_{X_2}^{Z_2}$. □

6.1.35. Lemma. Let the frequency domain i/s/o system $\Sigma_1$ be the $\Omega$-compression of the frequency domain i/s/o system $\Sigma_2$ along $Z_1$. For $i = 1, 2$ we denote the exactly $\Omega$-reachable subspace of $\Sigma_i$ by $\mathcal{R}_i^{\Omega, \text{exact}}$, the $\Omega$-reachable subspace of $\Sigma_i$ by $\mathcal{R}_i^{\Omega}$, and the $\Omega$-unobservable subspace of $\Sigma_i$ by $\mathcal{U}_i^{\Omega}$. Then

$$\mathcal{R}_i^{\Omega, \text{exact}} = P_{X_i}^{Z_i} \mathcal{R}_i^{\Omega, \text{exact}}, \quad \mathcal{R}_i^{\Omega} = P_{X_i}^{Z_i} \mathcal{R}_i^{\Omega}, \quad \mathcal{U}_i^{\Omega} = \mathcal{U}_2^{\Omega} \cap X_1.$$

In particular, if $\Sigma_2$ is $\Omega$-controllable or $\Omega$-observable, then so is $\Sigma_1$.

Proof. The proof is analogous to the proof of Lemma 1.5.32. □

6.1.36. Definition. Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two frequency domain i/s/o systems (with the same input and output spaces), where $X_1$ is a closed subspace of $X_2$, and let $\Omega$ be an open set in $\mathbb{C}$. We call $\Sigma_1$ an $\Omega$-restriction of $\Sigma_2$ to $X_1$ if the following two condition holds for all $\lambda \in \Omega$.

(i) Every $\Omega$-trajectory of $\Sigma_1$ is also an $\Omega$-trajectory of $\Sigma_2$.

(ii) If $(\hat{x}_2, \hat{y}; x^0, \hat{u})$ is an $\Omega$-trajectory of $\Sigma_2$ with $x^0 \in X_1$, then $\hat{x}_2(\lambda) \in X_1$ for all $\lambda \in \Omega$, and $(\hat{x}_2, \hat{y}; x^0, \hat{u})$ is also an $\Omega$-trajectory of $\Sigma_1$.

6.1.37. Lemma. Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two frequency domain i/s/o systems (with the same input and output spaces), where $X_1$ is a closed subspace of $X_2$. Let $\Omega$ be an open set in $\mathbb{C}$, and let $Z_1$ be a direct complement to $X_1$ in $X$. Then the following conditions are equivalent:

(i) $\Sigma_1$ is an $\Omega$-restriction of $\Sigma_2$ to $X_1$.

(ii) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by the embedding operator $X_1 \hookrightarrow X_2$.

(iii) $X_1$ is a strongly $\Omega$-invariant subspace for $\Sigma_2$, and $\Sigma_1$ is the $\Omega$-compression of $\Sigma_2$ onto $X_1$ along $Z_1 + Z_2$.

Proof. The proof is analogous to the proof of Lemma 1.5.36. □

6.1.38. Definition. Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two frequency domain i/s/o systems (with the same input and output spaces), where $X_1$ is a closed subspace of $X_2$, let $Z_1$ be a direct complement to $X_1$ in $X_2$, and let $\Omega$ be an open set in $\mathbb{C}$. We call $\Sigma_1$ an $\Omega$-projection of $\Sigma_2$ onto $X_1$ along $Z_1$ if the following two condition holds for all $\lambda \in \Omega$:

(i) If $(\hat{x}_2, \hat{y}; x^0, \hat{u})$ is an $\Omega$-trajectory of $\Sigma_2$, then $(P_{X_1}^{Z_1} \hat{x}_2, \hat{y}; P_{X_1}^{Z_1} x^0, \hat{u})$ is an $\Omega$-trajectory of $\Sigma_1$. 
(ii) If \((\hat{x}_1, \hat{y}_1; x^0_1, \hat{u})\) is an \(\Omega\)-trajectory of \(\Sigma_1\), then for each \(x^0_2 \in \mathcal{X}_2\) satisfying \(P^X_{\hat{X}_1} x^0_2 = x^0_1\) there exists an \(\Omega\)-trajectory \((\hat{x}_2, \hat{y}_2; x^0_2, \hat{u})\) of \(\Sigma_2\) satisfying \(P^X_{\hat{X}_1} \hat{x}_2 = x^1_1\).

6.1.39. Lemma. Let \(\Sigma_i = (\mathcal{S}_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}), \ i = 1, 2, \) be two frequency domain i/s/o systems (with the same input and output spaces), where \(\mathcal{X}_1\) is a closed subspace of \(\mathcal{X}_2\), let \(\mathcal{Z}_1\) be a direct complement to \(\mathcal{X}_1\) in \(\mathcal{X}_2\), and let \(\Omega\) be an open set in \(\mathbb{C}\). Then the following conditions are equivalent:

(i) \(\Sigma_1\) is an \(\Omega\)-projection of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\).

(ii) \(\Sigma_2\) and \(\Sigma_1\) are \(\Omega\)-intertwined by the projection operator \(P^X_{\hat{X}_1}\).

(iii) \(\mathcal{Z}_1\) is an unobservably \(\Omega\)-invariant subspace for \(\Sigma_2\), and \(\Sigma_1\) is the \(\Omega\)-compression of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\).

**Proof.** The proof is analogous to the proof of Lemmas 13.40. □

6.1.40. Lemma. Let \(\Omega\) be an open set in \(\mathbb{C}\), let \(\Sigma_i = (\mathcal{S}_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}), \ i = 1, 2, \) be two \(\Omega\)-resolvable frequency domain i/s/o systems (with the same input and output spaces), where \(\mathcal{X}_1\) is a closed subspace of \(\mathcal{X}_2\), and let \(\mathcal{Z}_1\) be a direct complement to \(\mathcal{X}_1\) in \(\mathcal{X}_2\). Then \(\Sigma_1\) is an \(\Omega\)-compression of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\) if and only if the following condition holds:

(i) For each \(x^0 \in \mathcal{X}_1\) and each analytic \(\mathcal{U}\)-valued function \(\hat{u}\) in \(\Omega\), if we denote the \(\Omega\)-trajectories of \(\Sigma_1\) and \(\Sigma_2\) with initial state \(x^0\) and input function \(\hat{u}\) given by Lemma 6.1.7 by \((\hat{x}_1, \hat{y}_1; x^0, \hat{u})\) respectively \((\hat{x}_2, \hat{y}_2; x^0, \hat{u})\), then \(\hat{x}_1 = P^X_{\hat{X}_1} \hat{x}_2\) and \(\hat{y}_1 = \hat{y}_2\).

**Proof.** Suppose first that \(\Sigma_1\) is the \(\Omega\)-compression of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\). Let \((\hat{x}_1, \hat{y}_1; x^0, \hat{u})\) and \((\hat{x}_2, \hat{y}_2; x^0, \hat{u})\) be the two trajectories in (i). By condition (i) in Definition 6.1.30 \(\Sigma_2\) has also an \(\Omega\)-trajectory \((\hat{x}_3, \hat{y}_3; x^0, \hat{u})\) satisfying \(\hat{x}_1 = P^X_{\hat{X}_1} \hat{x}_3\). By the uniqueness part of Lemma 6.1.7 \(\hat{x}_2 = \hat{x}_3\) and \(\hat{y}_2 = \hat{y}_3\), and hence condition (i) above holds.

Conversely, suppose that condition (i) above holds. This condition (and Lemma 6.1.7) implies that both condition (i) and (ii) in Definition 23.52 and hence (i) implies that \(\Sigma_1\) is the \(\Omega\)-compression of \(\Sigma_2\). □

6.1.41. Lemma. Let \(\Omega\) be an open set in \(\mathbb{C}\), let \(\mathcal{X}_2 = \mathcal{X}_1 \uplus \mathcal{Z}_1\), and let \(\Sigma_i = (\mathcal{S}_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}), \ i = 1, 2, \) be two \(\Omega\)-resolvable frequency domain i/s/o systems (with the same input and output spaces). Then the following claims are true:

(i) \(\Sigma_1\) is the \(\Omega\)-restriction of \(\Sigma_2\) onto \(\mathcal{X}_1\) if and only if

\[
\begin{bmatrix}
\mathcal{A}_1(\lambda) & \mathcal{B}_1(\lambda) \\
\mathcal{C}_1(\lambda) & \mathcal{D}_1(\lambda)
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}_2(\lambda) & \mathcal{B}_2(\lambda) \\
\mathcal{C}_2(\lambda) & \mathcal{D}_2(\lambda)
\end{bmatrix}
\begin{bmatrix}
\mathcal{X}_1 \\
\mathcal{U}
\end{bmatrix},
\]

for all \(\lambda \in \Omega\), or equivalently, if and only if
\[
\mathcal{A}_1(\lambda) = \mathcal{A}_2(\lambda)|_{\mathcal{X}_1}, \quad \mathcal{B}_1(\lambda) = \mathcal{B}_2(\lambda),
\]

\[
\mathcal{C}_1(\lambda) = \mathcal{C}_2(\lambda)|_{\mathcal{X}_1}, \quad \mathcal{D}_1(\lambda) = \mathcal{D}_2(\lambda)
\]

for all \(\lambda \in \Omega\).

(ii) \(\Sigma_1\) is the \(\Omega\)-projection of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\) if and only if

\[
\begin{bmatrix}
\mathcal{A}_1(\lambda) & \mathcal{B}_1(\lambda) \\
\mathcal{C}_1(\lambda) & \mathcal{D}_1(\lambda)
\end{bmatrix} \begin{bmatrix}
P^X_{\hat{X}_1} & 0 \\
0 & 1_\mathcal{U}
\end{bmatrix} = \begin{bmatrix}
P^X_{\hat{X}_1} & 0 \\
0 & 1_\mathcal{Y}
\end{bmatrix} \begin{bmatrix}
\mathcal{A}_2(\lambda) & \mathcal{B}_2(\lambda) \\
\mathcal{C}_2(\lambda) & \mathcal{D}_2(\lambda)
\end{bmatrix}.
\]

\[
(6.1.12a)
\]

\[
(6.1.12b)
\]

\[
(6.1.13a)
\]
for all \( \lambda \in \Omega \), or equivalently, if and only if
\[
(6.1.13b) \quad \widehat{A}_1(\lambda)P_{X_1}^{Z_1} = P_{X_1}^{Z_1}\widehat{A}_2(\lambda), \quad \widehat{B}_1(\lambda) = P_{X_1}^{Z_1}\widehat{B}_2(\lambda),
\]
\[
\widehat{C}_1(\lambda)P_{X_1}^{Z_1} = \widehat{C}_2(\lambda), \quad \widehat{D}_1(\lambda) = \widehat{D}_2(\lambda)
\]
for all \( \lambda \in \Omega \).

(iii) \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if
\[
(6.1.14a) \quad \begin{bmatrix} \widehat{A}_1(\lambda) & \widehat{B}_1(\lambda) \\ \widehat{C}_1(\lambda) & \widehat{D}_1(\lambda) \end{bmatrix} = \begin{bmatrix} P_{X_1}^{Z_1} & 0 \\ 0 & 1_y \end{bmatrix} \begin{bmatrix} \widehat{A}_2(\lambda) & \widehat{B}_2(\lambda) \\ \widehat{C}_2(\lambda) & \widehat{D}_2(\lambda) \end{bmatrix} \left[ \begin{array}{l} x_1 \\ u \end{array} \right]
\]
for all \( \lambda \in \Omega \), or equivalently, if and only if
\[
(6.1.14b) \quad \widehat{A}_1(\lambda) = P_{X_1}^{Z_1}\widehat{A}_2(\lambda)|_{x_1}, \quad \widehat{B}_1(\lambda) = P_{X_1}^{Z_1}\widehat{B}_2(\lambda),
\]
\[
\widehat{C}_1(\lambda) = \widehat{C}_2(\lambda)|_{x_1}, \quad \widehat{D}_1(\lambda) = \widehat{D}_2(\lambda)
\]
for all \( \lambda \in \Omega \).

In particular, in all the above cases \( \Sigma_1 \) is uniquely determined by \( \Sigma_2 \), the decomposition \( X_2 = X_1 + Z_1 \), and \( \Omega \).

**Proof.** The formulas in (i)–(iii) follow from Lemma 6.1.7 and Definitions 6.1.30 and so does the uniqueness of \( \Sigma_1 \). □

6.1.42. **Lemma.** Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( X_2 = X_1 + Z_1 \), and let \( \Sigma_i = (S_i; X_i, U, Y), \) \( i = 1, 2 \), be two \( \Omega \)-resolvable frequency domain i/s/o systems (with the same input and output spaces). Then

(i) conditions (i) and (ii) in Definition 6.1.30 are equivalent to each other,

(ii) conditions (i) and (ii) in Definition 6.1.30 are equivalent to each other, and

(iii) conditions (i) and (ii) in Definition 6.1.38 are equivalent to each other.

**Proof.** That the claims (ii) and (iii) holds follows from Lemmas 6.1.22, 6.1.37, 6.1.39 and 6.1.41. The proof of claim (i) is analogous to the proof of Lemma 6.1.41. □

6.1.43. **Corollary.** Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( X_2 = X_1 + Z_1 \), and let \( \Sigma_i = (S_i; X_i, U, Y), \) \( i = 1, 2 \), be two \( \Omega \)-resolvable frequency domain i/s/o systems (with the same input and output spaces). Then

(i) \( \Sigma_1 \) is the \( \Omega \)-restriction of \( \Sigma_2 \) to \( X_1 \) if and only if every \( \Omega \)-trajectory of \( \Sigma_1 \) is also an \( \Omega \)-trajectory of \( \Sigma_2 \),

(ii) \( \Sigma_1 \) is the \( \Omega \)-projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if \( (P_{X_1}^{Z_1}\hat{x}_2, \hat{y}; P_{X_1}^{Z_1}x_2, \hat{u}) \) is an \( \Omega \)-trajectory of \( \Sigma_1 \) whenever \( (\hat{x}_2, \hat{y}; x_2, \hat{u}) \) is an \( \Omega \)-trajectory of \( \Sigma_2 \).

**Proof.** This follows immediately from Lemma 6.1.42. □

6.1.44. **Theorem.** Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma = (S; X, U, Y) \) be an \( \Omega \)-resolvable frequency domain i/s/o system, and let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \).

(i) If \( \Sigma \) has an \( \Omega \)-restriction \( \Sigma_1 \) to \( X_1 \), then \( X_1 \) is strongly \( \Omega \)-invariant for \( \Sigma \). Conversely, if \( X_1 \) is strongly \( \Omega \)-invariant for \( \Sigma \), then \( \Sigma \) has a unique \( \Omega \)-resolvable \( \Omega \)-restriction to \( X_1 \). The i/s/o resolvent matrix of this restriction is given by 6.1.12.
Denote the reachable and unobservable subspaces of \( \Sigma \).

\[ \text{(6.1.16a)} \]

The i/s/o resolvent matrix of this restriction is given by \( (6.1.13) \).

**Proof.** By Lemmas 6.1.37 and 6.1.39 if \( \Sigma \) has an \( \Omega \)-restriction to \( \mathcal{X}_1 \) then \( \mathcal{X}_1 \) is strongly \( \Omega \)-invariant for \( \Sigma \). Conversely, if \( \mathcal{Z}_1 \) is unobservable \( \Omega \)-invariant for \( \Sigma \), then \( \Sigma \) has a unique \( \Omega \)-resolvable \( \Omega \)-projection onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

Suppose next that \( \mathcal{X}_1 \) is strongly \( \Omega \)-invariant for \( \Sigma \). By Theorem 5.2.23 the i/s/o resolvent matrix \( \hat{\Sigma} \) satisfies the resolvent identity \( (5.2.27) \) for all \( \lambda \in \rho(\Sigma) \).

\[ \text{(6.1.17a)} \]

Define \( \hat{\Sigma}_1(\lambda) \) by \( (6.1.12a) \) for all \( \lambda \in \Omega \). It follows from Lemma 6.1.16 that also \( \hat{\Sigma}_1 \) satisfies the resolvent identity \( (5.2.27) \) for all \( \lambda, \mu \in \Omega \). Thus, by Theorem 5.2.26 \( \rho(\Sigma) \subset \rho(\Sigma_1) \), and \( \hat{\Sigma}_1 \) is the restriction to \( \Omega \) of the i/s/o resolvent matrix of some frequency domain i/s/o system \( \Sigma_1 \), and it follows from Lemma 6.1.41 that \( \Sigma_1 \) is the \( \Omega \)-restriction of \( \Sigma \) to \( \mathcal{X}_1 \).

The uniqueness claim in (i) follows from Lemma 6.1.41.

The proof of the remaining part of (ii) is analogous to the proof given above. \( \square \)

**6.1.45. Remark.** In the above theorem we do not claim that the \( \Omega \)-restriction \( \Sigma_1 \) in (i) and \( \Omega \)-projection \( \Sigma_1 \) in (ii) are regular, even in the case where the i/s/o node \( \Sigma \) in Theorem 6.1.44 is regular. Sufficient conditions for this to be true are given in Lemma 6.3.3.

**6.1.7. The general structure of a frequency domain compression.**

**6.1.46. Lemma.** Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y}) \) be an \( \Omega \)-resolvable frequency domain i/s/o system with i/s/o resolvent matrix \( \hat{\Sigma} = \begin{bmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{bmatrix} \), and let \( \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}_1 \) be a direct sum decomposition of \( \mathcal{X} \). Let \( \Sigma_{\text{ext}} = (S_{\text{ext}};\mathcal{X},\begin{bmatrix} \mathcal{U} \mathcal{X}_1 \end{bmatrix},\begin{bmatrix} \mathcal{Y} \mathcal{X}_1 \end{bmatrix}) \) be the i/o extension of \( \Sigma \) with control operator equal to the embedding operator \( \mathcal{I}_{\mathcal{X}_1}: \mathcal{X}_1 \to \mathcal{X} \), observation operator \( P_{\mathcal{Z}_1}^{\mathcal{X}_1} \), and feedthrough operator zero, i.e.,

\[
\text{dom} (S_{\text{ext}}) = \begin{bmatrix} \text{dom}(S) \\ \mathcal{X}_1 \end{bmatrix},
\]

\[
S_{\text{ext}} \begin{bmatrix} x \\ u_1 \end{bmatrix} = \begin{bmatrix} S \begin{bmatrix} x \\ u_1 \end{bmatrix} + \begin{bmatrix} 0 \\ u_1 \end{bmatrix} \\ P_{\mathcal{Z}_1}^{\mathcal{X}_1} x \end{bmatrix}, \quad \begin{bmatrix} x \\ u_1 \end{bmatrix} \in \text{dom} (S_{\text{ext}}).
\]

**Denote the reachable and unobservable subspaces of** \( \Sigma_{\text{ext}} \) **by** \( \mathcal{R}_{\text{ext}}^\Omega \) **respectively** \( \mathcal{U}_{\text{ext}}^\Omega \).

(i) **There exists a (unique) minimal closed strongly\( \Omega \)-invariant subspace \( \mathcal{X}_{\text{min}}^\Omega \) for \( \Sigma \) which contains \( \mathcal{X}_1 \) (i.e., \( \mathcal{X}_{\text{min}}^\Omega \) is closed and strongly \( \Omega \)-invariant for \( \Sigma \), and \( \mathcal{X}_{\text{min}}^\Omega \) is contained in every other closed strongly \( \Omega \)-invariant subspace of \( \Sigma \) which contains \( \mathcal{X}_1 \)). This subspace has the following alternative descriptions:

\[
\text{(6.1.16a)} \quad \mathcal{X}_{\text{min}}^\Omega = \mathcal{X}_1 \cup \mathcal{R}_{\text{ext}}^\Omega,
\]

\[
\text{(6.1.16b)} \quad \mathcal{X}_{\text{min}}^\Omega = \mathcal{X}_1 \bigvee_{\lambda \in \Omega} \text{rng} \left( [\hat{\mathcal{A}}(\lambda)|\mathcal{X}_1] - [\hat{\mathcal{B}}(\lambda)] \right).
\]

(ii) **The space** \( \mathcal{X}_{\text{min}}^\Omega \) **has the direct sum decomposition** \( \mathcal{X}_{\text{min}}^\Omega = \mathcal{X}_1 + Z_{\text{min}}^\Omega \), **where**

\[
\text{(6.1.17a)} \quad Z_{\text{min}}^\Omega = \mathcal{X}_{\text{min}}^\Omega \cap \mathcal{Z}_1 = P_{\mathcal{Z}_1}^{\mathcal{X}_1} \mathcal{X}_{\text{min}}^\Omega = P_{\mathcal{Z}_1}^{\mathcal{X}_1} \mathcal{R}_{\text{ext}}^\Omega.
\]
This subspace also given by

\[ Z_{\text{min}}^\Omega = \bigvee_{\lambda \in \Omega} \text{rng} \left( P_{X_1}^\lambda \left[ \mathcal{A}(\lambda) |_{X_1} \quad \mathcal{B}(\lambda) \right] \right). \]

(iii) There exists a (unique) maximal unobservably \( \Omega \)-invariant subspace \( Z_{\text{max}}^\Omega \) for \( \Sigma \) which is contained in \( Z_1 \) (i.e., \( Z_{\text{max}}^\Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma \), and \( Z_{\text{max}}^\Omega \) contains every other unobservably \( \Omega \)-invariant subspace for \( \Sigma \) which is contained in \( Z_1 \)). This subspace has the following alternative descriptions:

\[ Z_{\text{max}}^\Omega = Z_1 \cap \Omega_{\text{ext}}^\Omega \]

Proof of (i): Define \( X_1 = X \ominus U \), then by Lemma 6.1.14, \( Z_{\text{ext}}^\Omega \) is strongly \( \Omega \)-invariant for \( \Sigma \). Consequently, \( \mathcal{A}(\lambda) X_1 \subset X_1 \), \( \mathcal{B}(\lambda) U \subset X_1 \), and by continuity, also \( \mathcal{A}(\lambda) X_{\text{min}}^\Omega \subset X_1 \) and \( \mathcal{B}(\lambda) U \subset X_1 \).

By Lemma 6.1.16, \( X_{\text{min}}^\Omega \) is strongly \( \Omega \)-invariant for \( \Sigma \). Thus, \( X_{\text{min}}^\Omega \subset X_1 \). If \( X_2 \) is any other closed strongly \( \Omega \)-invariant subspace for \( \Sigma \) which contains \( X_1 \), then by Lemma 6.1.16, for all \( \lambda \in \Omega \),

\[ \mathcal{A}(\lambda) X_1 \subset \mathcal{A}(\lambda) X_2 \subset X_2, \quad \mathcal{B}(\lambda) U \subset X_2, \]

By Lemma 6.1.14

\[ X_{\text{ext}}^\Omega = \bigvee_{\lambda \in \Omega} \text{rng} \left( [\mathcal{A}(\lambda) |_{X_1} \quad \mathcal{B}(\lambda)] \right), \]

and by taking the closed linear span over all \( \lambda \in \Omega \), \( X_1 \), and \( U \) and using Lemma 6.1.14, we get \( X_{\text{ext}}^\Omega \subset X_2 \). Since, in addition, \( X_1 \subset X_2 \) and \( X_2 \) is closed we therefore get \( X_{\text{min}}^\Omega = X_1 \lor X_{\text{ext}}^\Omega \subset X_2 \). This proves the minimality of \( Z_{\text{min}}^\Omega \).

Proof of (ii): The proof of (6.1.17) is proof is analogous to the proof of claim (ii) in Lemma 3.1.36. The alternative formula (6.1.17b) follows from (6.1.16b).

Proof of (iii): Define \( Z_{\text{max}}^\Omega \) as in (a), i.e., \( Z_{\text{max}}^\Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma \). Then it follows from Lemma 6.1.14 and (6.1.19) that also (6.1.18b) holds.

By Lemma 6.1.14, \( \Omega_{\text{ext}}^\Omega \) is unobservably \( \Omega \)-invariant subspace for \( \Sigma_{\text{ext}} \). This combined with Lemma 6.1.16 implies that for all \( \lambda \in \Omega \),

\[ \mathcal{A}(\lambda) \Omega_{\text{ext}}^\Omega \subset \Omega_{\text{ext}}^\Omega, \quad \mathcal{B}(\lambda) \Omega_{\text{ext}}^\Omega = \{0\}, \quad \mathcal{C}(\lambda) \Omega_{\text{ext}}^\Omega = \{0\}, \]
where the second condition can be rewritten in the form \( \hat{A}(\lambda)\mathcal{U}^\Omega_{\Sigma_{\text{ext}}} \subset Z_1 \). Consequently,
\[
\hat{A}(\lambda)\mathcal{Z}^\Omega_{\text{max}} = \hat{A}(\lambda) (Z_1 \cap \mathcal{U}^\Omega_{\Sigma_{\text{ext}}}) \subset \hat{A}(\lambda)\mathcal{U}^\Omega_{\Sigma_{\text{ext}}} \subset Z_1 \cap \mathcal{U}^\Omega_{\Sigma_{\text{ext}}} = \mathcal{Z}^\Omega_{\text{max}}.
\]
Since, in addition,
\[
\mathcal{E}(\lambda)\mathcal{Z}^\Omega_{\text{max}} = \mathcal{E}(\lambda) (Z_1 \cap \mathcal{U}^\Omega_{\Sigma_{\text{ext}}}) \subset \mathcal{E}(\lambda)\mathcal{U}^\Omega_{\Sigma_{\text{ext}}} = \{0\},
\]
this together with Lemma 6.1.16 implies that \( \mathcal{Z}^\Omega_{\text{max}} \) is unobservably \( \Omega \)-invariant for \( \Sigma \).

Trivially, \( \mathcal{Z}^\Omega_{\text{max}} \subset Z_1 \). If \( Z_2 \) is any other closed unobservably \( \Omega \)-invariant subspace for \( \Sigma \) which is contained in \( Z_1 \), then by Lemma 6.1.16 for all \( \lambda \in \Omega \),
\[
\hat{A}(\lambda)Z_2 \subset Z_2 \subset Z_1, \quad \mathcal{E}(\lambda)Z_2 = \{0\}.
\]
The first condition can be rewritten in the form \( P_{X_1}'\hat{A}(\lambda)Z_2 = \{0\} \). By Lemma 6.1.14
\[
\mathcal{U}^\Omega_{\Sigma_{\text{ext}}} = \bigcap_{\lambda \in \Omega} \ker \left( \left[ \begin{array}{c} P_{X_1}' \hat{A}(\lambda) \\ \mathcal{E}(\lambda) \end{array} \right] \right),
\]
and thus \( Z_2 \subset \mathcal{U}^\Omega_{\Sigma_{\text{ext}}} \). Since, in addition, \( Z_2 \subset Z_1 \) we therefore get \( Z_2 \subset Z_1 \cap \mathcal{U}^\Omega_{\Sigma_{\text{ext}}} = \mathcal{Z}^\Omega_{\text{max}} \). This proves the maximality of \( \mathcal{Z}^\Omega_{\text{min}} \).

6.1.47. Remark. The formulas for \( \mathcal{Z}^\Omega_{\text{min}} \) and \( \mathcal{Z}^\Omega_{\text{max}} \) in Lemma 6.1.46 look slightly more complicated than the corresponding formulas in Lemma 3.2.23. In that lemma \( \Sigma \) was assumed to be bounded, and \( \Omega \) was taken to be the unbounded component of the resolvent set of the main operator of \( A \). The reason why the formulas are simpler in Lemma 3.2.23 is that in that setting \( X_1 \subset \mathcal{R}_{\Sigma_{\text{ext}}} \) and \( \mathcal{U}^\Omega_{\Sigma_{\text{ext}}} \subset Z_1 \), so there is no need for taking the closed linear span of \( \mathcal{R}_{\Sigma_{\text{ext}}} \) and \( X_1 \) to \( \mathcal{R}_{\Sigma_{\text{ext}}} \), and there is no need to intersect \( \mathcal{U}^\Omega_{\Sigma_{\text{ext}}} \) with \( Z_2 \).

6.1.48. Theorem. Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma = (S; X, U, Y) \) be an \( \Omega \)-resolvable frequency domain i/s/o system, and let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \). Let \( X^\Omega_{\text{min}} \) be the minimal strongly \( \Omega \)-invariant subspace of \( \Sigma \) which contains \( X_1 \), let \( Z^\Omega_{\text{max}} \) be the maximal observably \( \Omega \)-invariant subspace of \( \Sigma \) which is contained in \( Z_1 \), and let \( Z^\Omega_{\text{min}} = X^\Omega_{\text{min}} \cap Z_1 \) (cf. Lemma 6.1.46). Then the following conditions are equivalent:

(i) \( \Sigma \) has a (unique) \( \Omega \)-resolvable \( \Omega \)-compression \( \Sigma_1 = (S_1; X, U, Y) \) onto \( X_1 \) along \( Z_1 \).
(ii) \( Z_1 \) contains some closed subspace \( Z^\Omega \) such that \( Z^\Omega \) is observably \( \Omega \)-invariant for \( \Sigma \) and \( X_1 + Z^\Omega \) is strongly \( \Omega \)-invariant for \( \Sigma \).
(iii) \( Z^\Omega_{\text{min}} \) is observably \( \Omega \)-invariant for \( \Sigma \).
(iv) \( X + Z^\Omega_{\text{max}} \) is strongly \( \Omega \)-invariant for \( \Sigma \).
(v) \( Z^\Omega_{\text{min}} \subset Z^\Omega_{\text{max}} \).

Two possible choices of the subspace \( Z^\Omega \) in (ii) are \( Z^\Omega = Z^\Omega_{\text{min}} \) and \( Z^\Omega = Z^\Omega_{\text{max}} \), and every possible subspace \( Z^\Omega \) in (ii) satisfies \( Z^\Omega_{\text{min}} \subset Z^\Omega \subset Z^\Omega_{\text{max}} \).

Proof. Throughout this proof we denote the i/s/o resolvent matrix of \( \Sigma \) by \( \hat{S} = \left[ \begin{array}{c|c} \hat{s} & \hat{S} \\ \hline \hat{s} & \hat{S} \end{array} \right] \).

(i) If (i) holds, then by Lemma 6.1.14(iii) combined with the resolvent identity 5.2.27 applied to both \( \hat{S} \) and \( \hat{S}_1 \) gives for all \( \lambda, \mu \in \Omega, \lambda \neq \mu \), and all \( x \in X_1 \) and
\[ u \in \mathcal{U}, \]
\[ P_{X_1}^{\hat{Z}_1} \hat{A}(\lambda) \left( \hat{A}(\mu)x + \hat{B}(\mu)u \right) = \frac{1}{\mu - \lambda} P_{X_1}^{\hat{Z}_1} \left( (\hat{A}(\lambda) - \hat{A}(\mu))x + (\hat{B}(\lambda) - \hat{B}(\mu))u \right) \]
\[ = \frac{1}{\mu - \lambda} \left( (\hat{A}_1(\lambda) - \hat{A}_1(\mu))x + (\hat{B}_1(\lambda) - \hat{B}_1(\mu))u \right) \]
\[ = \hat{A}_1(\lambda)(\hat{A}_1(\mu)x + \hat{B}_1(\mu)u) \]
\[ = P_{X_1}^{\hat{Z}_1} \hat{A}(\lambda) P_{X_1}^{\hat{Z}_1} \left( \hat{A}(\mu)x + \hat{B}(\mu)u \right), \]
\[ \hat{C}(\lambda) \left( \hat{A}(\mu)x + \hat{B}(\mu)u \right) = \frac{1}{\mu - \lambda} \left( (\hat{C}(\lambda) - \hat{C}(\mu))x + (\hat{D}(\lambda) - \hat{D}(\mu))u \right) \]
\[ = \frac{1}{\mu - \lambda} \left( (\hat{C}_1(\lambda) - \hat{C}_1(\mu))x + (\hat{D}_1(\lambda) - \hat{D}_1(\mu))u \right) \]
\[ = \hat{C}_1(\lambda)(\hat{A}_1(\mu)x + \hat{B}_1(\mu)u) \]
\[ = \hat{C}(\lambda) P_{X_1}^{\hat{Z}_1} \left( \hat{A}(\mu)x + \hat{B}(\mu)u \right), \]

and hence
\[ P_{X_1}^{\hat{Z}_1} \hat{A}(\lambda) P_{X_1}^{\hat{Z}_1} \left( \hat{A}(\mu)x + \hat{B}(\mu)u \right) = 0, \quad (6.1.20) \]
\[ \hat{C}(\lambda) P_{X_1}^{\hat{Z}_1} \left( \hat{A}(\mu)x + \hat{B}(\mu)u \right) = 0 \]

for all \( \mu, \lambda \in \Omega, \mu \neq \lambda, \) and all \( x \in X_1 \) and \( u \in \mathcal{U}. \) By letting \( \mu \to \lambda \) we get the same identity of \( \mu = \lambda. \) Taking the closed linear span over all \( x \in X_1, \) \( u \in \mathcal{U}, \) and \( \mu \in \Omega \) we get from \[ (6.1.17b), \]
\[ P_{X_1}^{\hat{Z}_1} \hat{A}(\lambda) z = 0, \quad \hat{C}(\lambda) z = 0, \quad \lambda \in \Omega, \quad z \in Z_{\text{min}}^{\Omega}. \]

Thus, for all \( z \in Z_{\text{min}}^{\Omega} \) and all \( \lambda \in \Omega \) we have \( \hat{A}(\lambda) z = (P_{X_1}^{\hat{Z}_1} + P_{X_1}^{\hat{Z}_1}) \hat{A}(\lambda) z \subset Z_{\text{min}}^{\Omega} \) and \( \hat{C}(\lambda) z = 0, \) consequently, by Lemma \[ 6.1.16, \] \( Z_{\text{min}}^{\Omega} \) is unobservable \( \Omega \)-invariant.

(iii) \Rightarrow (ii): This follows from Lemma \[ 6.1.46. \]

(ii) \Rightarrow (i): If (ii) holds, then for all \( \lambda \in \Omega, \)
\[ \hat{A}(\lambda)X_1 + \hat{B}(\lambda)\mathcal{U} \subset X_1 + Z^{\Omega}, \]

and hence, for all \( \mu, \lambda \in \Omega, \)
\[ \begin{bmatrix} P_{X_1}^{\hat{Z}_1} & 0 \\ 0 & 1_{Y} \end{bmatrix} \begin{bmatrix} \hat{A}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} P_{X_1}^{\hat{Z}_1} \begin{bmatrix} \hat{A}(\mu) \\ \hat{B}(\mu) \end{bmatrix} \begin{bmatrix} X_1 \\ \mathcal{U} \end{bmatrix} = 0. \]

Define \( \check{\hat{S}} = \begin{bmatrix} \hat{A}_1 & \hat{B}_1 \\ \hat{C}_1 & \hat{D}_1 \end{bmatrix} \) by \[ (6.1.14a). \] Then, using the resolvent identity \[ 5.2.27 \] for \( \hat{S} \) we get
\[ \check{\hat{S}}_1(\lambda) - \check{\hat{S}}_1(\mu) = \begin{bmatrix} P_{X_1}^{\hat{Z}_1} & 0 \\ 0 & 1_{Y} \end{bmatrix} \begin{bmatrix} \hat{A}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} \begin{bmatrix} \hat{A}(\mu) \\ \hat{B}(\mu) \end{bmatrix} \begin{bmatrix} X_1 \\ \mathcal{U} \end{bmatrix} = \begin{bmatrix} (\mu - \lambda) P_{X_1}^{\hat{Z}_1} & 0 \\ 0 & 1_{Y} \end{bmatrix} \begin{bmatrix} \hat{A}(\lambda) \\ \hat{C}(\lambda) \end{bmatrix} \begin{bmatrix} \hat{A}(\mu) \\ \hat{B}(\mu) \end{bmatrix} \begin{bmatrix} X_1 \\ \mathcal{U} \end{bmatrix} = \begin{bmatrix} (\mu - \lambda) \hat{A}_1(\lambda) \\ \hat{C}_1(\lambda) \end{bmatrix} \begin{bmatrix} \hat{A}_1(\mu) \\ \hat{B}_1(\mu) \end{bmatrix}. \]
This shows that \( \mathcal{S}_1 \) satisfies the resolvent identity \((6.2.27)\). By Theorem \(6.2.26\), \( \mathcal{S}_1 \) is the restriction to \( \Omega \) of the i/s/o resolvent matrix of an i/s/o node \( \Sigma_1 \), and it follows from Lemma \(6.1.41 \text{iii})\) that \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(i) \( \Rightarrow \) (iv): Assume that (i) holds. By Lemma \(6.1.16 \text{ii})\), in order to show that \( \mathcal{X}_1 + \mathcal{Z}_1^\Omega \) is strongly \( \Omega \)-invariant it suffices to show that for all \( x \) of the form \( x = x_1 + z_1 \) with \( x_1 \in \mathcal{X}_1 \) and \( z_1 \in \mathcal{Z}_1^\Omega \), for all \( u \in \mathcal{U} \), and for all \( \mu \in \Omega \) we have

\[
\hat{\mathbf{A}}(\mu)(x_1 + z_1) + \hat{\mathbf{B}}(\mu)u \in \mathcal{X}_1 + \mathcal{Z}_1^\Omega,
\]

or equivalently,

\[
P_{\mathcal{Z}_1^\Omega}(\hat{\mathbf{A}}(\mu)(x_1 + z_1) + \hat{\mathbf{B}}(\mu)u) \in \mathcal{Z}_1^\Omega.
\]

By \((6.1.18b)\), this is equivalent to the requirement that for all \( x_1, z_1 \), and \( u \) of the type above and for all \( \mu, \lambda \in \Omega \) we have

\[
P_{\mathcal{X}_1^\lambda}(\hat{\mathbf{C}}(\mu))P_{\mathcal{Z}_1^\lambda}(\hat{\mathbf{A}}(\lambda)(x_1 + z_1) + \hat{\mathbf{B}}(\mu)u) = 0,
\]

\[
\hat{\mathcal{C}}(\lambda)P_{\mathcal{Z}_1^\lambda}(\hat{\mathbf{A}}(\mu)(x_1 + z_1) + \hat{\mathbf{B}}(\mu)u) = 0.
\]

If \( z_1 = 0 \), then these two identities follow from \((6.1.20)\), so it only remains to prove that these two identities also hold when \( x_1 = 0 \) and \( u = 0 \). However, that this is true follows from the unobservable \( \Omega \)-invariance of \( \mathcal{Z}_1^\Omega \), established in Lemma \(6.1.46\).

Thus (i) \( \Rightarrow \) (iv).

(iv) \( \Rightarrow \) (ii): This follows from Lemma \(6.1.46\).

(iii) \( \Rightarrow \) (v): This follows from Lemma \(6.1.46\).

(v) \( \Rightarrow \) (iii): We know from Lemma \(6.1.46\) that \( \mathcal{X}_1 + \mathcal{Z}_1^\Omega \) is strongly \( \Omega \)-invariant and that \( \mathcal{Z}_1^\Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma \). By Lemma \(6.1.16\) the strong \( \Omega \)-invariance of \( \mathcal{X}_1 + \mathcal{Z}_1^\Omega \) gives \( \hat{\mathbf{A}}(\lambda)\mathcal{Z}_1^\Omega \subset \mathcal{X}_1 + \mathcal{Z}_1^\Omega \), \( \lambda \in \Omega \), whereas the unobservable \( \Omega \)-invariance of \( \mathcal{Z}_1^\Omega \) together with the condition \( \mathcal{Z}_1^\Omega \subset \mathcal{Z}_1^\lambda \) gives for all \( z_0 \in \mathcal{Z}_1^\Omega \) and all \( \lambda \in \Omega \)

\[
\hat{\mathbf{A}}(\lambda)z_0 \subset \hat{\mathbf{A}}(\lambda)\mathcal{Z}_1^\Omega \subset \mathcal{Z}_1^\Omega \subset \mathcal{Z}_1 \quad \hat{\mathcal{C}}(\lambda)z_0 = 0, \quad \lambda \in \Omega.
\]

Thus \( \mathcal{Z}_1^\Omega \subset \ker(\hat{\mathcal{C}}(\lambda)) \) and \( \hat{\mathbf{A}}(\lambda)\mathcal{Z}_1^\Omega \subset (\mathcal{X}_1 + \mathcal{Z}_1^\Omega) \cap \mathcal{Z}_1 = \mathcal{Z}_1^\Omega, \lambda \in \Omega \). By Lemma \(6.1.16\) \( \mathcal{Z}_1^\Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma \).

We have now completed the proof of the equivalence of the conditions (i)–(v).

From this proof also follows that (ii) holds with \( \mathcal{Z}_1^\Omega \) replaced by \( \mathcal{Z}_1^{\Omega_{\min}} \) and by \( \mathcal{Z}_1^{\Omega_{\max}} \), and the inclusions \( \mathcal{Z}_1^{\Omega_{\min}} \subset \mathcal{Z}_1 \subset \mathcal{Z}_1^{\Omega_{\max}} \) follow from Lemma \(6.1.46\). \(\square\)

6.1.49. Corollary. Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an \( \Omega \)-resolvable frequency domain i/s/o system, and let \( \mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1 \) be a direct sum decomposition of \( \mathcal{X} \). Then \( \Sigma \) has an \( \Omega \)-resolvable \( \Omega \)-compression \( \Sigma_1 = (\mathcal{S}_1; \mathcal{X}_1, \mathcal{U}_1, \mathcal{Y}_1) \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) with i/s/o resolvent matrix \( \hat{\mathcal{S}}_1 = \begin{bmatrix} \hat{\mathbf{A}}_1 & \hat{\mathbf{B}}_1 \end{bmatrix} \) if and only if \( \mathcal{Z}_1 \) has a direct sum decomposition \( \mathcal{Z}_1 = \mathcal{Z}_1^{\Omega} + \mathcal{Z}_c \) such that the i/s/o resolvent matrix \( \hat{\mathcal{S}} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \end{bmatrix} \) of \( \Sigma \) has the following structure with respect to the decomposition \( \mathcal{X} = \mathcal{Z}_1^{\Omega} + \mathcal{X}_1 + \mathcal{Z}_c \) of \( \mathcal{X} \) (where irrelevant entries have been denoted by *):

\[
(6.1.21) \quad \hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \hat{\mathcal{A}}(\lambda) & 0 & * & * \\ 0 & \hat{\mathcal{B}}(\lambda) & 0 & * \\ 0 & 0 & \hat{\mathcal{C}}(\lambda) & * \\ 0 & * & 0 & \hat{\mathcal{D}}(\lambda) \end{bmatrix}, \quad \lambda \in \Omega.
\]
Here $\mathcal{Z}^\Omega + \mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma$, $\mathcal{Z}^\Omega$ is unobservably $\Omega$-invariant of $\Sigma$, $\mathcal{A}_{\mathcal{Z}_2}$ is the $\Omega$-restriction of $\mathcal{A}$ to $\mathcal{Z}$, and $\mathcal{A}_{\mathcal{Z}_c}$ is the $\Omega$-projection of $\mathcal{A}$ onto $\mathcal{Z}_c$ along $\mathcal{X}_1 + \mathcal{Z}^\Omega$. The subspace $\mathcal{Z}^\Omega$ in this decomposition can be chosen to be the same as the subspace $\mathcal{Z}_1$ in condition (ii) in Theorem 6.1.48 and the subspace $\mathcal{Z}_c$ can be chosen to be an arbitrary direct complement to $\mathcal{Z}^\Omega$ in $\mathcal{Z}_1$. In particular, two possible choices of $\mathcal{Z}^\Omega$ are $\mathcal{Z}^\Omega = \mathcal{Z}_c^\Omega$ and $\mathcal{Z}^\Omega = \mathcal{Z}_c^\Omega$, where $\mathcal{Z}_c^\Omega$ and $\mathcal{Z}_c^\Omega$ are the subspaces defined in Lemma 6.1.46.

**Proof.** This follows from the equivalence of (i) and (ii) in Theorem 6.1.48 (take $\mathcal{Z}_c$ to be an arbitrary direct complement to $\mathcal{Z}^\Omega$ in $\mathcal{Z}_1$).

6.1.50. Theorem. Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma = (\mathcal{S}, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a $\Omega$-resolvable frequency domain i/s/o system, let $\mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1$ be a direct sum decomposition of $\mathcal{X}$, and suppose that $\Sigma_1 = (\mathcal{S}_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ is an $\Omega$-resolvable $\Omega$-compression of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$. Let $\mathcal{Z}^\Omega$ satisfy the conditions listed in (ii) in Theorem 6.1.48 and let $\mathcal{Z}_c$ be an arbitrary direct complement to $\mathcal{Z}^\Omega$ in $\mathcal{Z}_1$.

(i) Let $\Sigma_2$ be the $\Omega$-resolvable $\Omega$-restriction of $\Sigma$ to the strongly $\Omega$-invariant subspace $\mathcal{X}_1 + \mathcal{Z}^\Omega$ for $\Sigma$ given by Theorem 6.1.48(i). Then $\mathcal{Z}^\Omega$ is unobservably $\Omega$-invariant for $\Sigma_2$, and $\Sigma_1$ is the $\Omega$-projection onto $\mathcal{X}_1$ along $\mathcal{Z}^\Omega$ of $\Sigma_2$.

(ii) Let $\Sigma_3$ be the $\Omega$-resolvable $\Omega$-projection of $\Sigma$ onto $\mathcal{X}_1 + \mathcal{Z}_c$ along $\mathcal{Z}^\Omega$ given by Theorem 6.1.44(ii). Then $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma_3$, and $\Sigma_1$ is the $\Omega$-restriction to $\mathcal{X}_1$ of $\Sigma_3$.

**Proof.** (i) With the notation of 6.1.21 the i/s/o resolvent matrix of $\Sigma_2$ has the structure (with respect to the decomposition $\mathcal{Z}^\Omega + \mathcal{X}_1$ of the state space of $\Sigma_2$)

\[
\hat{\mathcal{S}}_2(\lambda) = \begin{bmatrix}
0 & \hat{\mathcal{A}}_1(\lambda) & \hat{\mathcal{B}}_1(\lambda) \\
0 & \hat{\mathcal{C}}_1(\lambda) & \hat{\mathcal{D}}_1(\lambda)
\end{bmatrix}, \quad \lambda \in \Omega.
\]

Thus by Lemma 6.1.16 $\mathcal{Z}^\Omega$ is unobservably $\Omega$-invariant for $\Sigma_2$. By Lemma 6.1.41 the projection of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}^\Omega$ given by Theorem 6.1.44(ii) is equal to $\Sigma_1$.

(ii) With the notation of 6.1.21 the i/s/o resolvent matrix of $\Sigma_3$ has the structure (with respect to the decomposition $\mathcal{X}_1 + \mathcal{Z}_c$ of the state space of $\Sigma_3$)

\[
\hat{\mathcal{S}}_3(\lambda) = \begin{bmatrix}
\hat{\mathcal{A}}_1(\lambda) & * \\
\hat{\mathcal{C}}_1(\lambda) & *
\end{bmatrix}, \quad \lambda \in \Omega.
\]

Thus by Lemma 6.1.16 $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma_4$. By Lemma 6.1.41 the restriction of $\Sigma_4$ to $\mathcal{X}_1$ given by Theorem 6.1.44(i) is equal to $\Sigma_1$.

6.1.51. Lemma. Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $\Sigma_1 = (\mathcal{S}_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ be two $\Omega$-resolvable frequency domain i/s/o systems (with the same input and output spaces) with $\mathcal{X}_2 = \mathcal{X}_1 + \mathcal{Z}_1$. Then the following two conditions are equivalent.

(i) $\Sigma_1$ is the $\Omega$-compression of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$.

(ii) $\mathcal{Z}_1$ contains some closed subspace $\mathcal{Z}^\Omega$ such that $\Sigma$ and $\Sigma_1$ are $\Omega$-intertwined by the operator $P_{\mathcal{X}_1}^{|\mathcal{X}_1 + \mathcal{Z}^\Omega}$.
Condition (ii) above holds for some particular subspace \( Z^\Omega \) if and only condition (ii) in Theorem \( \text{6.1.48} \) holds for the same subspace \( Z^\Omega \). Thus, in particular, two possible choices of the subspace \( Z^\Omega \) in (ii) are the subspaces \( Z^\Omega = Z^\Omega_{\min} \) and \( Z^\Omega = Z^\Omega_{\max} \) defined in Lemma \( \text{6.1.46} \) and every possible subspace \( Z^\Omega \) satisfies \( Z^\Omega_{\min} \subset Z^\Omega \subset Z^\Omega_{\max} \).

**Proof.** \( (i) \Rightarrow (ii) \): Let \( Z^\Omega \) satisfy condition (ii) in Theorem \( \text{6.1.48} \) and let \( \Sigma_2 \) be the frequency domain i/s/o system in Theorem \( \text{6.1.50}(i) \). Then by Lemma \( \text{6.1.41} \), \( \Sigma \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by the embedding operator \( X_1 \otimes Z^\Omega \rightarrow X \) and \( \Sigma_2 \) and \( \Sigma_1 \) are \( \Omega \)-intertwined by the projection operator \( P_{X_1}^{Z_{\Omega}} \). Thus by Lemma \( \text{6.1.25} \), \( \Sigma \) and \( \Sigma_1 \) are \( \Omega \)-intertwined by the composition of these two operators, which is the operator \( P_{X_1}^{Z_{\Omega}} = P_{X_1}^{Z_{\Omega}} \mid X_1 + Z^\Omega \).

\( (ii) \Rightarrow (i) \): If (ii) holds, then by Lemma \( \text{6.1.21} \), \( X_1 + Z^\Omega \) is strongly \( \Omega \)-invariant for \( \Sigma \), and \( Z^\Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma \). By Theorem \( \text{6.1.44} \), \( \Sigma \) has an \( \Omega \)-resolvable \( \Omega \)-compression \( \Sigma_3 \) onto \( X_1 \) along \( Z^\Omega \). By comparing the conditions (i)–(iv) in Lemmas \( \text{6.1.23} \) and \( \text{6.1.41} \) to each other and using the strong invariance of \( X_1 + Z^\Omega \) and the unobservable \( \Omega \)-invariance of \( Z^\Omega \) we find that \( \Sigma_3 = \Sigma_1 \). \( \Box \)

### 6.1.8. Results for connected frequency domains

Earlier in this chapter we have given various results related to \( \Omega \)-trajectories of frequency domain i/s/o systems. In the case where \( \Omega \) is connected it is possible to sharpen some of these results, as will be shown below.

**Lemma (cf. Lemma \( \text{6.1.11} \)).** Let \( \Omega^\circ \) be an open connected set in \( \mathbb{C} \), let \( \Sigma_i = (S_i; X_i, \mathcal{U}, Y_i) \) be two \( \Omega \)-resolvable frequency domain i/s/o systems with i/o resolvents \( \hat{\Sigma}_i \), \( i = 1, 2 \), let \( \Omega' \) be an arbitrary subset of \( \Omega^\circ \), which has a cluster point in \( \Omega^\circ \), and let \( \Omega \) be an open subset of \( \Omega^\circ \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent,

(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega^\circ \)-equivalent,

(iii) \( \hat{\Sigma}_1(\lambda) = \hat{\Sigma}_2(\lambda) \) for all \( \lambda \in \Omega \).

(iv) \( \hat{\Sigma}_1(\lambda) = \hat{\Sigma}_2(\lambda) \) for all \( \lambda \in \Omega^\circ \).

(v) \( \hat{\Sigma}_1(\lambda) = \hat{\Sigma}_2(\lambda) \) for all \( \lambda \in \Omega' \).

(vi) \( \hat{\Sigma}_1^{(n)}(\lambda_0) = \hat{\Sigma}_2^{(n)}(\lambda_0) \) for some \( \lambda_0 \in \Omega^\circ \) and all \( n \in \mathbb{N} \).

Thus, if \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent for some open subset \( \Omega \) of \( \Omega^\circ \), then \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent for every open subset \( \Omega \) of \( \Omega^\circ \).

**Proof.** (i) \( \Leftrightarrow \) (iii) and (ii) \( \Leftrightarrow \) (iv): See Lemma \( \text{6.1.11} \).

(iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v) \( \Leftrightarrow \) (vi): This follows from the analyticity of \( \hat{\Sigma}_1 \) and \( \hat{\Sigma}_2 \). \( \Box \)

**Lemma (cf. Lemma \( \text{6.1.16} \)).** Let \( \Omega^\circ \) be an open connected set in \( \mathbb{C} \), let \( \Sigma = (S; X, \mathcal{U}, Y) \) be an \( \Omega^\circ \)-resolvable frequency domain i/s/o system with i/o resolvent matrix \( \hat{\Sigma} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \), let \( Z \) be a closed subspace of \( X \), and let \( \Omega \) be an open subset of \( \Omega^\circ \).

(i) The following conditions are equivalent:

(a) \( Z \) is an \( \Omega \)-invariant subspace for \( \Sigma \);

(b) \( Z \) is an \( \Omega^\circ \)-invariant subspace for \( \Sigma \);

(c) \( \hat{\Sigma}(\lambda) Z \subset Z \) for all \( \lambda \in \Omega \);

(d) \( \hat{\Sigma}(\lambda) Z \subset Z \) for all \( \lambda \in \Omega^\circ \);

(e) \( \hat{\Sigma}(\lambda_0) Z \subset Z \) for some \( \lambda_0 \in \Omega^\circ \).
Thus, if $Z$ is an $\Omega$-invariant subspace for $\Sigma$ for some open subset $\Omega$ of $\Omega^o$, then $Z$ is an $\Omega$-invariant subspace for $\Sigma$ for every open subset $\Omega$ of $\Omega^o$.

(ii) The following conditions are equivalent:

(a) $Z$ is a strongly $\Omega$-invariant subspace for $\Sigma$;
(b) $Z$ is a strongly $\Omega^o$-invariant subspace for $\Sigma$;
(c) $\hat{A}(\lambda)Z \subset Z$ and $\text{rng} (\hat{B}(\lambda)) \subset Z$ for all $\lambda \in \Omega$;
(d) $\hat{A}(\lambda)Z \subset Z$ and $\text{rng} (\hat{B}(\lambda)) \subset Z$ for all $\lambda \in \Omega^o$;
(e) $\hat{A}(\lambda_0)Z \subset Z$ for some $\lambda_0 \in \Omega^o$ and $\text{rng} (\hat{B}(\lambda_1)) \subset Z$ for some $\lambda_1 \in \Omega^o$.

Thus, if $Z$ is a strongly $\Omega$-invariant subspace for $\Sigma$ for some open subset $\Omega$ of $\Omega^o$, then $Z$ is a strongly $\Omega$-invariant subspace for $\Sigma$ for every open subset $\Omega$ of $\Omega^o$.

(iii) The following conditions are equivalent:

(a) $Z$ is an unobservably $\Omega$-invariant subspace for $\Sigma$;
(b) $Z$ is an unobservably $\Omega^o$-invariant subspace for $\Sigma$;
(c) $\hat{A}(\lambda)Z \subset Z$ and $Z \subset \text{ker} (\hat{C}(\lambda))$ for all $\lambda \in \Omega$;
(d) $\hat{A}(\lambda)Z \subset Z$ and $Z \subset \text{ker} (\hat{C}(\lambda))$ for all $\lambda \in \Omega^o$;
(e) $\hat{A}(\lambda_0)Z \subset Z$ for some $\lambda_0 \in \Omega^o$ and $Z \subset \text{ker} (\hat{C}(\lambda_2))$ for some $\lambda_2 \in \Omega^o$.

Thus, if $Z$ is an unobservably $\Omega$-invariant subspace for $\Sigma$ for some open subset $\Omega$ of $\Omega^o$, then $Z$ is an unobservably $\Omega$-invariant subspace for $\Sigma$ for every open subset $\Omega$ of $\Omega^o$.

**Proof.** That (a) $\Leftrightarrow$ (c) and (b) $\Leftrightarrow$ (d) in all three parts (i)–(iii) follows from Lemma 6.1.16. It is also clear that (d) $\Rightarrow$ (e) in all three parts (i)–(iii). Thus it remains to show that (e) $\Rightarrow$ (d) in all three cases.

(i) (e) $\Rightarrow$ (i) (d): If $\hat{A}(\lambda_0)Z \subset Z$, then $\hat{A}(\lambda_0)^nZ \subset Z$ for all $n \in \mathbb{Z}^+$, and therefore $\hat{A}^n(\lambda_0)Z = (-1)^n \lambda_0^n \hat{A}(\lambda_0)^nZ \subset Z$ for all $n \in \mathbb{Z}^+$. By formula (A.3.6) with $G$ replaced by $\hat{A}(\lambda)\big|_Z$ this implies that $\hat{A}(\lambda)Z \subset Z$ for all $\lambda \in \Omega^o$.

(ii) (e) $\Rightarrow$ (ii) (d): By (i), if $\hat{A}(\lambda_0)Z \subset Z$ for some $\lambda_0 \in \Omega^o$, then $\hat{A}(\lambda)Z \subset Z$ for all $\lambda \in \Omega^o$. If, in addition, $\text{rng} (\hat{B}(\lambda_1)) \subset Z$ for some $\lambda_1 \in \Omega$, then follows from the resolvent identity (5.2.28) that $\text{rng} (\hat{B}(\lambda)) \subset Z$ for all $\lambda \in \Omega$.

(iii) (e) $\Rightarrow$ (iii) (d): The proof of this implication is analogous to the proof of the implication (ii) (e) $\Rightarrow$ (ii) (d) above. \qed

**6.1.54.Lemma (cf. Lemmas 6.1.14 and 6.1.18).** Let $\Omega^o$ be an open connected set in $\mathbb{C}$, let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an $\Omega^o$-resolvable frequency domain i/s/o system with s/s, i/s, and s/o resolvents $\hat{A}$, $\hat{B}$, and $\hat{C}$, respectively. Let $\Omega'$ be an arbitrary subset of $\Omega^o$ which has a cluster point in $\Omega^o$, let $\lambda_0 \in \Omega^o$, and let $\Omega$ be an open subset of $\Omega^o$. Denote the $\Omega$-reachable and $\Omega$-observable subspaces of $\Sigma$ by $\mathcal{R}_\Omega^\Omega$ respectively $\mathcal{R}_\Omega^\Omega$.

Then

$$
\mathcal{R}_\Omega^\Omega = \bigvee_{\lambda \in \Omega} \text{rng} (\hat{B}(\lambda)) = \bigvee_{\lambda \in \Omega^o} \text{rng} (\hat{B}(\lambda))
\begin{equation}
= \bigvee_{\lambda \in \Omega^o} \text{rng} (\hat{B}(\lambda)) = \bigvee_{n \in \mathbb{Z}^+} \text{rng} (\hat{A}(\lambda_0)^n \hat{B}(\lambda_0)),
\end{equation}
$$
Moreover, \( \mathcal{M}_{\Omega}^{1} \) is the minimal closed \( \Omega \)-invariant subspace for \( S \) which contains \( \text{rng}\ (\widehat{\mathcal{B}}(\lambda_0)) \), and \( \mathcal{M}_{\Omega}^{2} \) is the largest \( \Omega \)-invariant subspace for \( S \) which is contained in \( \ker(\mathcal{C}(\lambda_0)) \).

Thus, \( \mathcal{M}_{\Omega}^{1} \) and \( \mathcal{M}_{\Omega}^{2} \) do not depend on the choice of \( \Omega \), as long as \( \Omega \) is an open subset of \( \Omega^\circ \).

**Proof.** We know from Lemma \[6.1.14\] that the first equalities in (6.1.22) and (6.1.23) hold. The remaining equalities in (6.1.22) and (6.1.23) follow from the analyticity of the functions \( \widehat{\mathcal{B}} \) and \( \mathcal{C} \) (see Theorem 5.2.23(iv)), Lemma A.3.6, and (5.2.33).

That both \( \mathcal{M}_{\Omega}^{1} \) and \( \mathcal{M}_{\Omega}^{2} \) are \( \Omega \)-invariant subspaces of \( \widehat{\mathcal{A}} \) follows from Lemma \[6.1.16\]. It is also clear that \( \text{rng}\ (\widehat{\mathcal{B}}(\lambda)) \subset \mathcal{M}_{\Omega}^{1} \) and that \( \mathcal{M}_{\Omega}^{1} \subset \ker(\mathcal{C}(\lambda)) \) for all \( \lambda \in \Omega \). If \( Z \) is an arbitrary closed \( \Omega \)-invariant subspace for \( S \) with \( \text{rng}\ (\widehat{\mathcal{B}}(\lambda_0)) \subset Z \), then \( \widehat{\mathcal{A}}(\lambda_0)^n \widehat{\mathcal{B}}(\lambda_0) \subset Z \) for all \( n \geq 0 \), and it follows from (6.1.22) that \( \mathcal{M}_{\Omega}^{1} \subset Z \). Thus, \( \mathcal{M}_{\Omega}^{1} \) is the minimal closed \( \Omega \)-invariant subspace for \( S \) which contains \( \text{rng}\ (\widehat{\mathcal{B}}(\lambda_0)) \). Analogously, if \( Z \) is an arbitrary \( \Omega \)-invariant subspace for \( S \) which contains \( \ker(\mathcal{C}(\lambda_0)) \), then, for every \( z \in Z \) and every \( n \geq 0 \) we have \( \widehat{\mathcal{A}}(\lambda_0)^n z \subset Z \subset \ker(\mathcal{C}(\lambda_0)) \), and it follows from (6.1.23) that \( Z \subset \mathcal{M}_{\Omega}^{2} \). Thus, \( \mathcal{M}_{\Omega}^{2} \) is the largest \( \Omega \)-invariant subspace for \( S \) which is contained in \( \ker(\mathcal{C}(\lambda_0)) \). \( \square \)

**Lemma (cf. Lemma \[6.1.23\]).** Let \( \Omega^\circ \) be an open connected set in \( \mathbb{C} \), let \( \Sigma_i = (S_i; X_i; U, \mathcal{Y}) \), \( i = 1, 2 \), be two \( \Omega^\circ \)-resolvable frequency domain i/s/o systems with i/s/o resolvent matrices \( \widehat{\Sigma}_i = \begin{bmatrix} \widehat{\mathcal{A}} & \widehat{\mathcal{B}} \\ \widehat{\mathcal{C}} & \widehat{\mathcal{D}} \end{bmatrix} \), \( i = 1, 2 \), and let \( P \in \mathcal{ML}(X_1; X_2) \) be closed. Let \( \Omega \) be an open subset of \( \Omega^\circ \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \);
(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega^\circ \)-intertwined by \( P \);
(iii) conditions (i)–(iv) in Lemma \[6.1.23\] hold for all \( \lambda \in \Omega \);
(iv) conditions (i)–(iv) in Lemma \[6.1.23\] hold for all \( \lambda \in \Omega^\circ \);
(v) each one of conditions (i)–(iv) in Lemma \[6.1.23\] holds for some \( \lambda \in \Omega^\circ \) (which need not be the same in (i), (ii), (iii), and (iv)).

Thus, if \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \) for some open subset \( \Omega \) of \( \Omega^\circ \), then \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \) for every open subset \( \Omega \) of \( \Omega^\circ \).

**Proof.** (i) \( \Leftrightarrow \) (iii) and (ii) \( \Leftrightarrow \) (iv): See Lemma \[6.1.23\].

(iv) \( \Rightarrow \) (iii) \( \Rightarrow \) (v): This is clear.

(v) \( \Rightarrow \) (iv): Suppose that (v) holds, and let \( \Sigma_+ = \Sigma_2 \upharpoonright \Sigma_1 \) be the difference connection of \( \Sigma_2 \) and \( \Sigma_1 \). Then conditions (i)–(iii) in Lemma \[6.1.23\] are equivalent to the following conditions:

(i) \( \widehat{\mathcal{A}}(\lambda) \text{gph}(P) \subset \text{gph}(P) \);
(ii) \( \text{rng}\ (\widehat{\mathcal{B}}(\lambda)) \subset \text{gph}(P) \);
(iii) \( \text{gph}(P) \subset \ker(\mathcal{C}(\lambda)) \),

By Lemma \[6.1.53\] if each one of conditions (i′)–(iii′) holds for some \( \lambda \in \Omega^\circ \) (which need not be the same in (i′), (ii′), and (iii′)), then conditions (i′)–(iii′) hold for all \( \lambda \in \Omega^\circ \), or equivalently, conditions (i)–(iii) in Lemma \[6.1.23\] hold for all \( \lambda \in \Omega^\circ \).
This together with the resolvent identity (5.2.28) and the assumption that \( \mathcal{D}_1(\lambda) = \mathcal{D}_2(\lambda) \) for some \( \lambda \in \Omega^o \) implies that \( \mathcal{D}_1(\lambda) = \mathcal{D}_2(\lambda) \) for all \( \lambda \in \Omega^o \).

6.1.56. THEOREM (cf. Theorem 6.1.28). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), and let \( \Sigma_i = (S_i;X_i,U,Y) \), \( i = 1,2 \), be two \( \Omega^o \)-resolvable frequency domain i/s/o systems (with the same input and output spaces). Denote the i/s/o resolvent matrices of \( \Sigma_i \) by \( \hat{\mathcal{S}}_i = \left[ \hat{\mathcal{A}}_i, \hat{\mathcal{B}}_i, \hat{\mathcal{C}}_i, \hat{\mathcal{D}}_i \right] \), \( i = 1,2 \). Let \( \Omega' \) be an arbitrary subset of \( \Omega^o \) which has a cluster point in \( \Omega^o \), let \( \lambda_0 \in \Omega^o \), and let \( \Omega \) be an open subset of \( \Omega^o \). Furthermore, suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-invariant, and define \( \overline{P}_\Omega \) and \( P_{\Omega}^{\max} \) by (6.1.8) and (6.1.9). Then

\[
\text{gph} \left( P_{\Omega}^{\max} \right) = \bigcap_{\lambda \in \Omega^o} \ker \left( [\hat{\mathcal{C}}_2(\lambda) - \hat{\mathcal{C}}_1(\lambda)] \right) = \bigcap_{\lambda \in \Omega^o} \ker \left( [\hat{\mathcal{C}}_2(\lambda) - \hat{\mathcal{C}}_1(\lambda)] \right)
\]

Thus, \( \overline{P}_\Omega \) and \( P_{\Omega}^{\max} \) do not depend on the choice of \( \Omega \), as long as \( \Omega \) is an open subset of \( \Omega^o \) and \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-invariant.

PROOF. This follows from Theorem 6.1.28 and Lemma 6.1.54 applied to the system \( \Sigma_1 = \Sigma_2 \models \Sigma_1 \).

6.1.57. Lemma (cf. Lemma 6.1.41). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), let \( \Sigma_i = (S_i;X_i,U,Y) \), \( i = 1,2 \), be two \( \Omega^o \)-resolvable frequency domain i/s/o systems (with the same input and output spaces), and suppose that \( X_1 \) is a closed subspace of \( X_2 \) with a direct complement \( Z_1 \) in \( X_2 \). Let \( \Omega \) be an open subset of \( \Omega^o \).

(i) The following conditions are equivalent:

a) \( \Sigma_1 \) is the \( \Omega \)-restriction of \( \Sigma_2 \) to \( X_1 \);

b) \( \Sigma_1 \) is the \( \Omega^o \)-restriction of \( \Sigma_2 \) to \( X_1 \);

c) the equivalent conditions (6.1.12) hold for all \( \lambda \in \Omega \);

d) the equivalent conditions (6.1.12) hold for all \( \lambda \in \Omega^o \);

e) there exist \( \lambda_i \in \Omega^o \), \( i = 0,1,2,3 \), such that

\[
\hat{\mathcal{A}}_1(\lambda_0) = \hat{\mathcal{A}}_2(\lambda_0)|_{X_1}, \quad \hat{\mathcal{B}}_1(\lambda_1) = \hat{\mathcal{B}}_2(\lambda_1),
\]

\[
\hat{\mathcal{C}}_1(\lambda_2) = \hat{\mathcal{C}}_2(\lambda_2)|_{X_1}, \quad \hat{\mathcal{D}}_1(\lambda_3) = \hat{\mathcal{D}}_2(\lambda_3)
\]

Thus, if \( \Sigma_1 \) is the \( \Omega \)-restriction of \( \Sigma_2 \) to \( X_1 \) for some open subset \( \Omega \) of \( \Omega^o \), then \( \Sigma_1 \) is the \( \Omega \)-restriction of \( \Sigma_2 \) to \( X_1 \) for all open subset \( \Omega \) of \( \Omega^o \).

(ii) The following conditions are equivalent:

a) \( \Sigma_1 \) is the \( \Omega \)-projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \);

b) \( \Sigma_1 \) is the \( \Omega \)-projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \);

c) \( \Sigma_1 \) is the \( \Omega^o \)-projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \);

d) the equivalent conditions (6.1.13) hold for all \( \lambda \in \Omega^o \);
Lemma 6.1.41. It is also clear that (d) for all $P \subseteq \Sigma$ of

\[
\hat{A}_1(\lambda_0)P_{X_1} = P_{X_1}\hat{A}_2(\lambda_0), \quad \hat{B}_1(\lambda_1) = P_{X_1}\hat{B}_2(\lambda_1),
\]

\[
\hat{C}_1(\lambda_2)P_{X_1} = \hat{C}_2(\lambda_2), \quad \hat{D}_1(\lambda_3) = \hat{D}_2(\lambda_3)
\]

Thus, if $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ to $X_1$ for some open subset $\Omega$ of $\Omega^0$, then $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ to $X_1$ for all open subset $\Omega$ of $\Omega^0$.

Proof. That (a) $\leftrightarrow$ (c) and (b) $\leftrightarrow$ (d) in both parts (i) and (ii) follows from Lemma [6.1.41]. It is also clear that (d) $\Rightarrow$ (c) $\Rightarrow$ (e) in all three parts (i)–(iii). Thus it remains to show that (e) $\Rightarrow$ (d) in both cases.

(i)(e) $\Rightarrow$ (i)(d): If for some $\lambda_0 \in \Omega^0$ we have $\hat{A}_1(\lambda_0) = \hat{A}_2(\lambda_0)|_{X_1}$, then $\hat{A}_2(\lambda_0)X_1 \subseteq X_1$, and therefore

\[
\hat{A}_1(\lambda_0)^n = (\hat{A}_2(\lambda_0)|_{X_1})^n = \hat{A}_2(\lambda_0)|_{X_1}
\]

for all $n \in \mathbb{Z}^+$. The first identity in [5.2.33] gives $\hat{A}_1(\lambda_0) = \hat{A}_2(\lambda_0)|_{X_1}$ for all $n \in \mathbb{Z}^+$, since both $\hat{A}_1$ and $\hat{B}_2|_{X_1}$ are analytic functions in $\Omega^0$ and $\Omega^0$ is connected, this implies that $\hat{A}_1(\lambda) = \hat{A}_2(\lambda)|_{X_1}$ for all $\lambda \in \Omega^0$. If, in addition, $\hat{B}_1(\lambda_1) = \hat{B}_2(\lambda_1)$ for some $\lambda_1 \in \Omega^0$ and $\hat{C}_1(\lambda_2) = \hat{C}_2(\lambda_2)|_{X_1}$ for some $\lambda_2 \in \Omega^0$, then it follows from the second and third identities in [5.2.28] applied to both $\Sigma_1$ and $\Sigma_2$ that $\hat{A}_1(\lambda) = \hat{A}_2(\lambda)$ and $\hat{C}_1(\lambda) = \hat{C}_2(\lambda)|_{X_1}$ for all $\lambda \in \Omega^0$. Finally, if in addition $\hat{D}_1(\lambda_3) = \hat{D}_2(\lambda_3)$ for some $\lambda_3 \in \Omega^0$, then it follows from the last identity in [5.2.28] that $\hat{D}_1(\lambda) = \hat{D}_2(\lambda)$ for all $\lambda \in \Omega^0$.

(ii)(c) $\Rightarrow$ (ii)(d): If for some $\lambda \in \Omega^0$ we have $\hat{A}_1(\lambda)P_{X_1} = P_{X_1}\hat{A}_2(\lambda)$, then $P_{X_1}\hat{A}_2(\lambda)P_{X_1} = P_{X_1}\hat{A}_2(\lambda)$, and therefore

\[
\hat{A}_1(\lambda_0)^n = (\hat{A}_2(\lambda_0)^n) = (P_{X_1}\hat{A}_1(\lambda_0))^n = P_{X_1}\hat{A}_1(\lambda_0)^n.
\]

The remainder of the proof is analogous to the corresponding part of the proof of the implication (i)(d) $\Rightarrow$ (i)(c) given above.

6.1.58. Remark. It is also possible to prove Lemma [6.1.57] by appealing to Lemmas [6.1.37] [6.1.39] and [6.1.55].

6.1.59. Lemma (cf. Lemma [6.1.41]). Let $\Omega^0$ be an open connected set in $\mathbb{C}$, let $\Sigma_i = (S_i; X_i; U, Y)$, $i = 1, 2$, be two $\Omega^0$-resolvable frequency domain i/s/o systems (with the same input and output spaces), and suppose that $X_1$ is a closed subspace of $X_2$ with a direct complement $Z_1$ in $X_2$. Let $\Omega'$ be an arbitrary subset of $\Omega^0$ which has a cluster point in $\Omega^0$, and let $\Omega$ be an open subset of $\Omega^0$. Then the following conditions are equivalent:

(i) $\Sigma_1$ is the $\Omega$-compression of $\Sigma_2$ onto $X_1$ along $Z_1$;
(ii) $\Sigma_1$ is the $\Omega^0$-compression of $\Sigma_2$ onto $X_1$ along $Z_1$;
(iii) the equivalent conditions (6.1.14) hold for all $\lambda \in \Omega$;
(iv) the equivalent conditions (6.1.14) hold for all $\lambda \in \Omega^0$;
(v) The following conditions hold for all $\lambda \in \Omega'$ and some $\lambda_i \in \Omega^0$, $i = 1, 2, 3$:

\[
\hat{A}_1(\lambda) = P_{X_1}\hat{A}_2(\lambda)|_{X_1}, \quad \hat{B}_1(\lambda_1) = P_{X_1}\hat{B}_2(\lambda_1),
\]

\[
\hat{C}_1(\lambda_2) = \hat{C}_2(\lambda_2)|_{X_1}, \quad \hat{D}_1(\lambda_3) = \hat{D}_2(\lambda_3)
\]

(6.1.28)
The following conditions hold for all \( n \in \mathbb{Z}^+ \), and for some \( \lambda_i \in \Omega^p \), \( i = 0, 1, 2, 3 \):

\[
(6.1.29) \quad \hat{A}_1(\lambda_0)^n = P^Z_{X_1} \hat{A}_2(\lambda_0)^n|_{X_1}, \quad \hat{A}_1(\lambda_1)^n \hat{B}_1(\lambda_1) = P^Z_{X_1} \hat{A}_2(\lambda_1)^n \hat{B}_2(\lambda_1), \quad \hat{C}_1(\lambda_2) \hat{A}_1(\lambda_2)^n = \hat{C}_2(\lambda_2) \hat{A}_2(\lambda_2)^n|_{X_1}, \quad \hat{D}_1(\lambda_3) = \hat{D}_2(\lambda_3)
\]

Thus, if \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_2 \) to \( \mathcal{X}_1 \) for some open subset \( \Omega \) of \( \Omega^p \), then \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_2 \) to \( \mathcal{X}_1 \) for all open subset \( \Omega \) of \( \Omega^p \).

Proof. (i) \( \Leftrightarrow \) (iii) and (ii) \( \Leftrightarrow \) (iv): See Lemma 6.1.41.

(iii) \( \Leftrightarrow \) (iv) \( \Leftrightarrow \) (v): This follows from the analyticity of all the functions appearing in conditions (6.1.14) and using (5.2.33) we get (6.1.29) for all possible values of \( \lambda_i \in \Omega^p \), \( i = 1, 2, 3, 4 \), and all \( n \in \mathbb{Z}^+ \). Conversely, if (iv) holds, then by (5.2.33) and the analyticity of all the functions appearing in conditions (6.1.14) we get we find that \( \hat{A}_1(\lambda) = P^Z_{X_1} \hat{A}_2(\lambda)|_{X_1}, \hat{B}_1(\lambda) = P^Z_{X_1} \hat{B}_2(\lambda), \) and \( \hat{C}_1(\lambda) = \hat{C}_2(\lambda)|_{X_1} \) for all \( \lambda \in \Omega^p \). This combined with the condition \( \hat{D}_1(\lambda_3) = \hat{D}_2(\lambda_3) \) and the resolvent identity (5.2.28) implies that \( \hat{D}_1(\lambda) = \hat{D}_2(\lambda) \) for all \( \lambda \in \Omega^p \).

6.1.60. Lemma (cf. Lemma 6.1.46). Let \( \Omega^p \) be an open connected set in \( \mathbb{C} \), let \( \Sigma_2 = (\mathcal{X}_2; \mathcal{U}, Y) \) be an \( \Omega^p \)-resolvable frequency domain i/s/o system with i/s/o resolvent matrix \( \begin{bmatrix} \hat{A}_2 & \hat{B}_2 \end{bmatrix} \), and let \( \mathcal{X}_2 = \mathcal{X}_1 + \mathbb{Z}_1 \). Let \( \Omega \) be an arbitrary subset of \( \Omega^p \) which has a cluster point in \( \Omega^p \), let \( \lambda_0 \in \Omega^p \), and let \( \Omega \) be an open subset of \( \Omega^p \). Then the following claims are true:

(i) The subspace \( \mathcal{X}^\Omega_{\min} \) in Lemma 6.1.46 is given by

\[
(6.1.30) \quad \mathcal{X}^\Omega_{\min} = \mathcal{X}_1 \bigvee_{\lambda \in \Omega^p} \text{rng} \left( \begin{bmatrix} \hat{A}_2(\lambda)|_{X_1} & \hat{B}_2(\lambda) \end{bmatrix} \right)
\]

(ii) The subspace \( \mathcal{Z}^\Omega_{\min} \) in Lemma 6.1.46 is given by

\[
(6.1.31) \quad \mathcal{Z}^\Omega_{\min} = \bigvee_{\lambda \in \Omega^p} \text{rng} \left( \begin{bmatrix} P^X_{Z_1} \hat{A}_2(\lambda)|_{X_1} & \hat{B}_2(\lambda) \end{bmatrix} \right)
\]
(iii) The subspace $Z_{\Omega \max}^\Omega$ in Lemma 6.1.46 is given by

$$Z_{\Omega \max}^\Omega = \bigcap_{\lambda \in \Omega} \{ x \in Z_1 | \hat{A}(\lambda)x \in Z_1 \text{ and } \hat{C}(\lambda)x = 0 \}$$

$$= \bigcap_{\lambda \in \Omega^\circ} \{ x \in Z_1 | \hat{A}(\lambda)x \in Z_1 \text{ and } \hat{C}(\lambda)x = 0 \}$$

(6.1.32)

$$= \bigcap_{\lambda \in \Omega^\prime} \{ x \in Z_1 | \hat{A}(\lambda)x \in Z_1 \text{ and } \hat{C}(\lambda)x = 0 \}$$

$$= \bigcap_{n \in \mathbb{Z}^+} \{ x \in Z_1 | \hat{A}(\lambda_0)^{n+1}x \in Z_1 \text{ and } \hat{C}(\lambda_0)\hat{A}(\lambda_0)^n x = 0 \}.$$ 

Thus, $X_{\Omega \min}^\Omega$, $Z_{\Omega \min}^\Omega$, and $Z_{\Omega \max}^\Omega$ do not depend on the choice of $\Omega$, as long as $\Omega$ is an open subset of $\Omega^\circ$ and $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-invariant.

**Proof.** This follows from the analyticity of the functions $\hat{A}_2$, $\hat{B}_2$, and $\hat{C}_2$ (see Theorem 5.2.23(iv)), 5.2.33, and Lemma A.3.6. \hfill \Box
6.2. The Generators of $\Omega$-Intertwined and $\Omega$-Compressed I/S/O Systems (Feb 02, 2016)

In the preceding section we investigated a number of relationships between i/s/o systems in the frequency domain. It is also possible to describe the same relationships in terms of the system operators of the involved systems.

6.2.1. The system operator of an i/s/o $\Omega$-restriction.

6.2.1. Theorem. Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma = (S; X, U, Y)$ be an $\Omega$-resolvable i/s/o node, let $X_1$ be a closed subspace of $X$, and let $\Sigma_{\text{part}} = (S_{\text{part}}; X_1, U, Y)$ be the part of $\Sigma$ in $[X_1 \ U]$, i.e.,

$$\text{gph} \left( S_{\text{part}} \right) = \text{gph} \left( S \right) \cap \left\{ \left[ \begin{array}{c} z \\ y \\ x \\ u \end{array} \right] \in \left[ \begin{array}{c} Z \\ Y \\ X \\ U \end{array} \right] \bigg| \left[ \begin{array}{c} z \\ y \\ x \\ u \end{array} \right] \in S \right\}.$$  

Then the following conditions are equivalent:

(i) $X_1$ is strongly $\Omega$-invariant for $\Sigma$;

(ii) $\Sigma$ has a (unique) $\Omega$-resolvable $\Omega$-restriction to $X_1$;

(iii) $\Sigma_{\text{part}}$ is $\Omega$-resolvable.

If these equivalent conditions hold, then $\Sigma_{\text{part}}$ is the unique $\Omega$-resolvable $\Omega$-restriction of $\Sigma$ to $X_1$, and $S_{\text{part}}$ is single-valued whenever $S$ is single-valued.

The three equivalent conditions (i)–(iii) above are furthermore equivalent to each of the following conditions, where $\hat{S} = \left[ \begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array} \right]$ is the i/s/o resolvent matrix of $\Sigma$:

(iv) For all $\lambda \in \Omega$ we have

$$\text{gph} \left( S - \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] \right) \cap \left[ \begin{array}{c} X_1 \\ U \end{array} \right] = \text{gph} \left( S - \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] \right) \cap \left[ \begin{array}{c} X_1 \\ U \end{array} \right].$$

(v) For all $\lambda \in \Omega$, $\text{gph} \left( S_{\text{part}} - \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] \right)$ has the two equivalent representations

$$\text{gph} \left( S_{\text{part}} - \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] \right) = \text{rng} \left( \left[ \begin{array}{c} -1 \chi_i \\ \bar{S}(\lambda) \chi_i \\ 0 \end{array} \right] \right),$$

$$\text{gph} \left( S_{\text{part}} - \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] \right) = \text{ker} \left( \left[ \begin{array}{c} \bar{S}(\lambda) \chi_i \\ \bar{S}(\lambda) \chi_i \\ 0 \end{array} \right] \right).$$

(vi) $\left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] - \left[ \begin{array}{c} \chi_i \\ 0 \\ 0 \end{array} \right] S_{\text{part}}$ has an inverse in $B([X_1 \ U])$ for all $\lambda \in \Omega$.

(vii) $\left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] - \left[ \begin{array}{c} \chi_i \\ 0 \\ 0 \end{array} \right] S_{\text{part}}$ is surjective for all $\lambda \in \Omega$.

If, in addition, $\Omega$ is connected, then conditions (i)–(vii) above are also equivalent to the condition

(viii) $\Omega \cap \rho(S_{\text{part}}) \neq \emptyset$,

as well as to the conditions that one gets by replacing “for all $\lambda \in \Omega$” by “for some $\lambda \in \Omega$” in conditions (iv)–(vii) above.
Note that we do not claim that \( \text{dom}(S_{\text{part}}) \) is dense in \( \begin{bmatrix} X_1^* \\ Y_1^* \end{bmatrix} \) whenever \( \text{dom}(S) \) is dense in \( \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \). (This will be true under some additional assumptions; see Theorems 6.3.4 and 8.3.13 below.)

**Proof of Theorem 6.2.1** (i) \( \iff \) (ii): See Theorem 6.4.14.

(i) \( \iff \) (iv): By Lemma 5.2.13, \( \text{gph}(S - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) \) has the representation (5.2.11). This together with Lemma 6.1.16(ii) implies that (i) and (iv) are equivalent.

(i) \( \Rightarrow \) (v): Since \( S_{\text{part}} \) is the part of \( S \) in \( X_1^* \), it is also true that \( S_{\text{part}} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is the part of \( S - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) in \( X_1^* \). By Lemma 5.2.13, the left-hand side of (6.2.3a) is the range of the part of the operator on the right-hand side of (5.2.11) restricted to the subspace of vectors \( \begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \) which satisfy \( \mathbf{A}(\lambda)z + \mathbf{B}(\lambda)u \in \Omega \). If (i) holds, then by Lemma 6.1.16 this additional condition is redundant, i.e., it holds for all \( \begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \). Thus, \( \text{gph}(S_{\text{part}} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}) \) is the range of the part of the operator on the right-hand side of (5.2.11) in \( \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \). This gives (6.2.3a). The easy proof that (6.2.3a) is equivalent to (6.2.3b) is left to the reader.

(v) \( \Rightarrow \) (iv): This follows from Lemma 5.2.13 combined with (6.2.3) and the fact that \( S_{\text{part}} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is the part of \( S - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) in \( X_1 \).

(iii) \( \iff \) (v): This follows from Lemma 5.2.13.

(iii) \( \Rightarrow \) (vi): This follows from Lemma 5.2.19.

(vi) \( \Rightarrow \) (vii): This is obvious.

(vii) \( \Rightarrow \) (i): Condition (vii) is equivalent to the statement that for every \( \begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \) there exists some \( x \in X_1 \) such that \( z = \lambda x - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S_{\text{part}} \begin{bmatrix} z \\ u \end{bmatrix} \). Since \( S_{\text{part}} \) is the part of \( S \) in \( X_1 \), this is equivalent to the statement that for every \( \begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \) there exists some \( x \in X_1 \) such that \( z = \lambda x - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} z \\ u \end{bmatrix} \). By Theorem 5.2.17, \( \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) S \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} \mathbf{A}(\lambda) & \mathbf{B}(\lambda) \\ 0 & 1_\mathbb{R} \end{bmatrix} \in \mathcal{B}(\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}) \) for all \( \lambda \in \rho(\Sigma) \), and therefore \( x \) is determined uniquely by \( \begin{bmatrix} z \\ u \end{bmatrix} \) through the formula \( x = \mathbf{A}(\lambda)z + \mathbf{B}(\lambda)u \in X_1 \). Thus, (vii) is equivalent to the condition that for every \( \begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \) we have \( x = \mathbf{A}(\lambda)z + \mathbf{B}(\lambda)u \in X_1 \). Therefore by Lemma 6.1.16 (i) holds.

We have now proved that conditions (i)–(vii) are equivalent. Condition (v) together with Lemma 5.2.13 implies \( \Sigma_{\text{part}} \) is the \( \Omega \)-resolving \( i/s/o \) restriction of \( \Sigma \) to \( X_1 \), and that the main operator \( A_{\text{part}} \) of \( \Sigma_{\text{part}} \) is the part of \( A \) of the main operator \( A \) of \( \Sigma \). That \( S_{\text{part}} \) is single-valued whenever \( S \) is single-valued follows from (6.2.1).

All the additional claims for the case where \( \Omega \) is connected are proved in the same way, using Lemmas 6.1.16 and 6.4.11 replaced by Lemmas 6.1.53 and 6.1.57. \( \square \)

### 6.2.2. The system operator of an \( i/s/o \) \( \Omega \)-projection.

**6.2.2. Theorem.** Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an \( \Omega \)-resolvable \( i/s/o \) node, let \( \mathcal{Z}_1 \) be a closed subspace of \( \mathcal{X} \) with a direct complement \( \mathcal{X}_1 \) and let \( \Sigma_{\text{proj}} = (S_{\text{proj}}; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) be the static projection of \( \Sigma \) onto \( \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \) along \( \begin{bmatrix} Z_1 \\ 0 \\ 0 \end{bmatrix} \), i.e.,

\[
\text{gph}(S_{\text{proj}}) = \begin{bmatrix} P_{X_1}^{Z_1} & 0 & 0 & 0 \\ 0 & 1_Y & 0 & 0 \\ 0 & 0 & P_{X_1}^{Z_1} & 0 \\ 0 & 0 & 0 & 1_{\mathbb{R}} \end{bmatrix} \text{gph}(S).
\]

Then the following conditions are equivalent:
(i) $\mathcal{Z}_1$ is unobservably $\Omega$-invariant for $\Sigma$;
(ii) $\Sigma$ has a (unique) $\Omega$-resolvable $\Omega$-projection onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
(iii) $\Sigma_{proj}$ is $\Omega$-resolvable.

If these equivalent conditions hold, then $\Sigma_{proj}$ is the unique $\Omega$-resolvable $\Omega$-projection of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, and dom ($\Sigma_{proj}$) is dense in $[\mathcal{X}_1 \mathcal{U}]$ whenever dom ($\Sigma$) is dense in $[\mathcal{X}_1 \mathcal{U}]$.

The three equivalent conditions (i)–(iii) above are furthermore equivalent to each of the following conditions, where $\widehat{\mathcal{S}} = \begin{bmatrix} \widehat{\mathcal{A}} & \widehat{\mathcal{D}} \end{bmatrix}$ is the i/s/o resolvent matrix of $\Sigma$:

(iv) For all $\lambda \in \Omega$ we have

$$
(6.2.5) \quad \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \frac{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda)}{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1} \end{bmatrix} = \frac{1}{\text{rng}} \begin{bmatrix} -1 \chi \| X_1 \\ 0 \end{bmatrix} \frac{\begin{bmatrix} P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) \\ \chi \| X_1 \end{bmatrix}}{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}.
$$

(v) For all $\lambda \in \Omega$, $\text{gph}(S_{proj} - [\lambda \ 0])$ has the two equivalent representations

$$
(6.2.6a) \quad \text{gph}(S_{proj} - [\lambda \ 0]) = \text{rng} \left( \begin{bmatrix} -1 \chi \| X_1 \\ 0 \end{bmatrix} \frac{\widehat{\mathcal{A}}(\lambda) | X_1}{\frac{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}} \right)
$$

$$
(6.2.6b) \quad \text{gph}(S_{proj} - [\lambda \ 0]) = \text{ker} \left( \begin{bmatrix} -1 \chi \| X_1 \\ 0 \end{bmatrix} \frac{\widehat{\mathcal{A}}(\lambda) | X_1}{\frac{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}} \right).
$$

(vi) For all $\lambda \in \Omega$, the operator $[\lambda \ 0 \ 0] - S_{proj} \left[ \begin{bmatrix} 1 \chi \| X_1 \\ 0 \end{bmatrix} \right]$ has a bounded inverse.

(vii) For all $\lambda \in \Omega$, the operator $[\lambda \ 0 \ 0] - S_{proj} \left[ \begin{bmatrix} 1 \chi \| X_1 \\ 0 \end{bmatrix} \right]$ is injective.

If, in addition, $\Omega$ is connected, then conditions (i)–(vii) above are also equivalent to the condition

(viii) $\Omega \cap \rho(S_{proj}) \neq \emptyset$,

as well as to the conditions that one gets by replacing “for all $\lambda \in \Omega$” by “for some $\lambda \in \Omega$” in conditions (iv)–(vii) above.

Note that we do not claim that $S_{proj}$ is single-valued even if $S$ is single-valued.

(This will be true under some additional assumptions; see Theorems 6.3.4 and 8.3.15 below.)

**Proof of Theorem 6.2.2.** We begin by observing that since $S_{proj}$ is the projection of $S$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, it is also true that $S_{proj} - [\lambda \ 0 \ 0]$ is the projection of $S - [\lambda \ 0 \ 0]$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$. By [6.2.4] and Lemma 5.2.13

$$
(6.2.7) \quad \text{gph}(S_{proj} - [\lambda \ 0 \ 0]) = \text{rng} \left( \begin{bmatrix} -1 \chi \| X_1 \\ 0 \end{bmatrix} \frac{\widehat{\mathcal{A}}(\lambda) | X_1}{\frac{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}{P_{X_1}^{Z_1} \widehat{\mathcal{A}}(\lambda) | X_1}} \right)
$$
(i) ⇔ (ii): See Theorem 6.1.44

(i) ⇔ (v): If (i) holds, then by Lemma 6.1.16, $P_{\lambda_i,X_1}^Z\hat{\mathcal{A}}(\lambda)P_{\lambda_i}^Z = 0$ and $\hat{\mathcal{C}}(\lambda)P_{\lambda_i}^Z = 0$, and this implies that the right-hand side of (6.2.7) is equal to the right-hand side of (6.2.6a). The easy proof that (6.2.6a) and (6.2.6b) are equivalent is left to the reader. Conversely, if (v) holds, then it follows from (6.2.7) that

$$\text{rn}_g \left( \begin{bmatrix} -1_{X_1} & \hat{\mathcal{C}}(\lambda)_{|X_1} & 0 \\ P_{\lambda_i,X_1}^Z\hat{\mathcal{A}}(\lambda)_{|X_1} & P_{\lambda_i,X_1}^Z\hat{\mathcal{B}}(\lambda)_{|X_1} & 1_{\mathcal{U}} \end{bmatrix} \right) = \text{rn}_g \left( \begin{bmatrix} -1_{X_1} & \hat{\mathcal{C}}(\lambda)_{|X_1} & 0 \\ P_{\lambda_i,X_1}^Z\hat{\mathcal{A}}(\lambda)_{|X_1} & P_{\lambda_i,X_1}^Z\hat{\mathcal{B}}(\lambda)_{|X_1} & 1_{\mathcal{U}} \end{bmatrix} \right).$$

This implies that both $P_{\lambda_i,X_1}^Z\hat{\mathcal{A}}(\lambda)z$ and $\hat{\mathcal{C}}(\lambda)z$ are determined uniquely by $P_{\lambda_i}^Zz$, which means that both $P_{\lambda_i,X_1}^Z\hat{\mathcal{A}}(\lambda)P_{\lambda_i}^Z = 0$ and $\hat{\mathcal{C}}(\lambda)P_{\lambda_i}^Z = 0$. By Lemma 6.1.16 (i) holds.

(iii) ⇔ (v): This follows from Lemma 5.2.13.

(iv) ⇔ (v): By (6.2.7) the left-hand side of (6.2.5) is equal to $\text{gph} \left( S_{\text{proj}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right)$, and it follows from Lemma 5.2.13 that the right-hand side of (6.2.5) is equal to

$$\text{gph} \left( S_{\text{proj}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{rn}_g \left( \begin{bmatrix} -1_{X_1} & \hat{\mathcal{C}}(\lambda)_{|X_1} & 0 \\ P_{\lambda_i,X_1}^Z\hat{\mathcal{A}}(\lambda)_{|X_1} & P_{\lambda_i,X_1}^Z\hat{\mathcal{B}}(\lambda)_{|X_1} & 1_{\mathcal{U}} \end{bmatrix} \right).$$

Thus (6.2.5) is equivalent to (6.2.6a).

(iii) ⇒ (vi): This follows from Lemma 5.2.13

(vi) ⇒ (vii): This is obvious.

(vii) ⇒ (iv): Let $\begin{bmatrix} z \\ y \\ 0 \end{bmatrix} \in \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right)$ where $z \in X_1$, or equivalently, $\begin{bmatrix} z \\ y \end{bmatrix} \in (S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}) [0]$. Since $S_{\text{proj}}$ is the projection of $S$ onto $X_1$ along $Z_1$ it follows that $\begin{bmatrix} 0 \\ y \end{bmatrix} \in (S_{\text{proj}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}) \begin{bmatrix} P_{\lambda_i}^Zx \\ 0 \end{bmatrix}$, or equivalently, $\begin{bmatrix} 0 \\ y \end{bmatrix} \in (S_{\text{proj}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}) \begin{bmatrix} P_{\lambda_i}^Zx \\ 0 \end{bmatrix} + z_{\mathcal{U}}$. The injectivity assumption in (vi) then implies that $P_{\lambda_i}^Zx = 0$ and $y = 0$. Consequently,

$$\begin{bmatrix} z \\ y \end{bmatrix} \subset \text{gph} \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \cap \begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

and hence (vii) ⇒ (iv).

We have now proved that conditions (i)–(vii) are equivalent. Condition (v) together with Lemma 5.2.13 implies that $S_{\text{proj}}$ is the system operator of the $\Omega$-projection $\Sigma_1$ of $\Sigma_1$ onto $X_1$ along $Z_1$, and that the main operator $A_{\text{part}}$ of $\Sigma_1$ is the projection of the main operator $A$ of $\Sigma$ onto $X_1$ along $Z_1$. That dom $(S_{\text{proj}})$ is dense in $X_1$ whenever dom $(S)$ is dense in $X$ follows from 6.1.4. All the additional claims for the case where $\Omega$ is connected are proved in the same way, using Lemmas 6.1.16 and 6.1.41 replaced by Lemmas 6.1.53 and 6.1.57.

6.2.3. The system operator of an i/s/o $\Omega$-compression. According to Theorem 6.1.48 if $\Sigma = (S,X,U,Y)$ is an $\Omega$-resolvable frequency domain i/s/o system, then $\Sigma$ has an $\Omega$-resolvable $\Omega$-compression $\Sigma_1 = (S_1:X_1,U,Y)$ onto $X_1$ along $Z_1$ if and only if $Z_1$ contains some closed subspace $\mathcal{Z}^\Omega$ such that $\mathcal{Z}^\Omega$ is
unobservably Ω-invariant for Σ and X₁ + ZΩ is strongly Ω-invariant for Σ. This result can be reformulated as follows.

6.2.3. THEOREM. Let Ω be an open set in ℂ, let Σ = (S; X, U, Y) be an Ω-resolvable i/s/o node, and let X = X₁ + ZΩ + Z_c be a direct sum decomposition of X. Then the following conditions are equivalent:

(i) ZΩ is unobservably Ω-invariant for Σ and X₁ + ZΩ is strongly Ω-invariant for Σ.

(ii) If we denote the part of Σ in X₁ + ZΩ by Σ_{part} = (S_{part}; X₁ + ZΩ, U, Y) and the projection of Σ_{part} onto X₁ along ZΩ by (Σ_{part})_{proj} = ((S_{part})_{proj}; X₁, U, Y), then Ω ⊂ ρ(Σ_{part}) ∩ ρ((Σ_{part})_{proj}).

(iii) If we denote the projection of Σ onto X₁ + Z_c by Σ_{proj} = (S_{proj}; X₁ + Z_c, U, Y) and the part of Σ_{proj} in X₁ by (Σ_{proj})_{part} = ((Σ_{proj})_{part}; X₁, U, Y), then Ω ⊂ ρ(Σ_{proj}) ∩ ρ((Σ_{proj})_{part}).

If these equivalent conditions hold, then Σ has a unique Ω-resolvable Ω-compression onto X₁ along Z₁ := ZΩ + Z_c, and conversely, if Z₁ is a direct complement to X₁ in X and Σ has an Ω-resolvable Ω-compression onto X₁ along Z₁, then Z₁ can be decomposed into Z₁ = ZΩ + Z_c in such a way that the above equivalent conditions (i)–(iii) hold. The unique Ω-resolvable Ω-compression Σ₁ of Σ onto X₁ along Z₁ and is given by

\[ \Sigma_1 = (\Sigma_{part})_{proj} = (\Sigma_{proj})_{part} \]

The three equivalent conditions (i)–(iii) above are furthermore equivalent to each of the following conditions:

(iv) For all λ ∈ Ω the following two conditions hold:

\[ gph \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \cap \left[ \begin{bmatrix} X_1 + ZΩ \\ \frac{Y}{U} \end{bmatrix} \right] = gph \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \cap \left[ \begin{bmatrix} X_1 + ZΩ \\ \frac{Y}{U} \end{bmatrix} \right], \]

\[ \begin{bmatrix} p_{X_1 + Z_c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} gph \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \]

\[ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p_{X_1 + Z_c} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \left( gph \left( S - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \cap \left[ \begin{bmatrix} X_1 + Z_c \\ \frac{Y}{U} \end{bmatrix} \right] \right). \]

(v) the part of \[ \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \] in X₁ + ZΩ has an inverse in \( B \left( \begin{bmatrix} X_1 + ZΩ \\ \frac{Y}{U} \end{bmatrix} \right) \)

and the projection of \[ \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \] onto X₁ + Z_c along ZΩ has an inverse in \( B \left( \begin{bmatrix} X_1 + Z_c \\ \frac{Y}{U} \end{bmatrix} \right) \) for all \( \lambda \in \Omega \).

(vi) the part of \[ \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \] in X₁ + ZΩ is surjective and the projection of \[ \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \] onto X₁ + Z_c along ZΩ is injective for all \( \lambda \in \Omega \).

If, in addition, Ω is connected, then conditions (i)–(vi) above are also equivalent to the two conditions

(vii) \( \Omega \cap \rho(\Sigma_{part}) \cap \rho((\Sigma_{part})_{proj}) \neq \emptyset \),

(viii) \( \Omega \cap \rho(\Sigma_{proj}) \cap \rho((\Sigma_{proj})_{part}) \neq \emptyset \),

as well as to the conditions that one gets by replacing “for all \( \lambda \in \Omega \)” by “for some \( \lambda \in \Omega \)” in conditions (iv)–(vi) above.
Note that we do not claim that $S_1$ is single-valued or densely defined even if $S$ is single-valued and densely defined. (This will be true under some additional assumptions; see Theorems 6.3.6 and 8.3.16 below.)

**Proof of Theorem 6.2.3.** (i) $\Leftrightarrow$ (iv): The follows from the equivalence of conditions (i) and (iv) in Theorems 6.2.1 and 6.2.2.

(iv) $\Leftrightarrow$ (v): The follows from the equivalence of conditions (iv) and (vi) in Theorems 6.2.1 and 6.2.2.

(v) $\Leftrightarrow$ (vi): The follows from the equivalence of conditions (vi) and (vii) in Theorems 6.2.1 and 6.2.2.

(i) $\Rightarrow$ (ii): If (i) holds, then by Theorem 6.2.1 $\Sigma_{\text{part}}$ is the restriction of $\Sigma$ to $X_1 + Z^\Omega$ and $\Omega \subset \rho(\Sigma_{\text{part}})$. By Theorem 6.1.50, $Z^\Omega$ is $\Omega$-invariant, and by Theorem 6.2.2 ($\Sigma_{\text{part}}$)$_{\text{proj}}$ is the projection of $\Sigma_{\text{part}}$ onto $X_1$ along $Z^\Omega$. By Theorem 6.1.50, ($\Sigma_{\text{part}}$)$_{\text{proj}}$ is equal to the compression of $\Sigma$ onto $X_1$ along $Z^\Omega + Z_c$.

(ii) $\Rightarrow$ (i): If (ii) holds, then it follows from the equivalence of (i) and (iii) in Theorem 6.2.1 that $X_1 + Z^\Omega$ is strongly invariant for $\Sigma$, and that $\Sigma_{\text{part}}$ is the restriction of $\Sigma$ to $X_1 + Z^\Omega$. By applying the equivalence of (i) and (iii) in Theorem 6.2.1 to $\Sigma_{\text{part}}$ we find that $Z^\Omega$ is unobservably $\Omega$-invariant for $\Sigma_{\text{part}}$. This implies that $Z^\Omega$ is also unobservably $\Omega$-invariant for $\Sigma$.

(i) $\Leftrightarrow$ (iii): This proof is analogous to the proof given above. As above, as a part of this proof we see that also ($\Sigma_{\text{proj}}$)$_{\text{part}}$ is equal to the compression of $\Sigma$ onto $X_1$ along $Z^\Omega + Z_c$.

The additional claims for the case where $\Omega$ is connected also follow from the corresponding claims in Theorem 6.2.1 and 6.2.2.

**6.2.4. The system operators of $\Omega$-intertwined i/s/o systems.**

Let $\Omega$ be an open set in $\mathbb{C}$, let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$ be two $\Omega$-resolvable frequency domain i/s/o systems, and let $\Pi \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ be closed. Let $\Sigma_\Pi = (S_\Pi; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be the difference connection of $\Sigma_2$ and $\Sigma_1$ with state space $\mathcal{X} = [\mathcal{X}_2 \mid \mathcal{X}_1]$ and i/s/o resolvent matrix $\widehat{\mathcal{G}}_\Pi = \begin{bmatrix} \mathcal{G}_\Pi^+ & \mathcal{G}_\Pi^- \\ \mathcal{G}_\Pi^- & \mathcal{G}_\Pi^+ \end{bmatrix}$, and let $\Sigma = (S; \text{gph}(P), \mathcal{U}, \mathcal{Y})$ be the $\text{gph}(P)$-short circuit connection of $\Sigma_2$ and $\Sigma_1$ (see Definition 2.3.37). Then the following conditions are equivalent:

(i) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P$.

(ii) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-invariant, and $\text{gph}(P)$ is both a strongly $\Omega$-invariant and an unobservably $\Omega$-invariant subspace for $\Sigma_\Pi$.

(iii) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-invariant, and $\Omega \subset \rho(\Sigma_{\text{part}}) \cap \rho(\Sigma_{\text{proj}})$, where $\Sigma_{\text{part}} = (S_{\text{part}}; \text{gph}(P), \mathcal{U}, \mathcal{Y})$ is the part of $\Sigma_\Pi$ in $\text{gph}(P)$ and $\Sigma_{\text{proj}} = (S_{\text{proj}}; Z, \mathcal{U}, \mathcal{Y})$ is the projection of $\Sigma_\Pi$ along $\text{gph}(P)$ onto some direct complement $Z$ of $\text{gph}(P)$ in $\mathcal{X}$.

(iv) $\Sigma$ is $\Omega$-resolvable.

The three equivalent conditions (i)-(iv) above are furthermore equivalent to each of the following conditions:

(iv) For all $\lambda \in \Omega$ we have

\[
\text{gph} \left( S_{\Pi} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \cap \begin{bmatrix} \text{gph}(P) \end{bmatrix} = \text{gph} \left( S_{\Pi} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \cap \begin{bmatrix} \text{gph}(P) \end{bmatrix} \right). 
\]
(v) For all $\lambda \in \Omega$, $\text{gph} \left( S_{\text{part}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right)$ has the two equivalent representations

\[
\text{gph} \left( S_{\text{part}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{rng} \left( \begin{bmatrix} -1_{\text{gph}(P)} & 0 & 0 \\ 0 & 0 & 0 \\ \hat{\mathbf{A}}(\lambda)_{|\text{gph}(P)} & \hat{\mathbf{B}}(\lambda) & 1_{U} \end{bmatrix} \right),
\]

\[
\text{gph} \left( S_{\text{part}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) = \text{ker} \left( \begin{bmatrix} \hat{\mathbf{A}}(\lambda)_{|\text{gph}(P)} & 0 & 1_{\text{gph}(P)} & -\hat{\mathbf{B}}(\lambda) \\ 0 & 1_{Y} & 0 & 0 \end{bmatrix} \right).
\]

(vi) $\left[ \begin{array}{c} \lambda \\ 0 \\ 0 \\ 0 \end{array} \right] - \left[ \begin{array}{c} 1_{\text{gph}(P)} \\ 0 \\ 0 \\ 0 \end{array} \right] S_{\text{part}}$ has an inverse in $B\left( \left[ \begin{array}{c} \text{gph}(P) \\ U \end{array} \right] \right)$ and

\[
\text{gph} \left( S_{\text{part}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \subseteq \left[ \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} \right] \text{ for all } \lambda \in \Omega.
\]

(vii) $\left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] - \left[ \begin{array}{c} 1_{\text{gph}(P)} \Sigma \end{array} \right] S_{\text{part}}$ is surjective and

\[
\text{gph} \left( S_{\text{part}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right) \subseteq \left[ \begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} \right] \text{ for all } \lambda \in \Omega.
\]

If, in addition $\Omega$ is connected, then conditions (i)–(vii) above are equivalent to the condition

(viii) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-invariant, and $\Omega \cap \rho(\Sigma_{\text{part}}) \cap \rho(\Sigma_{\text{proj}}) \neq \emptyset$,

as well as to the conditions that one gets by replacing “for all $\lambda \in \Omega$” by “for some $\lambda \in \Omega$” in conditions (iv)–(vii) above.

PROOF. Let us denote the i/s/o resolvent matrices of $\Sigma_i$ by $\hat{\mathbf{S}}_i = \begin{bmatrix} \hat{\mathbf{A}}_i & \hat{\mathbf{B}}_1 \\ \hat{\mathbf{C}}_i & \hat{\mathbf{D}}_i \end{bmatrix}$, $i = 1, 2$.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): This follows from Lemmas 6.1.11 and 6.1.26 and Theorems 6.2.1 and 6.2.3.

(i) $\Leftrightarrow$ (iv): By Lemmas 5.2.13 and 5.2.35, $\Sigma_{\text{proj}}$ is $\Omega$-resolvable and $\text{gph} \left( S_{\text{proj}} - \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \right)$ has the representation (6.2.11) with $\Sigma$ replaced by $\Sigma_{\text{proj}}$. This together with Lemmas 6.1.16 and 6.1.26 implies that (i) and (iv) are equivalent.

(i) $\Leftrightarrow$ (v): Since both $\Sigma_1$ and $\Sigma_2$ are $\Omega$-resolvable, it follows from Lemma 5.2.13 that $\left[ \begin{array}{c} s_{1i} \\ s_{2i} \end{array} \right] \in (S_{\text{proj}} - \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right])$ if and only if $\left[ \begin{array}{c} s_{1i} \\ s_{2i} \end{array} \right] = \hat{\mathbf{S}}_{\text{proj}}(\lambda) \left[ \begin{array}{c} -z_{1i} \\ u \end{array} \right]$, $i = 1, 2$. Here $\left[ \begin{array}{c} -z_{1i} \\ u \end{array} \right]$ can be chosen to be an arbitrary vector in $\left[ \begin{array}{c} \lambda \\ U \end{array} \right]$, $i = 1, 2$. Thus, under the additional assumption that $z \in P_{Z_{1}}$, the above condition is equivalent to the four conditions

\[
\hat{\mathbf{A}}_1(\lambda)z \in P\hat{\mathbf{A}}_1(\lambda)z_1, \quad \hat{\mathbf{B}}_2(\lambda)u \in P\hat{\mathbf{B}}_1(\lambda)u,
\]

\[
\hat{\mathbf{C}}_2(\lambda)z = \hat{\mathbf{C}}_1(\lambda)z_1, \quad \hat{\mathbf{D}}_2(\lambda)u = \hat{\mathbf{D}}_1(\lambda)u.
\]

By Lemma 6.1.23 this is equivalent to (i).

(i) $\Rightarrow$ (vi): It follows from Theorem 6.2.1 and Lemma 6.1.26 that $\left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] - \left[ \begin{array}{c} 1_{\text{gph}(P)} \\ 0 \\ 0 \end{array} \right] S_{\text{part}}$ has an inverse in $B\left( \left[ \begin{array}{c} \text{gph}(P) \\ U \end{array} \right] \right)$. The second condition in (vi) follows from (v) (which we by now know is equivalent to (i)).

(vi) $\Rightarrow$ (vii): This is obvious.

(vii) $\Rightarrow$ (v): By Theorem 6.2.1 the first condition in (vii) implies that $\text{gph} \left( P \right)$ is strongly $\Omega$-invariant for $\Sigma$, and that $\text{gph} \left( S_{\text{part}} - \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] \right)$ has the two equivalent representations (6.2.3) with $\hat{\mathbf{X}}_1$ replaced by $\hat{\mathbf{P}}(\lambda)$. This combined with the second condition in (vii) implies that $\text{gph} \left( S_{\text{part}} - \left[ \begin{array}{c} \lambda \\ 0 \end{array} \right] \right)$ has the two equivalent representations (6.2.13).

The additional claims in the case where $\Omega$ is connected are proved in the same way, with Lemmas 6.1.16 and 6.1.23 replaced by Lemmas 6.1.33 and 6.1.35.
6.2.5. Compressions into \( \Omega \)-minimal i/s/o systems.

**Definition.** Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma = (S, X, U, Y) \) be an \( \Omega \)-resolvable frequency domain i/s/o system.

(i) \( \Sigma \) is called \( \Omega \)-minimal if \( \Sigma \) does not have any non-trivial \( \Omega \)-resolvable \( \Omega \)-compression.

(ii) By an \( \Omega \)-minimal compression of \( \Sigma \) we mean an \( \Omega \)-resolvable \( \Omega \)-compression \( \Sigma_1 \) of \( \Sigma \) which is \( \Omega \)-minimal (i.e., \( \Sigma_1 \) does not have any further non-trivial \( \Omega \)-solvable \( \Omega \)-compressions).

6.2.6. Remark. Observe that in the above definition, even in the case where the i/s/o node \( \Sigma \) is regular, we do not require the possible nontrivial \( \Omega \)-compressions to be regular, i.e., in order for \( \Sigma \) to be minimal it must not have any \( \Omega \)-compressible or non-regular nontrivial \( \Omega \)-compression. The reason for this that it is not known to what extent Theorem 6.2.7 below is valid for regular i/s/o nodes if we require an \( \Omega \)-compression of a regular \( \Omega \)-solvable i/s/o node to be regular.

6.2.7. Theorem. Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma = (S, X, U, Y) \) be an \( \Omega \)-resolvable frequency domain i/s/o system. Then \( \Sigma \) is \( \Omega \)-minimal if and only if \( \Sigma \) is both \( \Omega \)-controllable and \( \Omega \)-observable.

**Proof.** If \( \Sigma \) is not \( \Omega \)-observable, then \( \mathcal{U}_\Sigma \neq \{0\} \) where \( \mathcal{U}_\Sigma \) is the \( \Omega \)-unobservable subspace of \( \Sigma \). By Lemma 6.1.18 (iii), \( \mathcal{U}_\Sigma \) is unobservably \( \Omega \)-invariant for \( \Sigma \), and it follows from Theorem 6.1.44 (ii) that \( \Sigma \) has an \( \Omega \)-resolvable \( \Omega \)-projection along \( \mathcal{U}_\Sigma \) onto any direct complement to \( \mathcal{U}_\Sigma \). This is a nontrivial \( \Omega \)-compression of \( \Sigma \).

If \( \Sigma \) is not \( \Omega \)-controllable, then \( \mathcal{R}_\Sigma \neq X \) where \( \mathcal{R}_\Sigma \) is the \( \Omega \)-reachable subspace of \( \Sigma \). By Lemma 6.1.18 (ii), \( \mathcal{R}_\Sigma \) is strongly \( \Omega \)-invariant for \( \Sigma \), and it follows from Theorem 6.1.44 (i) that \( \Sigma \) has an \( \Omega \)-resolvable \( \Omega \)-restriction to \( \mathcal{R}_\Sigma \). This is a nontrivial \( \Omega \)-compression of \( \Sigma \).

Thus, \( \Omega \)-minimality of \( \Sigma \) implies both \( \Omega \)-controllability and \( \Omega \)-observability.

Conversely, suppose that \( \Sigma_1 = (S_1; X_1, U, Y) \) is an \( \Omega \)-compression of \( \Sigma \). If \( \Sigma \) is \( \Omega \)-observable, then \( Z_{\text{max}} = \{0\} \) where \( Z_{\text{max}} \) is the subspace defined in Lemma 6.1.46 and hence by Theorem 6.1.48, \( X_1 \) is strongly \( \Omega \)-invariant for \( \Sigma \). If furthermore \( \Sigma \) is \( \Omega \)-controllable, then by Lemma 6.1.14, \( X_1 = X \). Thus, if \( \Sigma \) is both \( \Omega \)-controllable and \( \Omega \)-observable, then \( \Sigma_1 = \Sigma \), and hence \( \Sigma \) does not have any nontrivial \( \Omega \)-compression.

6.2.8. Theorem. Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma = (S, X, U, Y) \) be an \( \Omega \)-resolvable i/s/o node. Then \( \Sigma \) has an \( \Omega \)-minimal \( \Omega \)-compression. Two families of such \( \Omega \)-compressions are described below, where we have denoted the \( \Omega \)-reachable and \( \Omega \)-unobservable subspaces of \( \Sigma \) by \( \mathcal{R}_\Sigma^\Omega \) respectively \( \mathcal{U}_\Sigma^\Omega \):

(i) Let \( X_1 \) be a direct complement to \( \mathcal{U}_\Sigma^\Omega \) in \( X \), and let \( X_\circ = \frac{\mathcal{R}}{X_1}^\Omega \), and let \( \Sigma_\circ = (S_\circ; X_\circ, U, Y) \), where \( S_\circ \) is the part in \( X_\circ \) of the projection of \( S \) onto \( X_1 \) along \( \mathcal{U}_\Sigma^\Omega \), i.e.,

\[
(6.2.14) \quad \text{gph}(S_\circ) = \left( \begin{bmatrix} P_{X_1}^\Omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & P_{X_1}^\Omega & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{gph}(S) \right) \cap \begin{bmatrix} X_\circ \\ Y \\ X_\circ \\ U \end{bmatrix}.
\]
Then $\Sigma_o$ is an $\Omega$-minimal $\Omega$-compression of $\Sigma$. One gets this minimal $\Omega$-compression by first $\Omega$-projecting $\Sigma$ onto $X_1$ along its $\Omega$-unobservable subspace $U^\Omega_2$, and then $\Omega$-restricting the resulting system to its $\Omega$-reachable subspace $X_o$.

(ii) Let $X_*$ be a direct complement to $R^\Omega_1 \cap U^\Omega_2$ in $R^\Omega_1$, and let $\Sigma_* = (S_*; X_*, U, Y)$, where $S_*$ is the projection onto $X_*$ along $R^\Omega_1 \cap U^\Omega_2$ of the part of $S$ in $R^\Omega_1$,

$$\text{gph}(S_*) = \begin{bmatrix} P_{X_*} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & P_{X_*} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left( \text{gph}(S) \cap \begin{bmatrix} R^\Omega_1 \\ Y \\ R^\Omega_1 \\ U \end{bmatrix} \right)$$

Then $\Sigma_*$ is an $\Omega$-minimal $\Omega$-compression of $\Sigma$. One gets this $\Omega$-compression by first $\Omega$-restricting $\Sigma$ to its $\Omega$-reachable subspace $R^\Omega_1$, and then $\Omega$-projecting the resulting system onto $X_*$ along its $\Omega$-unobservable $R^\Omega_2$.

**Proof.** (i) By Lemma 6.1.18 $U^\Omega_2$ in unobservably $\Omega$-invariant for $\Sigma$, and by Theorem 6.1.44 $\Sigma$ has an $\Omega$-resolvable $\Omega$-projection $\Sigma_1$ onto $X_1$ along $U^\Omega_2$. It follows from Lemma 6.1.14 that $\Sigma_1$ is $\Omega$-observable and that the $\Omega$-reachable subspace of $\Sigma_1$ is $X_*$. This subspace is strongly $\Omega$-invariant for $\Sigma_1$, and hence by Theorem 6.1.44 $\Sigma$ has an $\Omega$-resolvable $\Omega$-restriction $\Sigma_0$ to $X_*$. It follows from Lemma 6.1.14 that $\Sigma_0$ is both $\Omega$-observable and $\Omega$-controllable, and hence by Theorem 6.1.44 $\Sigma_0$ is $\Omega$-minimal. That the graph of the system operator of $\Sigma_0$ is given by (6.2.14) follows from Theorems 6.2.1 and 6.2.2.

(ii) The proof of (ii) is analogous to the proof of (i).

6.2.9. **Lemma.** The $\Omega$-minimal $\Omega$-compression of an $\Omega$-resolvable i/s/o system $\Sigma = (S; X, U, Y)$ is unique if and only at least one of conditions (i) and (ii) below holds:

(i) $\Sigma$ is $\Omega$-observable, i.e., $U^\Omega_2 = \{0\}$, where $U^\Omega_2$ is the $\Omega$-unobservable subspace of $\Sigma$.

(ii) the following equivalent conditions hold:

(a) $\Sigma$ has an $\Omega$-minimal $\Omega$-compression with state space $\{0\}$,
(b) the i/o resolvent of $\Sigma$ is a constant in $\Omega$.
(c) $R^\Omega_1 \subset U^\Omega_2$, where $R^\Omega_1$ is the $\Omega$-reachable subspace of $\Sigma$.

In case (i) the unique $\Omega$-minimal $\Omega$-compression $\Sigma_{\text{min}}$ is the part of $\Sigma$ in $R^\Omega_1$, i.e., $\Sigma_{\text{min}} = (S_{\text{min}}; R^\Omega_1, U, Y)$ where

$$\text{gph}(S_{\text{min}}) = \text{gph}(S) \cap \begin{bmatrix} R^\Omega_1 \\ Y \\ R^\Omega_1 \\ U \end{bmatrix}.$$ 

In case (ii) the unique minimal compression is $\Sigma_{\text{min}} = \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ; \{0\}, U, Y \right)$, where $D$ is the constant value of the i/s/o resolvent matrix of $\Sigma$ in $\Omega$. If neither (i) nor (ii) holds, then $\Sigma$ has an infinite number of minimal compressions.

**Proof.** The proof is analogous to the proof of Lemma 3.2.32. □
6.2.6. Frequency domain notions for \(\Omega\)-resolvable i/s/o nodes. In Definition 3.3.14 we transferred a number of dynamical notions for the continuous time and discrete time i/s/o systems induced by bounded i/s/o nodes into notions which applies to the node itself. The same idea can also be used to transfer notions defined in terms of the frequency domain i/s/o system induced by a resolvable i/s/o node into a notion which applies to the node itself.

6.2.10. Definition. Let \(\Omega\) be an open set in \(\mathbb{C}\), and let \(\Sigma = (S; X, U, Y)\) and \(\Sigma_i = (S_i; X_i, U_i, Y_i)\) be \(\Omega\)-resolvable i/s/o nodes. Denote the i/s/o resolvent matrices of \(\Sigma\) and \(\Sigma_i\) by 
\[
\hat{S} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \quad \text{and} \quad \hat{S}_i = \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix}, \quad i = 1, 2, \text{respectively.}
\]

(i) A closed subspace \(Z\) of \(X\) is strongly \(\Omega\)-invariant for \(\Sigma\) if \(\hat{A}(\lambda)Z \subset Z\) and \(\text{rng} \, (\hat{B}(\lambda)) \subset Z\) for all \(\lambda \in \Omega\), i.e., if \(Z\) is strongly \(\Omega\)-invariant for the frequency domain i/s/o system induced by \(\Sigma\) (see Lemma 6.1.16).

(ii) A closed subspace \(Z\) of \(X\) is unobservably \(\Omega\)-invariant for \(\Sigma\) if \(\hat{A}(\lambda)Z \subset Z\) and \(Z \subset \ker (\hat{C}(\lambda))\) for all \(\lambda \in \Omega\), i.e., if \(Z\) is strongly \(\Omega\)-invariant for the frequency domain i/s/o system induced by \(\Sigma\) (see Lemma 6.1.16).

(iii) The subspace \(R_{\Sigma}\) defined in (6.1.5) is called the \(\Omega\)-reachable subspace of \(\Sigma\) (this is the \(\Omega\)-reachable subspace of the frequency domain i/s/o system induced by \(\Sigma\)).

(iv) \(\Sigma\) is \(\Omega\)-controllable if the frequency domain i/s/o system induced by \(\Sigma\) is \(\Omega\)-controllable (i.e., the \(\Omega\)-reachable subspace of \(\Sigma\) is the full state space \(X\)).

(v) The subspace \(U_{\Sigma}\) defined in (6.1.6) is called the \(\Omega\)-unobservable subspace of \(\Sigma\) (this is the \(\Omega\)-unobservable subspace of the frequency domain i/s/o system induced by \(\Sigma\)).

(vi) \(\Sigma\) is \(\Omega\)-observable if the frequency domain i/s/o system induced by \(\Sigma\) is \(\Omega\)-observable (i.e., the \(\Omega\)-unobservable subspace of \(\Sigma\) is \(\{0\}\)).

(vii) \(\Sigma_1\) and \(\Sigma_2\) are externally \(\Omega\)-equivalent if \(\hat{D}_1(\lambda) = \hat{D}_2(\lambda)\) for all \(\lambda \in \Omega\), i.e., if the frequency domain i/s/o systems induced by \(\Sigma_1\) and \(\Sigma_2\) are externally \(\Omega\)-equivalent (see Lemma 6.1.11).

(viii) \(\Sigma_1\) and \(\Sigma_2\) are \(\Omega\)-intertwined by a closed \(P \in \mathcal{ML}(X_1; X_2)\) if the frequency domain i/s/o systems induced by \(\Sigma_1\) and \(\Sigma_2\) are intertwined by \(P\) (i.e., if conditions (i)–(iv) in Lemma 6.1.23 hold).

(ix) \(\Sigma_1\) is the \(\Omega\)-restriction to \(X_1\), or \(\Omega\)-projection or \(\Omega\)-compression onto \(X_1\) along \(Z_1\) of \(\Sigma_2\) if the frequency domain i/s/o system induced by \(\Sigma_1\) is the \(\Omega\)-restriction to \(X_1\), or \(\Omega\)-projection or \(\Omega\)-compression onto \(X_1\), respectively, along \(Z_1\) of the frequency domain i/s/o system induced by \(\Sigma_2\) (i.e., if conditions (6.1.12), (6.1.13), or (6.1.14) hold, respectively).

(x) \(\Sigma\) is \(\Omega\)-minimal if the frequency domain i/s/o system induced by \(\Sigma\) is \(\Omega\)-minimal, i.e., if \(\Sigma\) does not have any non-trivial \(\Omega\)-resolvable \(\Omega\)-compression.

If \(\Sigma\) is a bounded i/s/o node, then we can apply both Definition 3.3.14 and Definition 6.2.10 to \(\Sigma\). The following lemma describes the connection between the notions introduced in these two definitions.

6.2.11. Lemma. Let \(\Sigma = (S; X, U, Y)\) and \(\Sigma_i = (S_i; X_i, U_i, Y_i)\) be bounded i/s/o nodes. Then all the dynamical notions listed in Definition 3.3.14 are equivalent to the corresponding \(\Omega\)-dynamical notions listed in Definition 6.2.10 with \(\Omega = \rho_{\infty}(\Sigma)\).
(in those notions which refer only to $\Sigma$) or $\Omega$ equal to the unbounded component of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$ (in those notions that refer to $\Sigma_1$ and $\Sigma_2$).

**Proof.** The proof is analogous to the proof of Lemma 3.3.12. □

### 6.2.7. Dynamical properties of the resolvent family of bounded i/s/o nodes

At this point the reader may want to recall Definition 5.2.37, which explains what we mean by the resolvent family of bounded i/s/o nodes induced by a resolvable i/s/o node $\Sigma$.

#### 6.2.12. Lemma

Let $\Omega$ be an open set in $C$, let $\Sigma = (S; \mathcal{X}; \mathcal{U}; \mathcal{Y})$ and $\Sigma_i = (S_i; \mathcal{X}_i; \mathcal{U}_i; \mathcal{Y}_i), i = 1, 2$, be $\Omega$-resolvable i/s/o systems, and let $\Sigma^\lambda = (\hat{\mathcal{E}}(\lambda); \mathcal{X}; \mathcal{U}; \mathcal{Y})$, $\lambda \in \rho(\Sigma)$, and $\Sigma_i^\lambda = (\hat{\mathcal{E}}_i(\lambda); \mathcal{X}_i; \mathcal{U}_i; \mathcal{Y}_i), \lambda \in \rho(\Sigma_i), i = 1, 2$, be the resolvent families of bounded i/s/o nodes induced by $\Sigma$ respectively $\Sigma_i, i = 1, 2$. Then the following claims are true:

(i) A closed subspace $Z$ of $\mathcal{X}$ is strongly or unobservably $\Omega$-invariant for $\Sigma$ if and only if $Z$ is strongly respectively unobservably invariant for $\Sigma^\lambda$ for all $\lambda \in \Omega$.

(ii) $\mathcal{R}^\Omega_{\Sigma} = \bigvee_{\lambda \in \Omega} \mathcal{R}_{\Sigma^\lambda}$, where $\mathcal{R}^\Omega_{\Sigma^\lambda}$ is the $\Omega$-reachable subspace of $\Sigma$ and $\mathcal{R}_{\Sigma^\lambda}$ is the reachable subspace of $\Sigma^\lambda$.

(iii) $\mathcal{U}^\Omega_{\Sigma} = \bigcap_{\lambda \in \Omega} \mathcal{U}_{\Sigma^\lambda}$, where $\mathcal{U}^\Omega_{\Sigma^\lambda}$ is the $\Omega$-unobservable subspace of $\Sigma$ and $\mathcal{U}_{\Sigma^\lambda}$ is the unobservable subspace of $\Sigma^\lambda$.

(iv) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent if and only if $\Sigma_1^\lambda$ and $\Sigma_2^\lambda$ are externally equivalent for all $\lambda \in \Omega$.

(v) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by the closed $P \in \mathcal{M}(\mathcal{X}_1; \mathcal{X}_2)$ if and only if $\Sigma_1^\lambda$ and $\Sigma_2^\lambda$ are intertwined by $P$ for all $\lambda \in \Omega$.

(vi) $\Sigma_1$ is the $\Omega$-restriction to $\mathcal{X}_1$, or the $\Omega$-projection or $\Omega$-compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ of $\Sigma_2$ if and only if $\Sigma_1^\lambda$ is the restriction to $\mathcal{X}_1$, or the projection or compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ of $\Sigma_2^\lambda$, respectively, for all $\lambda \in \Omega$.

**Proof.**

(i) See Lemmas 3.2.2, 3.2.3 and 6.1.16.

(ii) By Lemma 5.2.4 and 6.1.14

$$\mathcal{R}^\Omega_{\Sigma} = \bigvee_{\lambda \in \Omega} \text{rng} (\hat{\mathcal{B}}(\lambda)) \subset \bigvee_{\lambda \in \Omega} \left( \bigvee_{n \in \mathbb{Z}^+} \text{rng} (\hat{\mathcal{A}}(\lambda)^n \hat{\mathcal{B}}(\lambda)) \right) = \bigvee_{\lambda \in \Omega} \mathcal{R}_{\Sigma^\lambda}. $$

The opposite inclusion follows from part (i) of Lemma 6.1.18.

(iii) By Lemma 3.2.4 and 6.1.14

$$\mathcal{U}^\Omega_{\Sigma} = \bigcap_{\lambda \in \Omega} \ker (\hat{\mathcal{E}}(\lambda)) \supset \bigcap_{\lambda \in \Omega} \left( \bigcap_{n \in \mathbb{Z}^+} \ker (\hat{\mathcal{E}}(\lambda)\hat{\mathcal{A}}(\lambda)^n) \right) = \bigcap_{\lambda \in \Omega} \mathcal{U}_{\Sigma^\lambda}. $$

Conversely, suppose that $x \in \mathcal{U}_{\Sigma^\lambda}$, i.e., suppose that $\hat{\mathcal{E}}(\lambda)x = 0$ for all $\lambda \in \Omega$. By differentiating this equality $n$ times with respect to $\lambda$ we get $\hat{\mathcal{E}}(\lambda)\hat{\mathcal{A}}(\lambda)^n x = 0$ for all $\lambda \in \Omega$ and all $n \in \mathbb{Z}^+$. Therefore, by (3.2.3), $x$ belongs to the unobservable subspace for $\Sigma^\lambda$. This being true for all $\lambda \in \Omega$, we have $x \in \bigcap_{\lambda \in \Omega} \mathcal{U}_{\Sigma^\lambda}$.

(iv) If $\Sigma_1^\lambda$ and $\Sigma_2^\lambda$ are externally equivalent for all $\lambda \in \Omega$, then it follows from Theorem 3.2.7 with $D_i$ replaced by $\widehat{D}_i(\lambda)$ that $\widehat{D}_1(\lambda) = \widehat{D}_2(\lambda)$ for all $\lambda \in \Omega$. Thus by Lemma 5.2.11, $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent. Conversely, suppose that $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent. Then by differentiating the identity $\widehat{D}_1(\lambda) = \widehat{D}_2(\lambda)$ $n$ times and using 5.2.33 we find that $\hat{\mathcal{C}}(\lambda)\hat{\mathcal{A}}(\lambda)^n \hat{\mathcal{B}}_1(\lambda) =
Let \( \Omega \) be an open subset of \( \mathbb{C} \). Thus, condition (iv) in Theorem 3.2.7 holds with \( \Sigma_i \) replaced by \( \Sigma_i^\lambda \), and consequently \( \Sigma_i^\lambda \) and \( \Sigma_i^\lambda \) are externally equivalent.

(v) See Lemmas 3.2.16 and 6.1.23.

(vi) The claims about \( \Omega \)-restrictions and \( \Omega \)-projections follow directly from Theorems 3.2.11 and 3.2.12 and Lemma 6.1.41. The proof of the claim about \( \Omega \)-equivalent.

Let \( \hat{\mathcal{E}}_2(\lambda) \hat{\mathcal{E}}_2(\lambda) \) for all \( n \in \mathbb{Z}^+ \) and all \( \lambda \in \Omega \). Thus, condition (iv) in Theorem 3.2.7 holds with \( \Sigma_i \) replaced by \( \Sigma_i^\lambda \), and consequently \( \Sigma_i^\lambda \) and \( \Sigma_i^\lambda \) are externally equivalent.

(v) See Lemmas 3.2.16 and 6.1.23.

(vi) The claims about \( \Omega \)-restrictions and \( \Omega \)-projections follow directly from Theorems 3.2.11 and 3.2.12 and Lemma 6.1.41. The proof of the claim about \( \Omega \)-equivalent.

There are certain claims that are missing from Lemma 6.2.12, namely claims about controllability, observability, and minimality. The reason for this omission is that in the setting of Lemma 6.2.12.

(i) \( \Omega \)-controllability of \( \Sigma \) does not imply controllability of \( \Sigma^\lambda \) for all \( \lambda \in \Omega \),

(ii) \( \Omega \)-observability of \( \Sigma \) does not imply observability of \( \Sigma^\lambda \) for all \( \lambda \in \Omega \),

(iii) \( \Omega \)-minimality of \( \Sigma \) does not imply minimality of \( \Sigma^\lambda \) for all \( \lambda \in \Omega \).

However, as we shall see below, these additional claims are true whenever \( \Omega \) is connected, or more generally, \( \Omega \) is contained in some connected component of \( \rho(\Sigma) \).

6.2.13. Lemma. Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), let \( \Sigma = (S; X, U, Y) \) and \( \Sigma_i = (S_i; X_i, U_i, Y_i) \), \( i = 1, 2 \), be three \( \Omega^o \)-resolvable i/s/o systems, and let \( \Sigma^\lambda = (\hat{\mathcal{E}}(\lambda); X, U, Y) \), \( \lambda \in \rho(\Sigma) \), and \( \Sigma_i^\lambda = (\hat{\mathcal{E}}_i(\lambda); X_i, U_i, Y_i) \), \( \lambda \in \rho(\Sigma_i) \), \( i = 1, 2 \), be the resolvent families of bounded i/s/o nodes induced by \( \Sigma \) respectively \( \Sigma_i \), \( i = 1, 2 \). Let \( \hat{\mathcal{E}}_2(\lambda) \hat{\mathcal{E}}_2(\lambda) \) for all \( n \in \mathbb{Z}^+ \) and all \( \lambda \in \Omega \). Thus, condition (iv) in Theorem 3.2.7 holds with \( \Sigma_i \) replaced by \( \Sigma_i^\lambda \), and consequently \( \Sigma_i^\lambda \) and \( \Sigma_i^\lambda \) are externally equivalent.

(i) For each closed subspace \( Z \) of \( X \) the following conditions are equivalent:

(a) \( Z \) is strongly or unobservably \( \Omega \)-invariant for \( \Sigma \);

(b) \( Z \) is strongly respectively unobservably invariant for \( \Sigma^\lambda \) for all \( \lambda \in \Omega^o \);

(c) \( Z \) is strongly respectively unobservably invariant for \( \Sigma^\lambda \) for some \( \lambda_0 \in \Omega^o \).

(ii) \( R^o_{\Sigma^\lambda} = R_{\Sigma^\lambda} \) for all \( \lambda \in \Omega^o \), where \( R^o_{\Sigma^\lambda} \) is the \( \Omega \)-reachable subspace of \( \Sigma \) and \( R_{\Sigma^\lambda} \) is the reachable subspace of \( \Sigma^\lambda \).

(iii) \( U^o_{\Sigma^\lambda} = U_{\Sigma^\lambda} \) for all \( \lambda \in \Omega^o \), where \( U^o_{\Sigma^\lambda} \) is the \( \Omega \)-unobservable subspace of \( \Sigma \) and \( U_{\Sigma^\lambda} \) is the unobservable subspace of \( \Sigma^\lambda \).

(iv) The following conditions are equivalent:

(a) \( \Sigma \) is \( \Omega \)-controllable or \( \Omega \)-observable;

(b) \( \Sigma^\lambda \) is controllable respectively observable for all \( \lambda \in \Omega^o \);

(c) \( \Sigma^\lambda_0 \) is controllable respectively observable for some \( \lambda_0 \in \Omega^o \).

(v) The following conditions are equivalent:

(a) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent;

(b) \( \Sigma_1^\lambda \) and \( \Sigma_2^\lambda \) are externally equivalent for all \( \lambda \in \Omega^o \);

(c) \( \Sigma_1^\lambda_0 \) and \( \Sigma_2^\lambda_0 \) are externally equivalent for some \( \lambda_0 \in \Omega^o \).

(vi) For each closed \( P \in \mathcal{M}(X_1; X_2) \) the following conditions are equivalent:

(a) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \);

(b) \( \Sigma_1^\lambda \) and \( \Sigma_2^\lambda \) are intertwined by \( P \) for all \( \lambda \in \Omega^o \);

(c) \( \Sigma_1^\lambda_0 \) and \( \Sigma_2^\lambda_0 \) are intertwined by \( P \) for some \( \lambda_0 \in \Omega^o \).

(vii) The following conditions are equivalent:

(a) \( \Sigma_1 \) is the \( \Omega \)-restriction to \( X_1 \), or the \( \Omega \)-projection or \( \Omega \)-compression onto \( X_1 \) along \( Z_1 \) of \( \Sigma_2 \);
(b) $\Sigma^\lambda_1$ is the restriction to $X_1$, or the projection or compression onto $X_1$ along $Z_1$ of $\Sigma^\lambda_2$, respectively, for all $\lambda \in \Omega^\circ$;
(c) $\Sigma^\lambda_0$ is the restriction to $X_1$, or the projection or compression onto $X_1$ along $Z_1$ of $\Sigma^\lambda_2$, respectively, for some $\lambda_0 \in \Omega^\circ$.

(viii) For each $\lambda \in \Omega^\circ$ the subspaces $X^\Omega_{\text{min}}$, $Z^\Omega_{\text{min}}$, and $Z^\Omega_{\text{max}}$ in Lemma 6.1.46 coincide with the corresponding subspaces $X_{\text{min}}$, $Z_{\text{min}}$, and $Z_{\text{max}}$ in Lemma 3.2.23 applied to $\Sigma^\lambda$.

(ix) For each $\lambda \in \Omega^\circ$ the $\Omega$-compressions of $\Sigma$ described in Theorem 6.1.48 have the same structure as the compressions of $\Sigma^\lambda$ described in Theorem 3.2.22.

(x) The following conditions are equivalent:
(a) $\Sigma$ is $\Omega$-minimal;
(b) $\Sigma^\lambda$ is minimal for all $\lambda \in \Omega^\circ$;
(c) $\Sigma^\lambda_0$ is minimal for some $\lambda_0 \in \Omega^\circ$.

(xi) For each $\lambda \in \Omega^\circ$ there is a one-to-one correspondence between the families of minimal $\Omega$-compressions of $\Sigma$ described in Theorem 6.2.8 and the families of minimal compressions of $\Sigma^\lambda$ described in Theorem 3.2.31.

(xii) The following conditions are equivalent:
(a) The minimal $\Omega$-compression of $\Sigma$ is unique;
(b) The minimal compression of $\Sigma^\lambda$ is unique for all $\lambda \in \Omega^\circ$;
(c) The minimal compression of $\Sigma^\lambda_0$ is unique for some $\lambda_0 \in \Omega^\circ$.

**Proof.** The proof of this lemma is straightforward. Below we only list the appropriate results that the proofs are based on, and leave the details to the readers.

(See also Lemma 6.2.12.)

(i) See Lemmas 3.2.2, 3.2.3, and 6.1.53.
(ii)–(iii) See Lemmas 3.2.4 and 6.1.54.
(iv) This follows from (ii) and (iii).
(v) See Theorem 3.2.7 and Lemma 6.1.52.
(vi) See Lemmas 3.2.16 and 6.1.55.
(vii) See Theorems 3.2.11 and 3.2.12 and Theorem 6.1.57 and Lemma 6.1.59.
(viii)–(xii) These claims follow from (i)–(vii). □
6.3. Frequency domain compressions of internally well-posed i/s/o systems

There is a special class of regular resolvable i/s/o systems for which we can strengthen some of the conclusions that have been presented above, namely the class of internally well-posed i/s/o systems.

6.3.1. Definition. By an \textit{internally well-posed i/s/o systems} we mean a regular resolvable i/s/o system $\Sigma$ whose main operator is the generator of a $C_0$ semigroup. This semigroup is called the \textit{evolution semigroup} of $\Sigma$.

6.3.2. Notation. If $\Sigma = (S; X, U, Y)$ is an internally well-posed i/s/o system, then we denote the component of $\rho(\Sigma)$ which contains some right half-plane by $\rho_+^\infty(\Sigma)$ (this set is equal to $\rho_+^\infty(A)$, where $A$ is the main operator of $\Sigma$.)

Of course, all the results presented in Sections 6.1 and 6.2 remain valid if all the i/s/o systems $\Sigma$ and $\Sigma_j$ are internally well-posed systems and we take $\Omega = \rho_+^\infty(\Sigma)$ and $\Omega_j = \rho_+^\infty(\Sigma_j)$. In the internally well-posed case we can strengthen some of the earlier conclusions. The main observation is the following.

6.3.3. Lemma. Let $\Sigma = (S; X, U, Y)$ be an internally well-posed i/s/o system with evolution semigroup $\mathfrak{A}$, and let $X = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{W}$.

(i) If $\Sigma_1 = (S_1; X_1, U, Y)$ is a $\rho_+^\infty(\Sigma)$-restriction of $\Sigma$ to $X_1$, then $\Sigma_1$ is internally well-posed, and the evolution semigroup of $\Sigma_1$ is the restriction of $\mathfrak{A}$ to $X_1$.

(ii) If $\Sigma_1 = (S_1; X_1, U, Y)$ is a $\rho_+^\infty(\Sigma)$-projection of $\Sigma$ onto $X_1$ along $Z_1$, then $\Sigma_1$ is internally well-posed, and the evolution semigroup of $\Sigma_1$ is the projection of $\mathfrak{A}$ onto $X_1$ along $Z_1$.

(iii) If $\Sigma_1 = (S_1; X_1, U, Y)$ is a $\rho_+^\infty(\Sigma)$-compression of $\Sigma$ onto $X_1$ along $Z_1$, then $\Sigma_1$ is internally well-posed, and the evolution semigroup of $\Sigma_1$ is the compression of $\mathfrak{A}$ onto $X_1$ along $Z_1$.

Proof. Proof of (i): Denote the s/s resolvent of $\Sigma$ by $\mathfrak{A}$. If $\Sigma$ has a $\rho_+^\infty(\Sigma)$-restriction to $X_1$, then it follows from Lemma 6.1.14 that $\mathfrak{A}(\lambda)|X_1 \subset X_1$ for all $\lambda \in \rho_+^\infty(\Sigma)$. By Theorem 4.1.26 $\mathfrak{A}$ has a $C_0$ semigroup restriction $\mathfrak{A}_1$ to $X_1$. Denote the generator of $\mathfrak{A}_1$ by $A_1$. Then by Theorem (4.1.26) $\mathfrak{A}(\lambda)|X_1 \subset X_1$, $\lambda \in \rho_+^\infty(A)$. This implies that $A_1$ is the main operator of the i/s/o system $\Sigma_1$ in (i). Consequently, $\Sigma_1$ is internally well-posed, and the evolution semigroup $\mathfrak{A}_1$ of $\Sigma_1$ is the restriction of $\mathfrak{A}$ to $X_1$.

Proofs of (ii) and (iii): The proofs of (ii) and (iii) are analogous to the proof of (i) given above. By combining Lemma 6.3.3 with the results presented in Sections 6.1 and 6.2 we get a number of additional results for internally well-posed i/s/o systems.

6.3.4. Theorem. Let $\Sigma = (S; X, U, Y)$ be an internally well-posed i/s/o system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X_2$.

(i) If $X_1$ is strongly $\rho_+^\infty(\Sigma)$-invariant for $\Sigma$, then $\Sigma$ has a unique internally well-posed $\rho_+^\infty(\Sigma)$-restriction to $X_1$.

(ii) If $Z_1$ is unobservably $\rho_+^\infty(\Sigma)$-invariant for $\Sigma$, then $\Sigma$ has a unique internally well-posed $\rho_+^\infty(\Sigma)$-projection onto $X_1$ along $Z_1$. 


6.3.5. **Theorem.** An internally well-posed i/s/o system $\Sigma$ is $\rho_{+\infty}(\Sigma)$-minimal if and only if it is both $\rho_{+\infty}(\Sigma)$-controllable and $\rho_{+\infty}(\Sigma)$-observable.

**Proof.** This follows from Theorem 6.2.7 and Lemma 6.3.3.

6.3.6. **Theorem.** Let $\Sigma = (S; X, U, Y)$ be an internally well-posed i/s/o system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. Denote $\Omega = \rho_{+\infty}(\Sigma)$, let $X_{\Omega_{\min}}$ be the minimal strongly $\Omega$-invariant subspace of $\Sigma$ which contains $X_1$, let $Z_{\Omega_{\max}}$ be the maximal unobservably $\Omega$-invariant subspace of $\Sigma$ which is contained in $Z_1$, and let $Z_{\Omega_{\min}} = X_{\Omega_{\min}} \cap Z_1$ (cf. Lemma 6.1.46). Then the following conditions are equivalent:

(i) $\Sigma$ has a (unique) internally well-posed $\Omega$-compression $\Sigma_1$ onto $X_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z_{\Omega}$ such that $Z_{\Omega}$ is unobservably $\Omega$-invariant for $\Sigma$ and $X + Z_{\Omega}$ is strongly $\Omega$-invariant for $\Sigma$.

(iii) $Z_{\Omega_{\min}}$ is unobservably $\Omega$-invariant for $\Sigma$.

(iv) $X + Z_{\Omega_{\max}}$ is strongly $\Omega$-invariant for $\Sigma$.

(v) $Z_{\Omega_{\min}} \subset Z_{\Omega_{\max}}$.

Two possible choices of the subspace $Z_{\Omega}$ in (ii) are to take either $Z_{\Omega} = Z_{\Omega_{\min}}$ or $Z_{\Omega} = Z_{\Omega_{\max}}$, and every possible subspace $Z_{\Omega}$ satisfies $Z_{\Omega_{\min}} \subset Z_{\Omega} \subset Z_{\Omega_{\max}}$.

**Proof.** This follows from Theorem 6.1.48 and Lemma 6.3.3.

6.3.7. **Theorem.** Let $\Sigma = (S; X, U, Y)$ be an internally well-posed i/s/o system, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. Denote $\Omega = \rho_{+\infty}(\Sigma)$, and suppose that $\Sigma_1 = (S_1; X_1, U, Y)$ is an internally well-posed $\Omega$-compression of $\Sigma$ onto $X_1$ along $Z_1$. Let $Z_{\Omega}$ satisfy the conditions listed in (ii) in Theorem 6.3.6, and let $Z_c$ be an arbitrary direct complement to $Z_{\Omega}$ in $Z_1$.

(i) Let $\Sigma_2$ be the internally well-posed $\Omega$-restriction of $\Sigma$ to the strongly $\Omega$-invariant subspace $X_1 + Z_{\Omega}$ for $\Sigma$ given by Theorem 6.3.4(i). Then $Z_{\Omega}$ is unobservably $\Omega$-invariant for $\Sigma_2$, and $\Sigma_1$ is the $\Omega$-projection onto $X_1$ along $Z_{\Omega}$ of $\Sigma_2$.

(ii) Let $\Sigma_3$ be the internally well-posed $\Omega$-projection of $\Sigma$ onto $X_1 + Z_c$ along $Z_{\Omega}$ given by Theorem 6.3.4(ii). Then $X_1$ is strongly $\Omega$-invariant for $\Sigma_3$, and $\Sigma_1$ is the $\Omega$-restriction to $X_1$ of $\Sigma_3$.

**Proof.** This follows from Theorem 6.1.50 and Lemma 6.3.3.

6.3.8. **Theorem.** Every internally well-posed i/s/o system $\Sigma$ has an internally well-posed $\rho_{+\infty}(\Sigma)$-minimal $\rho_{+\infty}(\Sigma)$-compression.

**Proof.** This follows from Theorem 6.2.8 and Lemma 6.3.3.
6.4. Ω-Intertwined and Ω-Compressed Multi-Valued Operators (Feb 02, 2016)

If we specialize the results that we have proved in this chapter to the case where \( \mathcal{U} = \mathcal{Y} = \{ \emptyset \} \), then we get results about intertwinements, compressions, and dilations of resolvents. These results can also be interpreted as frequency domain intertwinements, compressions, and dilations of the corresponding multi-valued operators induced by these resolvents. Since this is an important issue by itself we below list the conclusions that we have obtained so far for this special case. The proofs are omitted, since all the results that we list are special cases of the results that we have proved earlier in this chapter.

Throughout this section \( A \) and \( A_i \) are multi-valued operators in the \( H \)-spaces \( X \) and \( X_i \), and \( Z \) and \( Z_i \) are subspaces of \( X \) or \( X_i \), \( i = 1, 2, 3 \).

6.4.1. Frequency domain invariance.

6.4.1. Definition (cf. Definitions 5.2.8 and 6.1.6). Let \( A \in \mathcal{ML}(X) \), and let \( \Omega \) be an open set in \( \mathbb{C} \).

(i) We say that \( A \) is resolvable if \( \rho(A) \neq \emptyset \).

(ii) We say that \( A \) is \( \Omega \)-resolvable if \( A \) is resolvable and \( \Omega \subset \rho(A) \).

6.4.2. Definition (cf. Lemma 6.1.16). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( A \in \mathcal{ML}(X) \) be \( \Omega \)-resolvable. A subspace \( Z \) of \( X \) is an \( \Omega \)-invariant subspace for \( A \) if \((\lambda - A)^{-1}Z \subset Z \) for all \( \lambda \in \Omega \).

6.4.3. Lemma (cf. Lemma 6.1.17). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( A \in \mathcal{ML}(X) \) be \( \Omega \)-resolvable.

(i) If the subspace \( Z \) of \( X \) is \( \Omega \)-invariant for \( A \), then the closure of \( Z \) is also \( \Omega \)-invariant for \( A \).

(ii) If both \( Z_1 \) and \( Z_2 \) are \( \Omega \)-invariant for \( A \), then \( Z_1 + Z_1, Z_1 \setminus Z_2, \) and \( Z_1 \cap Z_2 \) are \( \Omega \)-invariant for \( A \).

6.4.2. Frequency domain intertwinements.

6.4.4. Definition (cf. Lemma 6.1.23). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( A_i \in \mathcal{ML}(X_i), i = 1, 2 \). We say that \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by \( P \in \mathcal{ML}(X_1; X_2) \) if \((\lambda - A_2)^{-1}x_2 \in P(\lambda - A_1)^{-1}x_1 \) for all \( x_2 \in Px_1 \) and all \( \lambda \in \Omega \).

Above we have used the convention that the conditions \( x_2 \in Px_1 \) and \((\lambda - A_2)^{-1}x_2 \in P(\lambda - A_1)^{-1}x_1 \) imply that \( x_1 \in \text{dom}(P) \) and \((\lambda - A_1)^{-1}x_1 \in \text{dom}(P) \).

6.4.5. Lemma (cf. Lemmas 6.1.21 and 6.1.24). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A_i \in \mathcal{ML}(X_i), i = 1, 2, \) and suppose that \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by some \( P \in \mathcal{ML}(X_1; X_2) \). Then the following claims are true:

(i) Both \( \text{dom}(P) \) and \( \ker(P) \) are \( \Omega \)-invariant subspaces for \( A_1 \).

(ii) Both \( \text{rng}(P) \) and \( \text{mul}(P) \) are \( \Omega \)-invariant subspaces for \( A_2 \).

(iii) \( A_1 \) and \( A_2 \) are also \( \Omega \)-intertwined by the closure of \( P \).

6.4.6. Lemma (cf. Lemma 6.1.25). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( A_i \in \mathcal{ML}(X_i), i = 1, 2, 3 \). If \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by the \( P_1 \in \mathcal{ML}(X_1; X_2) \) and \( A_2 \) and \( A_3 \) are \( \Omega \)-intertwined by \( P_2 \in \mathcal{ML}(X_2; X_3) \), then \( A_1 \) and \( A_3 \) are intertwined by \( P_3 := P_2P_1 \in \mathcal{ML}(X_1; X_3) \), and hence also by the closure of \( P_3 \).
6.4.7. Lemma (cf. Lemma 6.1.26). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A_i \in \mathcal{MC}(X_i) \), \( i = 1, 2 \), and let \( P \in \mathcal{MC}(X_1; X_2) \). Then \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by \( P \) if and only if \( \text{gph}(P) \) is an \( \Omega \)-invariant subspace for the cross product \( A_2 \times A_1 \) of \( A_2 \) and \( A_1 \).

6.4.3. Frequency domain compressions, restrictions, and projections.

6.4.8. Definition (cf. Lemma 6.1.41). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A_i \in \mathcal{MC}(X_i) \), \( i = 1, 2 \) be \( \Omega \)-resolvable, and suppose that \( X_1 \) is a closed subspace of \( X_2 \) with a direct complement \( Z_1 \) in \( X_2 \).

(i) \( A_1 \) is the \( \Omega \)-restriction of \( A_2 \) to \( X_1 \) if \( (\lambda - A_1)^{-1} = (\lambda - A_2)^{-1}|_{X_1} \) for all \( \lambda \in \Omega \). In this case we also call \( A_2 \) an extension of \( A_1 \).

(ii) \( A_1 \) is the \( \Omega \)-projection of \( A_2 \) onto \( X_1 \) along \( Z_1 \) if \( (\lambda - A_1)^{-1} P_{X_1}^{Z_1} (\lambda - A_2)^{-1} = P_{X_1}^{Z_1} (\lambda - A_2)^{-1}|_{X_1} \) for all \( \lambda \in \Omega \). In this case we also call \( A_2 \) an \( \Omega \)-dilation of \( A_1 \) along \( Z_1 \).

6.4.9. Lemma (cf. Lemma 6.1.34). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( A_i \in \mathcal{MC}(X_i) \), \( i = 1, 2, 3 \). If \( A_2 \) is the \( \Omega \)-compression of \( A_3 \) onto \( X_2 \) along \( Z_2 \), and if \( A_1 \) is the \( \Omega \)-compression of \( A_2 \) onto \( X_1 \) along \( Z_1 \), then \( A_1 \) is the \( \Omega \)-compression of \( A_3 \) onto \( X_1 \) along \( Z_1 + Z_2 \).

6.4.10. Lemma (cf. Lemmas 6.1.37 and 6.1.39). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A_i \in \mathcal{MC}(X_i) \), \( i = 1, 2 \), and suppose that \( X_1 \) is a closed subspace of \( X_2 \) with a direct complement \( Z_1 \) in \( X_2 \).

(i) The following conditions are equivalent:

(a) \( A_1 \) is the \( \Omega \)-restriction of \( A_2 \) to \( X_1 \).

(b) \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by the embedding operator \( X_1 \hookrightarrow X_2 \).

(c) \( X_1 \) is an \( \Omega \)-invariant subspace for \( A_2 \), and \( A_1 \) is the \( \Omega \)-compression of \( A_2 \) onto \( X_1 \) along \( Z_1 \).

(ii) Also the following conditions are equivalent:

(a) \( A_1 \) is the \( \Omega \)-projection of \( A_2 \) onto \( X_1 \) along \( Z_1 \).

(b) \( A_2 \) and \( A_1 \) are \( \Omega \)-intertwined by the projection operator \( P_{X_1}^{Z_1} \).

(c) \( Z_1 \) is an \( \Omega \)-invariant subspace for \( A_2 \), and \( A_1 \) is the \( \Omega \)-compression of \( A_2 \) onto \( X_1 \) along \( Z_1 \).

6.4.11. Theorem (cf. Theorem 6.1.44). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A \in \mathcal{MC}(X) \) be \( \Omega \)-resolvable, and let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \).

(i) \( X_1 \) is \( \Omega \)-invariant for \( A \) if and only if \( A \) has an \( \Omega \)-resolvable \( \Omega \)-restriction to \( X_1 \).

(ii) \( Z_1 \) is \( \Omega \)-invariant for \( A \) if \( A \) has an \( \Omega \)-resolvable \( \Omega \)-projection onto \( X_1 \) along \( Z_1 \).
6.4.12. **Lemma (cf. Lemma 6.1.46).** Let $\Omega$ be an open set in $\mathbb{C}$, let $A \in \mathcal{ML}(\mathcal{X})$ be $\Omega$-resolvable, and let $\mathcal{X} = \mathcal{X}_1 + Z_1$. Define

\begin{align}
Z_{\text{min}}^\Omega &:= \bigvee_{\lambda \in \Omega} \text{rng} \left( P_{Z_1}^\mathcal{X}_1 (\lambda - A)^{-1} |_{\mathcal{X}_1} \right), \\
Z_{\text{max}}^\Omega &:= \bigwedge_{\lambda \in \Omega} \ker \left( P_{X_1}^\mathcal{X}_1 (\lambda - A)^{-1} |_{Z_1} \right).
\end{align}

Then $\mathcal{X}_1 + Z_{\text{min}}^\Omega$ is the minimal closed $\Omega$-invariant subspace for $A$ which contains $\mathcal{X}_1$, and $Z_{\text{max}}^\Omega$ is the maximal $\Omega$-invariant subspace for $A$ which is contained in $Z_1$.

6.4.13. **Theorem (cf. Theorem 6.1.48).** Let $\Omega$ be an open set in $\mathbb{C}$, let $A \in \mathcal{ML}(\mathcal{X})$ be $\Omega$-resolvable, and let $\mathcal{X} = \mathcal{X}_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Then the following conditions are equivalent:

(i) $A$ has a (unique) $\Omega$-resolvable $\Omega$-compression onto $\mathcal{X}_1$ along $Z_1$.
(ii) $Z_1$ contains some closed subspace $Z^\Omega$ such that both $Z^\Omega$ and $\mathcal{X}_1 + Z^\Omega$ are $\Omega$-invariant for $A$.
(iii) The subspace $Z_{\text{min}}^\Omega$ defined in (6.4.1) is $\Omega$-invariant for $A$.
(iv) $\mathcal{X}_1 + Z_{\text{max}}^\Omega$ is $\Omega$-invariant for $A$, where $Z_{\text{max}}^\Omega$ is defined as in (6.4.2).
(v) $Z_{\text{min}}^\Omega \subset Z_{\text{max}}^\Omega$.

Two possible choices of the subspace $Z^\Omega$ in (ii) are $Z^\Omega = Z_{\text{min}}^\Omega$ and $Z^\Omega = Z_{\text{max}}^\Omega$, and every possible subspace $Z^\Omega$ satisfies $Z_{\text{min}}^\Omega \subset Z^\Omega \subset Z_{\text{max}}^\Omega$.

6.4.14. **Corollary (cf. Corollary 6.1.49).** Let $\Omega$ be an open set in $\mathbb{C}$, let $A \in \mathcal{ML}(\mathcal{X})$ be $\Omega$-resolvable, and let $\mathcal{X} = \mathcal{X}_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$. Then $A$ has an $\Omega$-resolvable $\Omega$-compression $A_1$ onto $\mathcal{X}_1$ along $Z_1$ if and only if $Z_1$ has a direct sum decomposition $Z_1 = Z^\Omega + Z_c$ such that the resolvent of $A$ has the following structure with respect to the decomposition $\mathcal{X} = Z^\Omega + \mathcal{X}_1 + Z_c$ of $\mathcal{X}$ (where irrelevant entries have been denoted by $*$):

\begin{equation}
(\lambda - A)^{-1} = \begin{bmatrix}
(\lambda - A_{Z^\Omega})^{-1} & * & * \\
0 & (\lambda - A_1)^{-1} & * \\
0 & 0 & (\lambda - A_{Z_c})^{-1}
\end{bmatrix}, \quad \lambda \in \Omega,
\end{equation}

Here both $Z^\Omega$ and $\mathcal{X}_1 + Z_c$ are $\Omega$-invariant for $A$, $A_{Z^\Omega}$ is the part of $A$ in $Z^\Omega$, and $A_{Z_c}$ is the projection of $A$ onto $Z_c$ along $\mathcal{X}_1 + Z^\Omega$. The subspace $Z^\Omega$ in this decomposition can be chosen to be the same as the subspace $Z^\Omega$ in condition (ii) in Theorem 6.4.13, and the subspace $Z_c$ can be chosen to be an arbitrary direct complement to $Z^\Omega$ in $Z_1$. In particular, two possible choices of $Z^\Omega$ are $Z^\Omega = Z_{\text{min}}^\Omega$ and $Z^\Omega = Z_{\text{max}}^\Omega$, where $Z_{\text{min}}^\Omega$ and $Z_{\text{max}}^\Omega$ are defined by (6.4.1) and (6.4.2).

6.4.15. **Theorem (cf. Theorem 6.1.50).** Let $\Omega$ be an open set in $\mathbb{C}$, let $A_i \in \mathcal{ML}(\mathcal{X}_i), i = 1, 2$, with $\mathcal{X}_2 = \mathcal{X}_1 + Z_1$, and suppose that $A_1$ is the $\Omega$-compression of $A_2$ onto $\mathcal{X}_1$ along $Z_1$. Let $Z^\Omega$ satisfy the conditions listed in (ii) in Theorem 6.4.13 and let $Z_c$ be an arbitrary direct complement to $Z^\Omega$ in $Z_1$.

(i) Let $A_3$ be the $\Omega$-resolvable $\Omega$-restriction of $A_2$ to the $\Omega$-invariant subspace $\mathcal{X}_1 + Z^\Omega$ of $\mathcal{X}_2$ given by Theorem 6.4.11(i). Then $Z^\Omega$ is $\Omega$-invariant for $A_3$, and $A_1$ is the $\Omega$-projection onto $\mathcal{X}_1$ along $Z^\Omega$ of $A_3$.
(ii) Let $A_4$ be the $\Omega$-resolvable $\Omega$-projection of $A_2$ onto $\mathcal{X}_1 + Z_c$ along $Z_c$ of $\mathcal{X}_2$ given by Theorem 6.4.11(ii). Then $X_1$ is $\Omega$-invariant for $A_4$, and $A_1$ is the $\Omega$-restriction to $X_1$ of $A_4$. 

\[ \text{\ } \]
6.4.16. Theorem (cf. Theorem 6.1.5, 1). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( A_i \in \mathcal{L}(\mathcal{X}_i) \), \( i = 1, 2 \), with \( \mathcal{X}_2 = \mathcal{X}_1 + \mathcal{Z}_1 \). Then the following two conditions are equivalent:

(i) \( A_1 \) is the \( \Omega \)-compression of \( A_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(ii) \( \mathcal{Z}_1 \) contains some closed subspace \( \mathcal{Z}^\Omega \) such that \( A_2 \) and \( A_1 \) are \( \Omega \)-intertwined by the operator \( P = P_{\mathcal{X}_1}^{\mathcal{Z}_1} \) with \( \text{dom}(P) = \mathcal{X}_1 + \mathcal{Z}_1 \).

Condition (ii) above holds for some particular subspace \( \mathcal{Z}^\Omega \) if and only condition (ii) in Theorem 6.4.13 holds for the same subspace \( \mathcal{Z}^\Omega \). Thus, in particular, two possible choices of the subspace \( \mathcal{Z}^\Omega \) in (ii) are \( \mathcal{Z}_1 = \mathcal{Z}_1^{\text{min}} \) and \( \mathcal{Z}_1 = \mathcal{Z}_1^{\text{max}} \) defined in (6.4.1) and (6.4.1), and every possible subspace \( \mathcal{Z}^\Omega \) satisfies \( \mathcal{Z}_1^{\text{min}} \subset \mathcal{Z}_1^\Omega \subset \mathcal{Z}_1^{\text{max}} \).

6.4.4. Results for connected frequency domains.

6.4.17. Lemma (cf. Lemma 6.1.53). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), let \( A \in \mathcal{L}(\mathcal{X}) \) be \( \Omega^o \)-resolvable, let \( \Omega \) be an open subset of \( \Omega^o \), and let \( \mathcal{Z} \) be a closed subspace of \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( \mathcal{Z} \) is an \( \Omega \)-invariant subspace for \( A \);

(ii) \( \mathcal{Z} \) is an \( \Omega^o \)-invariant subspace for \( A \);

(iii) \( (\lambda_0 - A)^{-1} \mathcal{Z} \subset \mathcal{Z} \) for some \( \lambda_0 \in \Omega^o \);

Thus, if \( \mathcal{Z} \) is an \( \Omega \)-invariant subspace for \( A \) for some open subset \( \Omega \) of \( \Omega^o \), then \( \mathcal{Z} \) is an \( \Omega \)-invariant subspace for \( A \) for every open subset \( \Omega \) of \( \Omega^o \).

6.4.18. Lemma (cf. Lemma 6.1.55). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), let \( A_i \in \mathcal{B}(\mathcal{X}_i) \) be \( \Omega^o \)-resolvable, \( i = 1, 2 \), let \( \Omega \) be an open subset of \( \Omega^o \), and let \( P \in \mathcal{L}(\mathcal{X}_1; \mathcal{X}_2) \). Then the following conditions are equivalent:

(i) \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by \( P \);

(ii) \( A_1 \) and \( A_2 \) are \( \Omega^o \)-intertwined by \( P \);

(iii) \( (\lambda_0 - A_2)^{-1} x_2 \in P(\lambda_0 - A_1)^{-1} x_1 \) for all \( x_2 \in P x_1 \) and some \( \lambda_0 \in \Omega^o \);

Thus, if \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by \( P \) for some open subset \( \Omega \) of \( \Omega^o \), then \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by \( P \) for every open subset \( \Omega \) of \( \Omega^o \).

Above we have used the convention that the conditions \( x_2 \in P x_1 \) and \( (\lambda - A_2)^{-1} x_2 \in P(\lambda - A_1)^{-1} x_1 \) imply that \( x_1 \in \text{dom}(P) \) and \( (\lambda - A_1)^{-1} x_1 \in \text{dom}(P) \).

6.4.19. Lemma (cf. Lemmas 6.1.57 and 6.1.59). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), let \( A_i \in \mathcal{L}(\mathcal{X}_i) \) be \( \Omega^o \)-resolvable, \( i = 1, 2 \), and suppose that \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \) with a direct complement \( \mathcal{Z}_1 \) in \( \mathcal{X}_2 \). Let \( \Omega \) be an open subset of \( \Omega^o \) and let \( \Omega^\circ \) be an arbitrary subset of \( \Omega^o \) which has a cluster point in \( \Omega^o \).

(i) The following conditions are equivalent:

(a) \( A_1 \) is the \( \Omega \)-restriction of \( A_2 \) to \( \mathcal{X}_1 \);

(b) \( A_1 \) is the \( \Omega^o \)-restriction of \( A_2 \) to \( \mathcal{X}_1 \);

(c) \( (\lambda_0 - A_1)^{-1} = (\lambda_0 - A_2)^{-1} |_{\mathcal{X}_1} \) for some \( \lambda_0 \in \Omega^o \).

Thus, if \( A_1 \) is the \( \Omega \)-restriction of \( A_2 \) to \( \mathcal{X}_1 \) for some open subset \( \Omega \) of \( \Omega^o \), then \( A_1 \) is the \( \Omega \)-restriction of \( A_2 \) to \( \mathcal{X}_1 \) for all open subset \( \Omega \) of \( \Omega^o \).

(ii) The following conditions are equivalent:

(a) \( A_1 \) is the \( \Omega \)-projection of \( A_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \);

(b) \( A_1 \) is the \( \Omega^o \)-projection of \( A_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \);

(c) \( (\lambda_0 - A_1)^{-1} P_{\mathcal{X}_1}^{\mathcal{Z}_1} = P_{\mathcal{X}_1}^{\mathcal{Z}_1} (\lambda_0 - A_2)^{-1} \) for some \( \lambda_0 \in \Omega^o \).

Thus, if \( A_1 \) is the \( \Omega \)-projection of \( A_2 \) to \( \mathcal{X}_1 \) for some open subset \( \Omega \) of \( \Omega^o \), then \( A_1 \) is the \( \Omega \)-projection of \( A_2 \) to \( \mathcal{X}_1 \) for all open subset \( \Omega \) of \( \Omega^o \).
Thus, if \( A_1 \) is the \( \Omega \)-compression of \( A_2 \) to \( X_1 \) for some open subset \( \Omega \) of \( \Omega^o \), then \( A_1 \) is the \( \Omega \)-compression of \( A_2 \) to \( X_1 \) for all open subset \( \Omega \) of \( \Omega^o \).

6.4.20. Lemma (cf. Lemma 6.1.60). Let \( \Omega^o \) be an open connected set in \( C \), let \( A \in \mathcal{M}(X) \) be \( \Omega^o \)-resolvable, and let \( X = X_1 + Z_1 \). Let \( \Omega^e \) be an arbitrary subset of \( \Omega^o \) which has a cluster point in \( \Omega^o \), let \( \lambda_0 \in \Omega^o \), and let \( \Omega \) be an open subset of \( \Omega^o \). Then the subspaces \( Z_{\Omega}^\Omega \) and \( Z_{\Omega}^\Omega \) defined in (6.4.1) and (6.4.2) are given by

\[
Z_{\Omega}^\Omega_{\min} = \bigvee_{\lambda \in \Omega} \text{rg} \left( P_{X_1}^Z (\lambda - A)^{-1}|_{X_1} \right)
\]
\[
= \bigvee_{\lambda \in \Omega^o} \text{rg} \left( P_{X_1}^Z (\lambda - A)^{-1}|_{X_1} \right)
\]
\[
= \bigvee_{\lambda \in \Omega^e} \text{rg} \left( P_{X_1}^Z (\lambda - A)^{-1}|_{X_1} \right)
\]
\[
(6.4.4)
\]
\[
Z_{\Omega}^\Omega_{\max} = \bigcap_{\lambda \in \Omega} \ker \left( P_{X_1}^Z (\lambda - A)^{-1}|_{Z_1} \right)
\]
\[
= \bigcap_{\lambda \in \Omega^o} \ker \left( P_{X_1}^Z (\lambda - A)^{-1}|_{Z_1} \right)
\]
\[
= \bigcap_{\lambda \in \Omega^e} \ker \left( P_{X_1}^Z (\lambda - A)^{-1}|_{Z_1} \right)
\]
\[
(6.4.5)
\]

Thus, \( Z_{\Omega}^\Omega_{\min} \) and \( Z_{\Omega}^\Omega_{\max} \) do not depend on the choice of \( \Omega \), as long as \( \Omega \) is an open subset of \( \Omega^o \).

6.4.5. Restrictions and projections of multi-valued operators.

6.4.21. Definition. Let \( A \in \mathcal{M}(X) \), and let \( Z \) be a closed subspace of \( X \). By the part of \( A \) in \( Z \) we mean the multi-valued operator \( A_{\text{part}} \) whose graph is given by

\[
gph(A_{\text{part}}) = gph(A) \cap \left[ \frac{Z}{Z} \right] = \left\{ \begin{bmatrix} z \\ x \end{bmatrix} \in \left[ \frac{Z}{Z} \right] : z \in Ax \right\}.
\]

6.4.22. Theorem (cf. Theorem 6.2.1). Let \( \Omega \) be an open set in \( C \), let \( A \in \mathcal{M}(X) \) be \( \Omega \)-resolvable, and let \( X_1 \) be a closed subspace of \( X \). Then the following conditions are equivalent:

(i) \( X_1 \) is \( \Omega \)-invariant for \( A \);

(ii) \( A \) has a (unique) \( \Omega \)-resolvable \( \Omega \)-restriction to \( X_1 \);
(iii) the part $A_{\text{part}}$ of $A$ in $X_1$ is $\Omega$-resolvable (cf. Definition 6.4.21).

If these equivalent conditions hold, then $A_{\text{part}}$ is the unique $\Omega$-resolvable $\Omega$-restriction of $A$ to $X_1$, and $A_{\text{part}}$ is single-valued whenever $A$ is single-valued.

The three equivalent conditions (i)–(iii) above are furthermore equivalent to each of the following conditions, where $\hat{X}$ is the resolvent of $A$:

(iv) $\text{gph} (\lambda - A) \cap \bigl[ \frac{X_1}{\hat{X}} \bigr] \subset \bigl[ \frac{X_1}{\hat{X}} \bigr]$ for all $\lambda \in \Omega$.

(v) For all $\lambda \in \Omega$, $\text{gph}(\lambda - A_{\text{part}})$ has the two equivalent representations

\begin{align}
(6.4.7a) & \quad \text{gph} (\lambda - A_{\text{part}}) = \text{rng} \left( \left[ \frac{X_1}{\hat{X}} \right] \right) \\
(6.4.7b) & \quad \text{gph} (\lambda - A_{\text{part}}) = \ker \left( \left[ \frac{X_1}{\hat{X}} \right] \right) .
\end{align}

(vi) $\lambda - A_{\text{part}}$ has an inverse in $\mathcal{B}(X_1)$ for all $\lambda \in \Omega$.

(vii) $\lambda - A_{\text{part}}$ is surjective for all $\lambda \in \Omega$.

If, in addition, $\Omega$ is connected, then conditions (i)–(vii) above are also equivalent to the condition

(viii) $\Omega \cap \rho(A_{\text{part}}) \neq \emptyset$,

as well as to the conditions that one gets by replacing “for all $\lambda \in \Omega$” by “for some $\lambda \in \Omega$” in conditions (iv)–(vii) above.

Note that we do not claim that $\text{dom}(A_{\text{part}})$ is dense in $X_1$ whenever $\text{dom}(A)$ is dense in $X$. (This will be true under some additional assumptions; see Theorem 4.1.26)

6.4.23. Definition. Let $X$ be an $H$-spaces, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X_1$. By the projection onto $X_1$ along $Z_1$ of $A \in \mathcal{B}(X)$ we mean the multi-valued operator $(A)_{\text{proj}}$ whose graph is given by

\begin{equation}
(6.4.8) \quad \text{gph}((A)_{\text{proj}}) = \left[ \begin{array}{cc} \{WZ_1 \} & 0 \\
0 & P_{X_1} \end{array} \right] \text{gph}(A).
\end{equation}

6.4.24. Theorem (cf. Theorem 6.2.2). Let $\Omega$ be an open set in $\mathbb{C}$, let $A \in \mathcal{MC}(X)$ be $\Omega$-resolvable, and let $Z_1$ be a closed subspace of $X$ with a direct complement $X_1$. Then the following conditions are equivalent:

(i) $Z_1$ is $\Omega$-invariant for $A$.

(ii) $A$ has a (unique) $\Omega$-resolvable $\Omega$-projection onto $X_1$ along $Z_1$;

(iii) the projection $A_{\text{proj}}$ of $A$ onto $X_1$ along $Z_1$ is $\Omega$-resolvable (cf. Definition 6.4.23).

If these equivalent conditions hold, then $A_{\text{proj}}$ is the unique $\Omega$-resolvable $\Omega$-projection of $A$ onto $X_1$ along $Z_1$, and $\text{dom}(A_{\text{proj}})$ is dense in $X_1$ whenever $\text{dom}(A)$ is dense in $X$.

The three equivalent conditions (i)–(iii) above are furthermore equivalent to each of the following conditions, where $\hat{X}$ is the resolvent of $A$:

(iv) $\text{gph} (\lambda - A) \cap \bigl[ \frac{Z_1}{\hat{X}} \bigr] \subset \bigl[ \frac{Z_1}{\hat{X}} \bigr]$ for all $\lambda \in \Omega$.

(v) For all $\lambda \in \Omega$, $\text{gph}(\lambda - A_{\text{proj}})$ has the two equivalent representations

\begin{align}
(6.4.9a) & \quad \text{gph} (\lambda - A_{\text{proj}}) = \text{rng} \left( \left[ \frac{Z_1}{\hat{X}} \right] \right) \\
(6.4.9b) & \quad \text{gph} (\lambda - A_{\text{proj}}) = \ker \left( \left[ \frac{Z_1}{\hat{X}} \right] \right) .
\end{align}

(vi) For all $\lambda \in \Omega$, the operator $\lambda - A_{\text{proj}}$ has a bounded inverse.
(vii) For all \( \lambda \in \Omega \), the operator \( \lambda - A_{\text{proj}} \) is injective.

If, in addition, \( \Omega \) is connected, then conditions (i)–(vii) above are also equivalent to the condition

(viii) \( \Omega \cap \rho(A_{\text{proj}}) \neq \emptyset \),

as well as to the conditions that one gets by replacing “for all \( \lambda \in \Omega \)” by “for some \( \lambda \in \Omega \)” in conditions (iv)–(vii) above.

Note that we do not claim that \( A_{\text{proj}} \) is single-valued whenever \( A \) is single-valued. (This will be true under some additional assumptions; see Theorem 4.1.32)

6.4.25. Theorem (cf. Theorem 6.2.3). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A \in \mathcal{ML}(X) \) be \( \Omega \)-resolvable, and let \( X = X_1 + Z^\Omega + Z_c \) be a direct sum decomposition of \( X_1 \). Then the following conditions are equivalent:

(i) both \( Z^\Omega \) and \( X_1 + Z^\Omega \) are \( \Omega \)-invariant for \( A \).
(ii) If we denote the part of \( A \) in \( X_1 + Z^\Omega \) by \( A_{\text{part}} \) and the projection of \( A_{\text{part}} \) onto \( X_1 \) along \( Z^\Omega \) by \( (A_{\text{part}})_{\text{proj}} \), then \( \Omega \subset \rho(A_{\text{part}}) \cap \rho((A_{\text{part}})_{\text{proj}}) \).
(iii) If we denote the projection of \( A \) onto \( X_1 \) along \( Z_c \) by \( A_{\text{proj}} \) and the part of \( A_{\text{proj}} \) in \( X_1 \) by \( (A_{\text{proj}})_{\text{part}} \), then \( \Omega \subset \rho(A_{\text{proj}}) \cap \rho((A_{\text{proj}})_{\text{part}}) \).

If these equivalent conditions hold, then \( A \) has a unique \( \Omega \)-resolvable \( \Omega \)-compression onto \( X_1 \) along \( Z^\Omega + Z_c \), and conversely, if \( Z_1 \) is a direct complement to \( X_1 \) in \( X \) and \( A \) has an \( \Omega \)-resolvable \( \Omega \)-compression \( A_1 \) onto \( X_1 \) along \( Z_1 \), then \( Z_1 \) can be decomposed into \( Z_1 = Z^\Omega + Z_c \) in such a way that the above equivalent conditions (i)–(iii) hold. The unique \( \Omega \)-resolvable \( \Omega \)-compression \( A_1 \) of \( A \) onto \( X_1 \) along \( Z_1 \) is equal to \( A_1 = (A_{\text{part}})_{\text{proj}} = (A_{\text{proj}})_{\text{part}} \).

The three equivalent conditions (i)–(iii) above are furthermore equivalent to each of the following conditions, where \( \lambda \) is the resolvent of \( A \):

(iv) \( \text{gph}(\lambda - A) \cap \left[ \frac{X_1 + Z^\Omega}{X_1} \right] \subset \left[ \frac{X}{X_1 + Z^\Omega} \right] \) and
    \( \text{gph}(\lambda - A) \cap \left[ \frac{Z^\Omega}{X} \right] \subset \left[ \frac{X}{X_1 + Z^\Omega} \right] \) for all \( \lambda \in \Omega \).
(v) the part of \( \lambda - A \) in \( X_1 + Z^\Omega \) has an inverse in \( B(X_1 + Z^\Omega) \) and the projection of \( \lambda - A \) onto \( X_1 + Z_c \) along \( Z^\Omega \) has an inverse in \( B(X_1 + Z_c) \) for all \( \lambda \in \Omega \).
(vi) the part of \( \lambda - A \) in \( X_1 + Z^\Omega \) to \( X_1 + Z^\Omega \) is surjective and the projection of \( \lambda - A \) onto \( X_1 + Z_c \) along \( Z^\Omega \) is injective for all \( \lambda \in \Omega \).

If, in addition, \( \Omega \) is connected, then conditions (i)–(vii) above are also equivalent to the two conditions

(vii) \( \rho(A_{\text{part}}) \cap \rho((A_{\text{part}})_{\text{proj}}) \neq \emptyset \),
(viii) \( \rho(A_{\text{proj}}) \cap \rho((A_{\text{proj}})_{\text{part}}) \neq \emptyset \),

as well as to the conditions that one gets by replacing “for all \( \lambda \in \Omega \)” by “for some \( \lambda \in \Omega \)” in conditions (iv)–(vi) above.

6.4.26. Theorem (cf. Theorem 6.2.4). Let \( \Omega \) be an open set in \( \mathbb{C} \), let \( A_i \in \mathcal{ML}(X_i) \), \( i = 1, 2 \), and let \( P \in \mathcal{ML}(X_1; A_2) \). Let \( X = \left[ \frac{X_2}{X_1} \right] \), and let \( A = A_2 \times A_1 \) be the cross product of \( A_2 \) and \( A_1 \) (cf. Definition 5.2.32). Then the following conditions are equivalent:

(i) \( A_1 \) and \( A_2 \) are \( \Omega \)-intertwined by \( P \).
(ii) \( \text{gph}(P) \) is an \( \Omega \)-invariant subspace for \( A \).
(iii) \( \Omega \subset \rho(A_{\text{part}}) \), where \( A_{\text{part}} \) is the part of \( A \) in \( \text{gph}(P) \).
(iv) $\Omega \subseteq \rho(A_{\text{proj}})$, where $A_{\text{proj}}$ is the projection of $A$ along $\text{gph}(P)$ onto some direct complement of $\text{gph}(P)$ in $X$.

The four equivalent conditions (i)–(iv) above are furthermore equivalent to each of the following conditions:

(v) $\text{gph}(\lambda - A) \cap \left[ \frac{\text{gph}(P)}{X} \right] \subseteq \left[ \frac{\text{gph}(P)}{\text{gph}(P)} \right]$ for all $\lambda \in \Omega$.

(vi) For all $\lambda \in \Omega$, $\text{gph}(\lambda - A_{\text{part}})$ has the two equivalent representations

\[(6.4.10a) \quad \text{gph}(\lambda - A_{\text{part}}) = \text{rng}\left( \frac{1}{\text{gph}(P)} \left( (\lambda - A)^{-1}_{\text{gph}(P)} \right) \right)\]

\[(6.4.10b) \quad \text{gph}(\lambda - A_{\text{part}}) = \ker\left( (\lambda - A)^{-1}_{\text{gph}(P)} - 1_{\text{gph}(P)} \right)\).

(vii) $\lambda - A_{\text{part}}$ is surjective for all $\lambda \in \Omega$.

If, in addition $\Omega$ is connected, then conditions (i)–(vii) above are equivalent to the conditions

(viii) $\Omega \cap \rho(A_{\text{part}}) \neq \emptyset$,

(ix) $\Omega \cap \rho(A_{\text{proj}}) \neq \emptyset$,

as well as to the conditions that one gets by replacing “for all $\lambda \in \Omega$” by “for some $\lambda \in \Omega$” in conditions (v)–(vi) above.
6.5. Notes and Comments (Feb 02, 2016)

The class of internally well-posed i/s/o systems has been studied in, e.g., Staffans [2005] and ????? Usually these systems are studied in the time domain setting, and not in the frequency domain setting as we have done here. Note that we do not claim that the $\rho_{+\infty}(\Sigma)$-restrictions, $\rho_{+\infty}(\Sigma)$-projections, and $\rho_{+\infty}(\Sigma)$-compressions in Lemma 3.2.32 for internally well-posed i/s/o systems are equivalent to the time domain restrictions, compressions, and restrictions introduced in Definitions 2.5.28, 2.5.33, and 2.5.37. As a matter of fact, the $\rho_{+\infty}(\Sigma)$-restrictions are actually equivalent to the time domain restrictions described in Lemma 2.5.48, but it is not know to what extent the same is true about $\rho_{+\infty}(\Sigma)$-restrictions and $\rho_{+\infty}(\Sigma)$-compressions. See ??? for further details.
In Chapter 6 we studied i/s/o systems in the frequency domain, assuming that the systems had a nonempty resolvent set, and introduced frequency domain versions of the notions of, e.g., the reachable and unobservable subspaces, controllability and observability, strong and unobservable invariance, external equivalence, intertwinements, compressions, dilations, restrictions, extensions, and projections. In this chapter we present the analogous theory for s/s systems with nonempty resolvent sets. Some of the s/s results are proved directly in the s/s setting, but most of them are reduced to the corresponding i/s/o results by means of i/s/o representations of the given s/s systems.
7.1. Frequency Domain State/Signal Systems (Jan 02, 2016)

In this section we first introduce the notion of a frequency domain trajectory induced by a s/s node, and then we use this notion to define and study various frequency domain notions and properties of s/s systems.

7.1.1. Introduction to frequency domain s/s systems. The equation (1.1.1) describes the time domain evolution of a s/s system Σ. From the time domain equation (1.1.1) we get the frequency domain equation (1.6.2) by taking (formal) Laplace transforms as explained in the connection with (1.6.2).

It is possible to introduce the notion of a frequency domain trajectory induced by a s/s node by replacing (1.1.1) by (1.6.2), and at the same time replacing the time domain interval I by some open subset Ω of the complex (frequency domain) plane C, in the same way as we did for i/s/o nodes.

7.1.1. Definition (cf. Definition 6.1.1). Let Σ = (V; X, W) be an s/s node with characteristic node bundle Ė (see Definition 1.6.1).

(i) By a (frequency domain) Ω-trajectory generated by Σ, where Ω is some nonempty open subset of C, we mean a triple (̂x, ̂w; x₀) where ̂x and ̂w are analytic functions defined on Ω with values in X respectively W, and x₀ is a constant in X, which satisfy the equivalent equations

\[
\begin{bmatrix}
\lambda ̂x(\lambda) - x₀ \\
 ̂x(\lambda) \\
 ̂w(\lambda)
\end{bmatrix} \in V,
\]

(7.1.1a)

\[
\begin{bmatrix}
x₀ \\
 ̂x(\lambda) \\
 ̂w(\lambda)
\end{bmatrix} \in Ė(\lambda),
\]

(7.1.1b)

for all \( \lambda \in \Omega \). The different components of such a trajectory are called as follows: \( ̂x \) is the state component, \( ̂w \) is the signal component, and \( x₀ \) is the initial state of the trajectory \((̂x, ̂w; x₀)\).

(ii) By the s/s (input/state/output) frequency domain system induced by the s/s node Σ we mean the node Σ itself together with sets of all Ω-trajectories generated by Σ. We use the same notation Σ = (V; X, W) for the frequency domain s/s system as for the s/s node, and alternatively write “Ω-trajectories of the frequency domain s/s system Σ” instead of “Ω-trajectories generated by the s/s node Σ”.

(iii) When the i/s/o node Σ is closed, or regular, or resolvable, then we also call the frequency domain i/s/o system Σ closed, or regular, or resolvable, respectively. (See Definitions 1.1.3, 1.1.9, and 5.3.1.)

Without any further assumptions on Σ and Ω there may not exist any nonzero trajectories. However, as will be shown in Lemma 7.1.10 below, if Σ is resolvable and if Ω is a suitable subset of \( \rho(Σ) \), then there exists a rich class of Ω-trajectories of Σ (which is parameterized by the initial state of the trajectory and the input function of some i/s/o representation of Σ).

7.1.2. Remark. By the argument at the beginning Section 1.6.1 that was used to motivate the theory presented in that section, in the case where Ω is a right-half plane the set of frequency domain Ω-trajectories generated by an s/s node Σ can...
be interpreted as formal Laplace transforms of time domain future trajectories of Σ.

7.1.3. **Lemma** (cf. Lemma 6.1.4). Let Σ = (V; X, W) be a frequency domain s/s system, and let Ω₁ and Ω₂ be open sets in C.

(i) If Ω₁ ⊂ Ω₂, then the restriction of an Ω₂-trajectory of Σ to Ω₁ is an Ω₁-trajectory of Σ.

(ii) Let (x₁, w₁; x₀₁) be two Ω₁-trajectories, i = 1, 2, of Σ which coincide on Ω₁ ∩ Ω₂ (if Ω₁ ∩ Ω₂ ≠ ∅). Define (x, w; x₀) by

\[ (\hat{x}(\lambda), \hat{w}(\lambda); x₀) = (\hat{x}_i(\lambda), \hat{w}_i(\lambda); x₀_i), \quad \lambda \in \Omega, \quad i = 1, 2. \]

Then (x, w; x₀) is an Ω-trajectory of Σ, where Ω := Ω₁ ∩ Ω₂.

**Proof.** This follows immediately from Definition 7.1.1.

7.1.4. **Lemma** (cf. Lemma 6.1.5). Let Σ = (V; X, W) be a frequency domain s/s system, and let Ω be an open set in C. Then the following conditions are equivalent:

(i) the initial state of a trajectory x₀ of an Ω-trajectory (x, w; x₀) of Σ is determined uniquely by x and w.

(ii) V satisfies condition (ii) in Definition 6.1.9

In particular, these conditions are true if Σ is regular.

**Proof.** This follows immediately from Definition 7.1.1.

For a resolvable frequency domain s/s system Σ satisfying Ω ⊂ ρ(Σ) it is possible to say more about the nature of Ω-trajectories of Σ. Since this class of systems will play a central role in this chapter we introduce the following terminology.

7.1.5. **Definition** (cf. Definitions 6.1.6 and 6.1.7). Let Ω be a (nonempty) open set in C. By an Ω-resolvable s/s node or frequency domain s/s system we mean a resolvable s/s node respectively frequency domain s/s system Σ satisfying Ω ⊂ ρ(Σ).

7.1.6. **Lemma.** Let Ω be an open set in C, and let Σ = (V; X, W) be an Ω-resolvable frequency domain s/s system with state-signal/state map ˆΣ, signal/state map ˆB, state/signal map ˆC, and characteristic signal bundle ˆF (see Definitions 6.1.7 and 3.3.29). Then the following claims hold:

(i) (x, w; x₀) is an Ω-trajectory of Σ if and only if x₀ is a constant, w is an analytic W-valued function in Ω satisfying w(λ) ∈ ˆC(λ)x₀, λ ∈ Ω, and

\[ \hat{x}(\lambda) = ˆΣ(\lambda) \left[ \frac{x₀}{w(\lambda)} \right], \quad \lambda \in \Omega. \]

In particular, this implies that ˆx is analytic in Ω, and that ˆx(λ) is determined uniquely by x₀ and w(λ), λ ∈ Ω.

(ii) (x, w; 0) is an Ω-trajectory of Σ (with zero initial state) if and only if w is an analytic W-valued function in Ω satisfying w(λ) ∈ ˆF(λ), λ ∈ Ω, and

\[ \hat{x}(\lambda) = ˆB(\lambda)w(\lambda), \quad \lambda \in \Omega. \]

In particular, this implies that ˆx is analytic in Ω, and that ˆx(λ) is determined uniquely by x₀ and w(λ), λ ∈ Ω.

(iii) ˆw is the signal component of some Ω-trajectory (x, w; x₀) of Σ if and only if ˆw is an analytic W-valued function in Ω satisfying ˆw(λ) ∈ rng (ˆC(λ)), λ ∈ Ω.

(iv) ˆw is the signal component of some Ω-trajectory (x, w; 0) (with initial state zero) if and only if ˆw is an analytic W-valued function in Ω satisfying ˆw(λ) ∈ ˆF(λ), λ ∈ Ω.
Then the following claims are true.

7.1.7. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a frequency domain s/s system, and \( \Sigma_i/s/o = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o representation of \( \Sigma \), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then \( (\hat{x}, \hat{w}; x^0) \) is an \( \Omega \)-trajectory of \( \Sigma \) if and only if \( (\hat{x}, P_Y \hat{w}; x^0, P_U \hat{w}) \) is an \( \Omega \)-trajectory of \( \Sigma_i/s/o \).

Proof. This follows from Definitions 7.1.10 and 7.1.11.

7.1.8. Remark. In the sequel we shall make frequent (sometimes implicit) use of the following connections between the resolvent set of a s/s node \( \Sigma \) and the resolvent sets of its i/s/o representations \( \Sigma_i/s/o \) established in Theorem 5.3.9, namely that

1. \( \rho(\Sigma) \) is the union of \( \rho(\Sigma_i/s/o) \) as \( \Sigma_i/s/o \) varies over all i/s/o representations of \( \Sigma \).

In particular, this implies that

2. \( \Sigma \) is resolvable if and only if \( \Sigma \) has a resolvable i/s/o representation, and
3. \( \rho(\Sigma_i/s/o) \subset \rho(\Sigma) \) for every i/s/o representation \( \Sigma_i/s/o \) of \( \Sigma \).

7.1.9. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable s/s node. By an frequency i/s/o-admissible domain for \( \Sigma \) we mean a (nonempty) open subset \( \Omega \) of \( \mathbb{C} \) such that \( \Omega \subset \rho(\Sigma_i/s/o) \) for some i/s/o representation \( \Sigma_i/s/o \) of \( \Sigma \).

7.1.10. Lemma (cf. Lemma 6.1.7). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable frequency domain s/s system, and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \). Then the following claims are true.

1. If \( \Sigma_i/s/o = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is an \( \Omega \)-resolvable i/s/o representation of \( \Sigma \) with i/s/o resosent function \( \hat{\Sigma} = \begin{bmatrix} \hat{S} & \hat{\mathcal{U}} \\ \hat{\mathcal{W}} & \hat{D} \end{bmatrix} \), then for every \( x^0 \in \mathcal{X} \) and for every analytic \( \mathcal{U} \)-valued function \( \hat{u} \) in \( \Omega \) the frequency domain s/s system \( \Sigma \) has a unique \( \Omega \)-trajectory \( (\hat{x}, \hat{w}, x^0) \) satisfying \( P_Y \hat{w} = \hat{u} \). This trajectory is given by \( (\hat{x}, \hat{u} + \hat{y}; x^0) \), where \( \hat{x} \) and \( \hat{y} \) are given by (6.1.2).

2. \( \Sigma \) is determined uniquely by the set of all \( \Omega \)-trajectories of \( \Sigma \) evaluated at some point \( \lambda \in \Omega \).

Proof. This follows from Lemmas 5.3.28 and 6.1.7.

7.1.11. The frequency domain behavior and external equivalence.

7.1.12. Definition (cf. Definition 6.1.8). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a frequency domain s/s system, and let \( \Omega \) be an open set in \( \mathbb{C} \). By the \( \Omega \)-behavior \( \mathcal{W}_\Sigma^{(\Omega)} \) of \( \Sigma \) we mean the set of all analytic \( \mathcal{W} \)-valued functions \( \hat{w} \) in \( \Omega \) for which there exists some analytic \( \mathcal{X} \)-valued function \( \hat{x} \) in \( \Omega \) such that \( (\hat{x}, \hat{w}; 0) \) is an \( \Omega \)-trajectory of \( \Sigma \) (with initial state \( x^0 = 0 \)).

7.1.13. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a frequency domain s/s system, and \( \Sigma_i/s/o = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o representation of \( \Sigma \), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then \( \hat{w} \) belongs to the \( \Omega \)-behavior of \( \Sigma \) if and only if \( \begin{bmatrix} P_Y \hat{w} \\ P_U \hat{w} \end{bmatrix} \) belongs to the \( \Omega \)-behavior of \( \Sigma_i/s/o \).

Proof. This follows from Definitions 6.1.8 and 7.1.11 and Lemma 7.1.7.
7.1.13. Lemma (cf. Lemma 6.1.9). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a frequency domain s/s system with characteristic signal bundle \( \hat{\mathcal{F}} \) and \( \Omega \)-behavior \( \mathcal{W}_\Sigma^\Omega \).

(i) If \( \Omega \) is an open set in \( \mathbb{C} \), then every \( \hat{\omega} \in \hat{\mathcal{W}}_\Sigma \) is a \( \mathcal{W} \)-valued function in \( \Omega \) satisfying

\[
\hat{\omega}(\lambda) \in \hat{\mathcal{F}}(\lambda), \quad \lambda \in \Omega.
\]

(ii) If \( \Sigma \) is resolvable and \( \Omega \) is a frequency i/s/o-admissible domain for \( \Sigma \), then an analytic \( \mathcal{W} \)-valued function \( \hat{\omega} \) belongs to \( \mathcal{W}_\Sigma^\Omega \) if and only if (7.1.2) holds.

Proof. That (i) holds follows from Definitions 6.1.1, 7.1.1, and 7.1.11. That (ii) holds follows from Lemma 7.1.6(iv) and Definition 7.1.11. \( \Box \)

7.1.14. Definition (cf. Definition 6.1.10). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \) be two frequency domain s/s systems (with the same signal space), and let \( \Omega \) be an open set in \( \mathbb{C} \). We say that \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent if they have the same \( \Omega \)-behavior.

7.1.15. Lemma. Let \( \Sigma_j = (V_j; \mathcal{X}_j, \mathcal{W}), \ j = 1, 2, \) be two frequency domain s/s system (with the same signal space), let \( \Sigma_{i/s/o}^j = (S_j; \mathcal{X}_j, \mathcal{U}, \mathcal{Y}) \) be i/s/o representations of \( \Sigma_j, \ j = 1, 2, \) (with the same input and output spaces), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent if and only if \( \Sigma_{1/s/o}^1 \) and \( \Sigma_{2/s/o}^2 \) are externally \( \Omega \)-equivalent.

Proof. This follows from Definitions 6.1.8 and 7.1.11 and Lemma 7.1.7. \( \Box \)

In Lemma 7.1.15 it is assumed that the two i/s/o representations \( \Sigma_{i/s/o}^1 \) and \( \Sigma_{i/s/o}^2 \) have the same input and output spaces. This condition is necessary due to the fact that, by definition, two frequency domain i/s/o systems cannot be externally \( \Omega \)-equivalent unless they have the same input and output spaces. Below we shall also present some other related results where we allow the two i/s/o representations to have different input and output spaces. For these results to be valid we need to impose some conditions on the resolvent sets of the two s/s nodes \( \Sigma_1 \) and \( \Sigma_2 \).

The following definition is a modification of Definition 7.1.9 to the case where we are studying two or more frequency domain s/s systems at the same time.

7.1.16. Definition. Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), \ i = 1, \ldots, n, \) be resolvable frequency domain s/s systems (with the same signal space) satisfying \( \bigcap_{i=1}^n \rho(\Sigma_i) \neq \emptyset \).

(i) By a separately frequency i/s/o-admissible domain for \( \Sigma_j = (V_j; \mathcal{X}_j, \mathcal{W}), \ j = 1, \ldots, n \) we mean a (nonempty) open subset \( \Omega \) of \( \bigcap_{j=1}^n \rho(\Sigma_j) \) such that for each \( j = 1, \ldots, n \) there exists an \( \Omega \)-resolvable i/s/o representation \( \Sigma_{i/s/o}^j = (S_j; \mathcal{X}_j, \mathcal{U}_j, \mathcal{Y}_j) \) of \( \Sigma_j \) (note that these i/s/o representations may have different input and output spaces).

(ii) By a jointly frequency i/s/o-admissible domain for \( \Sigma_j = (V_j; \mathcal{X}_j, \mathcal{W}), \ j = 1, \ldots, n \) we mean a (nonempty) open subset \( \Omega \) of \( \bigcap_{j=1}^n \rho(\Sigma_j) \) with the following property: there exists an i/o representation \( (\mathcal{U}, \mathcal{Y}) \) of \( \mathcal{W} \) such that the corresponding i/s/o representations \( \Sigma_{i/s/o}^j = (S_j; \mathcal{X}_j, \mathcal{U}, \mathcal{Y}) \) of \( \Sigma_j, \ j = 1, \ldots, n, \) are \( \Omega \)-resolvable.
7.1.17. Remark. If \( \Omega \) is a separately i/s/o-admissible domain for the frequency domain s/s systems \( \Sigma_1 \) and \( \Sigma_2 \), then it follows from, e.g., Remark 7.1.8 that \( \rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset \). Conversely, if \( \rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset \), then there exists (infinitely many) separately frequency i/s/o-admissible domains for \( \Sigma_1 \) and \( \Sigma_2 \) (this, too, follows from Remark 7.1.8). However, the condition \( \rho(\Sigma_1) \cap \rho(\Sigma_2) \neq \emptyset \) does not imply that the set of jointly frequency i/s/o-admissible domains for \( \Sigma_1 \) and \( \Sigma_2 \) is nonempty, as the following counterexample shows: Let \( \Sigma_1 \) and \( \Sigma_2 \) be the s/s nodes given in Example 1.4.5 and Example 1.4.6. Then \( \Sigma_1 \) and \( \Sigma_2 \) are time reflections of each other. It follows from Examples 5.3.19 and 5.3.20, \( \rho(\Sigma_1) = \rho(\Sigma_2) = C \setminus j\mathbb{R} \), and that there does not exist any jointly frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \). However, both \( \mathbb{C}^+ \) and \( \mathbb{C}^- \) are separately frequency i/s/o-admissible domains for \( \Sigma_1 \) and \( \Sigma_2 \), and an open set \( \Omega \) is a separately frequency i/s/o-admissible domain \( \Omega \) for \( \Sigma_1 \) and \( \Sigma_2 \) if and only if \( \Omega \) is contained in either \( \mathbb{C}^+ \) or in \( \mathbb{C}^- \).

7.1.18. Lemma (c. Lemma 6.1.11). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \) be two resolvable frequency domain s/s systems (with the same signal space) with signal bundles \( \hat{\mathcal{S}}_i \), \( i = 1, 2 \), and suppose that \( \Omega \) is a separately i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \). Then \( \Sigma_1 \) and \( \Sigma_1 \) are externally \( \Omega \)-equivalent if and only if \( \hat{\mathcal{S}}_1(\lambda) = \hat{\mathcal{S}}_2(\lambda) \) for all \( \lambda \in \Omega \).

Proof. This follows from Lemma 7.1.13 and Definition 7.1.14.

7.1.19. Lemma. Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two \( \Omega \)-resolvable frequency domain s/s system (with the same signal space), and suppose that the characteristic signal bundles \( \hat{\mathcal{S}}_1 \) and \( \hat{\mathcal{S}}_2 \) of \( \Sigma_1 \) respectively \( \Sigma_2 \) satisfy \( \hat{\mathcal{S}}_1(\lambda) = \hat{\mathcal{S}}_2(\lambda) \) for all \( \lambda \in \Omega \) (this is, in particular, true of \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent). Then the following conditions are equivalent:

(i) \( \Omega \) is a frequency i/s/o-admissible domain for \( \Sigma_1 \);
(ii) \( \Omega \) is a frequency i/s/o-admissible domain for \( \Sigma_2 \);
(iii) \( \Omega \) is a separately frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \);
(iv) \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \).

Proof. Clearly (iv) implies (iii) which implies both (ii) and (i), so to prove the lemma it suffices to prove that (i) \( \Rightarrow \) (iv) (because the implication (ii) \( \Rightarrow \) (iv) then follows if we interchange \( \Sigma_1 \) and \( \Sigma_2 \)).

Let \( \Sigma_{I/s/o}^1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) be an i/s/o representation of \( \Sigma_1 \) satisfying \( \Omega \subset \rho(\Sigma_{I/s/o}^1) \). By Theorem 5.3.6 and Definition 5.3.8 a point \( \lambda \in \Omega \) is frequency domain i/s/o-admissible for \( \Sigma_1 \) if and only if \( \lambda \) is frequency domain i/s/o-admissible for \( \Sigma_2 \). This implies that if we denote the i/s/o representation of \( \Sigma_2 \) corresponding to the i/o representation \( (\mathcal{U}, \mathcal{Y}) \) of \( \mathcal{W} \) by \( \Sigma_{I/s/o}^2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}) \), then \( \rho(\Sigma_{I/s/o}^2) \cap \Omega = \rho(\Sigma_{I/s/o}^1) \cap \Omega \), and thus \( \Omega \subset \rho(\Sigma_{I/s/o}^2) \). Thus \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \).

7.1.3. Frequency domain controllability and observability.

7.1.20. Definition (cf. Definition 6.1.12). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a frequency domain s/s system, and let \( \Omega \) be an open set in \( \mathbb{C} \).

(i) A state vector \( x_0 \in \mathcal{X} \) is called exactly \( \Omega \)-reachable for \( \Sigma \) if there exists an \( \Omega \)-trajectory \( (\hat{x}, \hat{w}; 0) \) of \( \Sigma \) (with zero initial state) such that \( x_0 = \hat{x}(\lambda) \) for some \( \lambda \in \Omega \).
(ii) An Ω-trajectory \((\hat{x}, \hat{w}; x^0)\) of \(\Sigma\) with initial state \(x^0\) is called \(\Omega\)-unobservable if \(\hat{w} = 0\).

(iii) A state vector \(x^0 \in \mathcal{X}\) is called \(\Omega\)-unobservable for \(\Sigma\) if there exists an \(\Omega\)-unobservable \(\Omega\)-trajectory \((\hat{x}, 0; x^0)\) of \(\Sigma\) with this initial state.

It is easy to see that the sets of all \(\Omega\)-unobservable states is a subspace of \(\mathcal{X}\).

7.1.21. DEFINITION (cf. Definition 6.1.13). Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a frequency domain s/s system, let \(\Omega\) be an open set in \(\mathbb{C}\).

(i) The linear span of all exactly \(\Omega\)-reachable states of \(\Sigma\) is called the exactly \(\Omega\)-reachable subspace of \(\Sigma\) and it is denoted by \(\mathcal{R}_\Sigma^{\Omega,\text{exact}}\).

(ii) The closure of \(\mathcal{R}_\Sigma^{\Omega,\text{exact}}\) is called the (approximately) \(\Omega\)-reachable subspace of \(\Sigma\) and it is denoted by \(\mathcal{R}_\Sigma^{\Omega}\).

(iii) The subspace of all \(\Omega\)-unobservable states of \(\Sigma\) is called the \(\Omega\)-unobservable subspace of \(\Sigma\), and it is denoted by \(\mathcal{U}_\Sigma^{\Omega}\).

(iv) \(\Sigma\) is \(\Omega\)-controllable if \(\mathcal{R}_\Sigma^{\Omega} = \mathcal{X}\), and \(\Sigma\) is \(\Omega\)-observable if \(\mathcal{U}_\Sigma^{\Omega} = \{0\}\).

7.1.22. LEMMA. Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a frequency domain s/s system, let \(\Sigma_{i/s/o}\) be an i/s/o representation of \(\Sigma\), and let \(\Omega\) be an open set in \(\mathbb{C}\). Then

(i) \(\Sigma\) and \(\Sigma_{i/s/o}\) have the same exactly \(\Omega\)-reachable states and the same \(\Omega\)-unreachable states. Moreover, an \(\Omega\)-trajectory \((\hat{x}, 0; x^0)\) of \(\Sigma\) is \(\Omega\)-unobservable for \(\Sigma\) if and only if the corresponding \(\Omega\)-trajectory \((\hat{x}, 0; x^0, 0)\) of \(\Sigma_{i/s/o}\) is \(\Omega\)-unobservable for \(\Sigma_{i/s/o}\).

(ii) The corresponding statements are also true for the exactly \(\Omega\)-reachable subspaces, the \(\Omega\)-reachable subspaces, and the \(\Omega\)-unreachable subspaces of \(\Sigma\) and \(\Sigma_{i/s/o}\), i.e.,

\[
\mathcal{R}_\Sigma^{\Omega,\text{exact}} = \mathcal{R}_{\Sigma_{i/s/o}}^{\Omega,\text{exact}}, \quad \mathcal{R}_\Sigma^{\Omega} = \mathcal{R}_{\Sigma_{i/s/o}}^{\Omega}, \quad \mathcal{U}_\Sigma^{\Omega} = \mathcal{U}_{\Sigma_{i/s/o}}^{\Omega}.
\]

(iii) \(\Sigma\) is \(\Omega\)-controllable or \(\Omega\)-observable if and only if \(\Sigma_{i/s/o}\) is \(\Omega\)-controllable or \(\Omega\)-observable respectively.

PROOF. This follows from Definitions 6.1.12, 6.1.13, 7.1.20 and 7.1.21 and Lemma 7.1.7. □

7.1.23. COROLLARY. Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a frequency domain s/s system, let \(\Sigma_{i/s/o}^j = (S_j; \mathcal{X}_j, \mathcal{U}_j, \mathcal{Y}_j), j = 1, 2\), be two i/s/o representations of \(\Sigma\) (with possible different input and output spaces), and let \(\Omega\) be an open set in \(\mathbb{C}\). Then

\[
\mathcal{R}_\Sigma^{\Omega,\text{exact}} = \mathcal{R}_{\Sigma_{i/s/o}^1}^{\Omega,\text{exact}}, \quad \mathcal{R}_\Sigma^{\Omega} = \mathcal{R}_{\Sigma_{i/s/o}^1}^{\Omega}, \quad \mathcal{U}_\Sigma^{\Omega} = \mathcal{U}_{\Sigma_{i/s/o}^1}^{\Omega}.
\]

PROOF. This follows from Lemma 7.1.22 since both sides in these equalities are equal to \(\mathcal{R}_\Sigma^{\Omega}\) respectively. □

7.1.24. LEMMA (cf. Lemma 6.1.14). Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a resolvable frequency domain s/s system with characteristic node bundle \(\mathcal{E}\), signal/state map \(\mathcal{B}\), and state/signal map \(\mathcal{E}\), and let \(\Omega\) be a frequency i/s/o-admissible domain for \(\Sigma\). Then with the notation introduced in Definition 7.1.21

\[
\mathcal{R}_\Sigma^{\Omega,\text{exact}} = \text{span}_{\lambda \in \Omega} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{E}(\lambda) \cup \{0\} \\ \mathcal{X} \cup \mathcal{W} \end{bmatrix},
\]

\[
= \text{span}_{\lambda \in \Omega \cap \text{rng} (\mathcal{B}(\lambda))}.
\]
7.1. FREQUENCY DOMAIN S/S SYSTEMS

(7.1.5) \[ R_\Omega^\Sigma = \bigvee_{\lambda \in \Omega} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \left( \mathcal{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} \right) \]

\[ = \bigvee_{\lambda \in \Omega} \text{rng} (\hat{\mathcal{B}}(\lambda)), \]

(7.1.6) \[ U_\Omega^\Sigma = \bigcap_{\lambda \in \Omega} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left( \mathcal{E}(\lambda) \cap \begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix} \right) \]

\[ = \bigcap_{\lambda \in \Omega} \ker (\hat{\mathcal{C}}(\lambda)) . \]

Proof. This follows from Lemmas 6.3.21, 6.3.28, 6.1.14, and 7.1.22. \[ \square \]

7.1.4. Frequency domain invariance.

7.1.25. Definition (cf. Definition 6.1.15). Let \( \Sigma = (V; \mathcal{X}, W) \) be a frequency domain s/s system, and let \( \Omega \) be an open set in \( \mathbb{C} \).

(i) A subspace \( Z \) of \( \mathcal{X} \) is strongly \( \Omega \)-invariant for \( \Sigma \) if every \( \Omega \)-trajectory \( (\hat{x}, \hat{w}; x^0) \) of \( \Sigma \) with initial state \( x^0 \in Z \) satisfies \( \hat{x}(\lambda) \in Z \) for all \( \lambda \in \Omega \).

(ii) A subspace \( Z \) of \( \mathcal{X} \) is unobservably \( \Omega \)-invariant for \( \Sigma \) if for every \( x^0 \in Z \) there exists an \( \Omega \)-unobservable \( \Omega \)-trajectory \( (\hat{x}, 0; x^0) \) of \( \Sigma \) satisfying \( \hat{x}(\lambda) \in Z \) for all \( \lambda \in \Omega \).

7.1.26. Lemma. Let \( \Sigma \) be a frequency domain s/s system, let \( \Sigma_{i/s/o} \) be an i/s/o representation of \( \Sigma \), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then

(i) A subspace \( Z \) of \( \mathcal{X} \) is strongly \( \Omega \)-invariant for \( \Sigma \) if and only if \( Z \) is strongly \( \Omega \)-invariant for \( \Sigma_{i/s/o} \), and

(ii) A subspace \( Z \) of \( \mathcal{X} \) is unobservably \( \Omega \)-invariant for \( \Sigma \) if and only if \( Z \) is unobservably \( \Omega \)-invariant for \( \Sigma_{i/s/o} \).

Proof. This follows from Definitions 6.1.15 and 7.1.25 and Lemma 7.1.7. \[ \square \]

7.1.27. Lemma (cf. Lemma 6.1.16). Let \( \Sigma \) be a resolvable frequency domain s/s system with characteristic node bundle \( \mathcal{E} \), let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \), and let \( Z \) be a subspace of \( \mathcal{X} \).

(i) \( Z \) is a strongly \( \Omega \)-invariant subspace for \( \Sigma \) if and only if the following three equivalent conditions hold for all \( \lambda \in \Omega \):

(7.1.7a) \[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \left( \mathcal{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} \right) \subset Z , \]

(7.1.7b) \[ \mathcal{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} = \mathcal{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} , \]

(7.1.7c) \[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \left( \mathcal{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} \right) \subset Z \text{ and } \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left( \mathcal{E}(\lambda) \cap \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix} \right) = Z . \]
(ii) \( \mathcal{Z} \) is an unobservably \( \Omega \)-invariant subspace for \( \Sigma \) if and only if the following three equivalent conditions hold for all \( \lambda \in \Omega \):

\[
(7.1.8a) \quad [1_Z \ 0 \ 0] \left( \mathcal{E}(\lambda) \cap \left[ \frac{Z}{\{0\}} \right] \right) = Z,
\]

\[
(7.1.8b) \quad \begin{bmatrix} \rho_{x_i}^{\Omega} & 0 & 0 \\ 0 & \rho_{x_i}^{\Omega} & 0 \\ 0 & 0 & 1_{W} \end{bmatrix} \mathcal{E}(\lambda) = \begin{bmatrix} \rho_{x_i}^{\Omega} & 0 & 0 \\ 0 & \rho_{x_i}^{\Omega} & 0 \\ 0 & 0 & 1_{W} \end{bmatrix} \left( \mathcal{E}(\lambda) \cap \left[ \frac{X_i}{W} \right] \right),
\]

\[
(7.1.8c) \quad [1_Z \ 0 \ 0] \left( \mathcal{E}(\lambda) \cap \left[ \frac{Z}{\{0\}} \right] \right) = Z \quad \text{and} \quad [0 \ 1_{X} \ 0] \left( \mathcal{E}(\lambda) \cap \left[ \frac{X}{\{0\}} \right] \right) \subset Z;
\]

**Proof.** The proof of (i) is a simplified version of the proofs of Lemma 3.4.5 (in the case of claim (i)) and Lemma 3.4.8 (in the case of claim (ii)) with references to Proposition 2.5.49 and Lemmas 3.2.2, 3.2.3, and 3.4.1 replaced by references to Theorem 5.3.6 and Lemmas 5.3.28, 6.1.16, and 7.1.26. \( \square \)

7.1.28. **Lemma (cf. Lemma 6.1.17).** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable frequency domain s/s system, and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \). If a subspace \( \mathcal{Z} \) of \( \mathcal{X} \) is a strongly \( \Omega \)-invariant or unobservable \( \Omega \)-invariant subspace for \( \Sigma \), then the closure \( \overline{\mathcal{Z}} \) of \( \mathcal{Z} \) in \( \mathcal{X} \) is also strongly \( \Omega \)-invariant respectively unobservable \( \Omega \)-invariant for \( \Sigma \).

**Proof.** If the invariance conditions in in Lemmas 7.1.27 hold for some subspace \( \mathcal{Z} \) of \( \mathcal{X} \), then they also hold if we replace \( \mathcal{Z} \) by \( \overline{\mathcal{Z}} \). \( \square \)

7.1.29. **Lemma (cf. Lemma 6.1.18).** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable frequency domain s/s system, and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \).

(i) There exists a minimal strongly \( \Omega \)-invariant subspace \( \mathcal{R}_{\Omega}^{\Sigma} \) for \( \Sigma \), i.e., \( \mathcal{R}_{\Omega}^{\Sigma} \) is strongly \( \Omega \)-invariant for \( \Sigma \) and \( \mathcal{R}_{\Omega}^{\Sigma} \) is contained in every other strongly \( \Omega \)-invariant subspace for \( \Sigma \). The subspace \( \mathcal{R}_{\Omega}^{\Sigma} \) satisfies \( \mathcal{R}_{\Omega}^{\Sigma,\text{exact}} \subset \mathcal{R}_{\Omega}^{\Sigma} \subset \mathcal{R}_{\Omega}^{\Sigma,\text{exact}} \), and hence \( \mathcal{R}_{\Omega}^{\Sigma} = \mathcal{R}_{\Omega}^{\Sigma} \).

(ii) The \( \Omega \)-reachable subspace \( \mathcal{R}_{\Omega, r}^{\Sigma} \) is the minimal closed strongly \( \Omega \)-invariant subspace for \( \Sigma \), i.e., \( \mathcal{R}_{\Omega, r}^{\Sigma} \) is strongly \( \Omega \)-invariant for \( \Sigma \) and \( \mathcal{R}_{\Omega, r}^{\Sigma} \) is contained in every other closed strongly \( \Omega \)-invariant subspace for \( \Sigma \).

(iii) The \( \Omega \)-unobservable subspace \( \mathcal{U}_{\Omega}^{\Sigma} \) is the maximal unobservable \( \Omega \)-invariant subspace for \( \Sigma \), i.e., \( \mathcal{U}_{\Omega}^{\Sigma} \) is unobservable \( \Omega \)-invariant, and \( \mathcal{U}_{\Omega}^{\Sigma} \) contains every other unobservable \( \Omega \)-invariant subspace for \( \Sigma \).

**Proof.** This follows from Lemmas 6.1.18, 7.1.27 and 7.1.26. \( \square \)

We remark that a more explicit description of the subspace \( \mathcal{R}_{\Omega}^{\Sigma} \) in part (i) of Lemma 7.1.29 can be obtained from Lemma 6.1.18 since (by the same argument which proves Lemma 7.1.29) this subspace coincides with the corresponding subspace for \( \Sigma_{i/s/o} \), where \( \Sigma_{i/s/o} \) is an arbitrary i/s/o representation of \( \Sigma \) satisfying \( \Omega \subset \rho(\Sigma_{i/s/o}) \).

Also note that (7.1.6) implies that \( \mathcal{U}_{\Omega}^{\Sigma} \) is closed whenever \( \Omega \) is a frequency i/s/o-admissible subspace for \( \Sigma \).

7.1.5. Frequency domain intertwinnings.

7.1.30. **Definition (cf. Definition 6.1.19).** Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i), i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), and let \( \Omega \) be an
open set in \( \mathbb{C} \). We say that \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2) \) if the following two conditions holds:

(i) If \((\hat{x}_1, \hat{w}; x_1^0)\) is an \( \Omega \)-trajectory of \( \Sigma_1 \) with \( x_1^0 \in \text{dom}(P) \), then for every \( x_2^0 \in P \hat{x}_1^0 \) there exists an \( \Omega \)-trajectory \((\hat{x}_2, \hat{w}; x_2^0)\) of \( \Sigma_2 \) satisfying \( \hat{x}_2(\lambda) \in P \hat{x}_1(\lambda) \) for all \( \lambda \in \Omega \).

(ii) Condition (i) above also holds if we interchange \( \Sigma_1 \) and \( \Sigma_2 \) and replace \( P \) by \( P^{-1} \). In other words, if \((\hat{x}_2, \hat{w}; x_2^0)\) is an \( \Omega \)-trajectory of \( \Sigma_2 \) with \( x_2^0 \in \text{rng}(P) \), then for every \( x_1^0 \in P^{-1} x_2^0 \) there exists an \( \Omega \)-trajectory \((\hat{x}_1, \hat{w}; x_1^0)\) of \( \Sigma_1 \) satisfying \( \hat{x}_1(\lambda) \in P^{-1} \hat{x}_2(\lambda) \) for all \( \lambda \in \Omega \).

7.1.31. DEFINITION (cf. Definition 6.1.20). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), and let \( \Omega \) be an open set in \( \mathbb{C} \).

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-pseudo-similar if they are \( \Omega \)-intertwined by a closed single-valued injective linear operator \( P: \mathcal{X} \to \mathcal{X}_1 \) with dense domain and dense range, called the \( \Omega \)-pseudo-similarity operator.

(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-similar if they are intertwined by a bounded linear operator \( P \in \mathcal{B}(\mathcal{X}_1;\mathcal{X}_2) \) with a bounded inverse \( P^{-1} \in \mathcal{B}(\mathcal{X}_2;\mathcal{X}_1) \), called the \( \Omega \)-similarity operator.

7.1.32. LEMMA (cf. Lemma 6.1.21). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems, let \( \Omega \) be an open set in \( \mathbb{C} \), and denote the exactly \( \Omega \)-reachable subspaces and the \( \Omega \)-unobservable subspaces of \( \Sigma_i \) by \( \mathcal{R}_{\Sigma_i}^{\Omega, \text{exact}} \) respectively \( \mathcal{U}_{\Sigma_i}^\Omega \), \( i = 1, 2 \). If \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by some \( P \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2) \), then the following claims hold:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent.

(ii) \( \text{dom}(P) \) is strongly \( \Omega \)-invariant for \( \Sigma_1 \). In particular, \( \mathcal{R}_{\Sigma_1}^{\Omega, \text{exact}} \subset \text{dom}(P) \).

(iii) \( \text{rng}(P) \) is strongly \( \Omega \)-invariant for \( \Sigma_2 \). In particular, \( \mathcal{R}_{\Sigma_2}^{\Omega, \text{exact}} \subset \text{rng}(P) \).

(iv) \( \ker(P) \) is unobservable \( \Omega \)-invariant for \( \Sigma_1 \). In particular, \( \ker(P) \subset \mathcal{U}_{\Sigma_1}^\Omega \).

(v) \( \mu(P) \) is unobservable \( \Omega \)-invariant for \( \Sigma_2 \). In particular, \( \mu(P) \subset \mathcal{U}_{\Sigma_2}^\Omega \).

PROOF. The proof is analogous to the proof of Lemma 1.5.27.

7.1.33. LEMMA. Let \( \Sigma_j = (V_j; \mathcal{X}_j, \mathcal{W}) \) be two frequency domain s/s systems, let \( \Sigma_{ij/s/o} = (S_j; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be two i/s/o representations of \( \Sigma_j \), \( j = 1, 2 \) (with the same input and output spaces), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2) \) if and only if \( \Sigma_{1/s/o} \) and \( \Sigma_{2/s/o} \) are \( \Omega \)-intertwined by \( P \). The same statement is also true if we replace “\( \Omega \)-intertwined by \( P \)” by “\( \Omega \)-pseudo-similar” or by “\( \Omega \)-similar”.

PROOF. This follows from Definitions 6.1.19 and 7.1.30 and Lemma 7.1.7.

The restriction in Lemma 7.1.33 that the two i/s/o representations \( \Sigma_1 \) and \( \Sigma_2 \) have the same input and output spaces cannot be avoided, due to the fact that two frequency domain i/s/o systems cannot be \( \Omega \)-intertwined (in the sense of Definition 7.1.30) unless they have the same input and output spaces.

7.1.34. LEMMA. Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \) be two resolvable frequency domain s/s systems (with the same signal space), let \( \Omega \) be a separately frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \), and let \( P \in \mathcal{ML}(\mathcal{X}_1;\mathcal{X}_2) \). Denote the characteristic node
bundles of Σi by Ėi and the signal/state and signal-state/state resolvents of Σi by Ėi, respectively Ėi, i = 1, 2. Then the following conditions are equivalent:

(i) Σ1 and Σ2 are Ω-intertwined by P;
(ii) The following two conditions hold for all λ ∈ Ω:
   (a) If x0 2 ∈ Px1 0 and wλ ∈ W, then \[ \begin{bmatrix} \dot{x}^0_{2,1} \\ x_{1,1} \end{bmatrix} \in \hat{E}_1(\lambda) \] for some (unique) x1,1 ∈ X1 if and only if \[ \begin{bmatrix} \dot{x}^0_{2,1} \\ x_{1,1} \end{bmatrix} \in \hat{E}_2(\lambda) \] for some (unique) x2,1 ∈ X2;
   (b) the vectors x1,1 and x2,1 in (a) satisfy \[ \begin{bmatrix} \dot{x}^0_{2,1} \\ x_{1,1} \end{bmatrix} \in \text{gph}(P).
(iii) The following two conditions hold for all λ ∈ Ω:
   (a) \[ \hat{E}_2(\lambda)x^0_1 = \hat{E}_1(\lambda)x^0_1 \] whenever x^0_1 ∈ Px_1 0;
   (b) \[ \hat{E}_2(\lambda)x^0_1 \in P\hat{E}_1(\lambda) \begin{bmatrix} x^0_1 \\ \lambda \end{bmatrix} \] whenever x^0_1 ∈ Px_1 0 and wλ ∈ \hat{E}_2(\lambda)x^0_1 = \hat{E}_1(\lambda)x^0_1.
(Recall that \[ \begin{bmatrix} x^0_1 \\ \lambda \end{bmatrix} \in \text{dom}(\hat{E}_1(\lambda)) \] if and only if wλ ∈ \hat{E}_1(\lambda)x^0_1, i = 1, 2.)

Suppose, in addition that P is closed, and let Σ = (V; X, W) be the s/s node with state space X = gph(P) and generating subspace

\[
V = \left\{ \begin{bmatrix} z_2 \\ z_1 \\ w \end{bmatrix} \in \text{gph}(P), \begin{bmatrix} \dot{z}_1 \\ z_1 \\ w \end{bmatrix} \in V_1 \text{ and } \begin{bmatrix} \dot{z}_2 \\ z_2 \\ w \end{bmatrix} \in V_2 \right\}.
\]

Then the conditions (i) and (ii) above are also equivalent to the condition

(iv) Σ1, Σ2, and Σ are externally Ω-equivalent, and Ω is a separately frequency i/s/o-admissible domain for Σ1, Σ2, and Σ.

In this case Σ and Σ1 are Ω-intertwined by the bounded operator \( P_{X_1}^{V_1} |_{\text{gph}(P)} \), and Σ and Σ2 are Ω-intertwined by the bounded operator \( P_{X_2}^{V_2} |_{\text{gph}(P)} \).

**Proof.** (i) ⇒ (ii): Let \( \Sigma^1_{i/s/o} = (S_1; X_1, U, Y) \) be an i/s/o representation of Σ. If (i) holds, then by Lemma 7.1.32 Σ1 and Σ2 are externally Ω-equivalent, and hence by Lemma 7.1.19 Σ2 has an i/s/o representation \( \Sigma^2_{i/s/o} = (S_2; X_2, U, Y) \) with the same input and output spaces. By Lemma 7.1.33 \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are Ω-intertwined by P. It then follows from Lemma 6.1.26 and the representation formula (5.3.8) for Ė1 and Ė2 that (ii) holds.

(ii) ⇒ (i): If condition (ii) holds, the the characteristic signal bundles \( \hat{E}_1 \) of Σi, i = 1, 2, satisfy \( \hat{E}_1(\lambda) = \hat{E}_2(\lambda) \) for all \( \lambda \in \Omega \), and by Lemma 7.1.18 Σ1 and Σ2 are externally Ω-equivalent. It therefore follows from Lemma 7.1.19 that Ω is a jointly frequency i/s/o-admissible domain for Σ1 and Σ2. In particular, there exists i/s/o representations \( \Sigma^j_{i/s/o} = (S_j; X_j, U, Y) \) of \( \Sigma_j \), j = 1, 2, with the same input and output spaces. Condition (ii) combined together with the representation formula (5.3.8) for Ė1 and Ė2 and Lemma 3.2.16 implies that \( \Sigma^1_{i/s/o} \) and \( \Sigma^2_{i/s/o} \) are intertwined by P. By Lemma 7.1.33 also Σ1 and Σ2 are intertwined by P.

(i) ⇔ (iii): This follows from Lemmas 5.3.21

(i) ⇒ (iv): Suppose that (i) holds. Clearly V is closed since gph(P), V1, and V2 are closed, and hence Σ is a s/s node. It follows from (7.1.9) that the characteristic
node bundle $\hat{\mathcal{E}}$ of $\Sigma$ is given by

\[(7.1.10)\]

\[
\hat{\mathcal{E}}(\lambda) = \left\{ \left[ \begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \right] \in \left[ \begin{bmatrix} \text{gph} (P) \\ \text{gph} (P) \end{bmatrix} \right] \left| \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} \in \hat{\mathcal{E}}_1(\lambda) \text{ and } \begin{bmatrix} z_2 \\ x_2 \end{bmatrix} \in \hat{\mathcal{E}}_2(\lambda) \right. \right\}, \quad \lambda \in \mathbb{C}.
\]

As in the proof of the implication (i) $\Rightarrow$ (ii) we fix some i/s/o representations $\Sigma_{i/s/o}^j = (S_j; \mathcal{X}_j, \mathcal{U}_j)$ of $\Sigma_j$ (with the same signal space), $j = 1, 2$, satisfying $\Omega \subset \rho(\Sigma_1) \cap \rho(\Sigma_2)$. Denote the i/s/o resolvent matrices of $\Sigma_{i/s/o}^j$ by $\hat{\mathcal{S}}_j = \left[ \begin{bmatrix} \hat{S}_j \hat{B}_j \\ \hat{C}_j \hat{D}_j \end{bmatrix}, \right.$ $j = 1, 2$. It follows from Lemma [6.1.23] that it is possible to define $\hat{\mathcal{S}}(\lambda) = \left[ \begin{bmatrix} \hat{A}_j(\lambda) \hat{B}_j(\lambda) \\ \hat{C}_j(\lambda) \hat{D}_j(\lambda) \end{bmatrix} \right]$ for $\lambda \in \Omega$ by

\[(7.1.11)\]

\[
\hat{\mathcal{A}}(\lambda) = \begin{bmatrix} \hat{A}_2(\lambda) \\ 0 \end{bmatrix}, \quad \hat{B}(\lambda) = \begin{bmatrix} \hat{B}_2(\lambda) \\ \hat{B}_1(\lambda) \end{bmatrix}, \quad \hat{\mathcal{D}}(\lambda) = \begin{bmatrix} \hat{D}_2(\lambda) \\ \hat{D}_1(\lambda) \end{bmatrix}.
\]

and that the operator defined in this way is a bounded linear operator from $\left[ \begin{bmatrix} \hat{A}(\lambda) \\ \hat{B}(\lambda) \end{bmatrix} \right]$ into $\left[ \begin{bmatrix} \hat{A}(\lambda) \\ \hat{B}(\lambda) \end{bmatrix} \right]$. Moreover, it follows from (7.1.10) and (5.3.8) for the systems $\Sigma_1$ and $\Sigma_2$ that (5.3.8) also holds for the system $\Sigma$ for all $\lambda \in \Omega$ with the above definition of $\hat{\mathcal{S}}(\lambda)$. Thus, by Theorem [5.3.6] $\Omega \subset \rho(S_{i/s/o})$. This implies that $\Omega$ is a jointly frequency i/s/o-admissible domain for $\Sigma_1$, $\Sigma_2$, and $\Sigma$. That these three systems are externally $\Omega$-equivalent follows from Lemma [7.1.18]

(iv) $\Rightarrow$ (i): Assume that (iv) holds. Arguing as above and using (7.1.10) together with the representations (5.3.8) for the systems $\Sigma_1$, $\Sigma_2$, and $\Sigma$ we find that (7.1.11) holds, where $\hat{\mathcal{S}}(\lambda) = \left[ \begin{bmatrix} \hat{A}(\lambda) \hat{B}(\lambda) \\ \hat{C}(\lambda) \hat{D}(\lambda) \end{bmatrix} \right]$ maps $\left[ \begin{bmatrix} \text{gph} (P) \end{bmatrix} \right]$ into $\left[ \begin{bmatrix} \text{gph} (P) \end{bmatrix} \right]$. This together with Lemmas [6.1.23] and [7.1.33] implies that (i) holds.

The final claim follows from Lemmas [6.1.23] and [7.1.33] and (7.1.11). \(\square\)

7.1.35. Lemma (cf. Lemma [6.1.24]). Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W})$ be two resolvable frequency domain s/s systems, and let $\Omega$ be a separately i/s/o-admissible domain for $\Sigma_1$ and $\Sigma_2$. If $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by some multi-valued operator $P$, then they are also $\Omega$-intertwined by the closure of $P$.

**Proof.** If Condition (ii) in Lemma [7.1.34] holds for $P$, then it also holds when $P$ is replaced by the closure of $P$. \(\square\)

7.1.36. Lemma (cf. Lemma [6.1.25]). Let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W})$, $i = 1, 2, 3$, be three frequency domain s/s systems (with the same signal space), and let $\Omega$ be an open set in $\mathbb{C}$.

(i) If $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P_1 \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ and $\Sigma_2$ and $\Sigma_3$ are $\Omega$-intertwined by $P_2 \in \mathcal{ML}(\mathcal{X}_2; \mathcal{X}_3)$, then $\Sigma_1$ and $\Sigma_3$ are $\Omega$-intertwined by $P_3 := P_2P_1 \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_3)$.

(ii) Suppose, in addition, that $\Omega$ is a separately frequency i/s/o-admissible domain for $\Sigma_1$ and $\Sigma_3$. Then $\Sigma_1$ and $\Sigma_3$ are also $\Omega$-intertwined by the closure of $P_3$.\(\square\)
Then the following claims hold:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by some \( P \in \mathcal{ML}(\Sigma_1; \Sigma_2) \) if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent.

(ii) Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are jointly externally \( \Omega \)-equivalent. Let \( \Sigma_{i/s/o}^j = (S_j; X_j, U, Y) \) be \( i/s/o \) representations of \( \Sigma_j \), and let \( \Sigma_{i/s/o} = \Sigma_{i/s/o}^1 \cap \Sigma_{i/s/o}^2 \). Then \( \Sigma \) is the difference connection of \( \Sigma^1 \) and \( \Sigma^2 \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \). Let \( \Sigma = (V; X, W) \) be the \( s/s \) system node with state space \( X = \left[ \begin{array} {c} X_1 \\ X_2 \end{array} \right] \) and generating subspace

\[
V = \left\{ \left[ \begin{array} {c} z_1 \\ x_1 \\ w \end{array} \right] \in \left[ \begin{array} {c} X_1 \\ X_2 \\ W \end{array} \right] \left[ \begin{array} {c} z_2 \\ x_2 \\ w \end{array} \right] \in V_1 \text{ and } \left[ \begin{array} {c} z_2 \\ x_2 \\ w \end{array} \right] \in V_2 \right\}.
\]

Then the following claims are true.

(a) There exists a unique minimal \( P^\Omega_{\min} \in \mathcal{ML}(\Sigma_1; \Sigma_2) \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \), i.e., there exists a unique \( P^\Omega_{\min} \in \mathcal{ML}(\Sigma_1; \Sigma_2) \) such that \( \text{gph}(P^\Omega_{\min}) \subset \text{gph}(P) \) for any other \( P \in \mathcal{ML}(\Sigma_1; \Sigma_2) \). The multi-valued operator \( P^\Omega_{\min} \) coincides with the operator \( P^\Omega_{\min} \) in Theorem 6.1.28 with the system \( \Sigma \) in Theorem 6.1.28 replaced by \( \Sigma_{i/s/o} \).

(b) The closure \( \overline{P^\Omega_{\min}} \) of \( P^\Omega_{\min} \) is the minimal closed multi-valued operator which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( P^\Omega_{\min} \) has the following alternative characterizations:

\[
\text{gph} \left( P^\Omega_{\min} \right) = \bigvee_{\lambda \in \Omega} \text{rng} \left( \begin{bmatrix} \mathcal{B}_i(\lambda) \\ \mathcal{B}_o(\lambda) \end{bmatrix} \right),
\]

where \( \mathcal{B}_i \) is the signal/state resolvents of \( \Sigma_i \), \( i = 1, 2 \).

(c) There exists a unique maximal \( P^\Omega_{\max} \in \mathcal{ML}(\Sigma_1; \Sigma_2) \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \), i.e., there exists a unique \( P^\Omega_{\max} \in \mathcal{ML}(\Sigma_1; \Sigma_2) \) such that \( \text{gph}(P^\Omega_{\max}) \subset \text{gph}(P^\Omega_{\max}) \) for any other \( P \) which \( \Omega \)-intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( P^\Omega_{\max} \) can be described in the following three equivalent ways:

\[
\text{gph} \left( P^\Omega_{\max} \right) \text{ coincides with the set of all possible initial states of all } \Omega \text{-trajectories of } \Sigma;
\]

\[
\text{gph} \left( P^\Omega_{\max} \right) \text{ coincides with the } \Omega \text{-unobservable subspace of } \Sigma_{i/s/o};
\]
(3) The graph of \( P_{\max}^\Omega \) is given by

\[
gph(P_{\max}^\Omega) = \bigcap_{\lambda \in \Omega} \ker \left( [\tilde{\mathcal{C}}_2(\lambda) - \tilde{\mathcal{C}}_1(\lambda)] \right),
\]

where \( \tilde{\mathcal{C}}_i \) is the (multi-valued) state/signal resolvent of \( \Sigma_i \), \( i = 1, 2 \).

In particular, \( P_{\max}^\Omega \) is closed.

Thus, if \( P \) is an arbitrary multi-valued operator which intertwines \( \Sigma_1 \) and \( \Sigma_2 \), then

\[
gph(P_{\min}^\Omega) \subset \gph(P) \subset \gph(P_{\max}^\Omega).
\]

Proof. Proof of (i): We know from Lemma 7.1.32 that if \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \), then they are externally \( \Omega \)-equivalent. The converse part of (i) follows from (ii), which we shall prove next. (The proof of (ii) does not use (i).)

Proof of (ii): Many of the claims in (ii) follow directly from Theorem 6.1.28 and Lemma 7.1.33. This includes the existence of the minimal and maximal \( \Omega \)-intertwining multi-valued operators \( P_{\min}^\Omega \) and \( P_{\max}^\Omega \), and the claim that \( P_{\min}^\Omega \) and \( P_{\max}^\Omega \) are \( \Omega \)-intertwining multi-valued operators. The same argument proves claim (ii)(a), and it also gives the characterization of \( \gph(P_{\min}^\Omega) \) as the \( \Omega \)-reachable subspace of \( \Sigma_1 \) and \( \Sigma_2 \). The alternative formulas (7.1.13) and (7.1.14) follow from the corresponding formulas in Theorem 6.1.28 and Lemmas 7.1.35 and 6.1.7.

From this follows that \( \hat{x}_i \) is an \( \Omega \)-trajectory of \( \Sigma \), and \( \hat{x}_i \) is the minimal closed \( \Omega \)-intertwining multi-valued operator. The same argument proves (ii)(b). It follows from (7.1.12) that (7.1.16)

\[
\hat{\mathcal{E}}(\lambda) = \left\{ \begin{bmatrix} \frac{z_1}{x_1} \\ \frac{z_2}{x_2} \\ w \end{bmatrix} \in \mathcal{X} \middle| \begin{bmatrix} z_1 \\ x_1 \\ w \end{bmatrix} \in \tilde{\mathcal{C}}_1(\lambda) \text{ and } \begin{bmatrix} z_2 \\ x_2 \\ w \end{bmatrix} \in \tilde{\mathcal{C}}_2(\lambda) \right\}, \quad \lambda \in \mathbb{C}.
\]

From this follows that \( \begin{bmatrix} \frac{z_2}{x_2} \\ \frac{z_1}{x_1} \end{bmatrix} \) and \( \begin{bmatrix} \frac{z_1}{x_1} \\ \frac{z_2}{x_2} \end{bmatrix} \) are \( \Omega \)-trajectories of \( \Sigma \), and \( \hat{\mathcal{E}}(\lambda) \) is the characteristic signal bundle of \( \Sigma_i \), \( i = 1, 2 \). Therefore by Lemma 7.1.18 \( \hat{\mathcal{S}}_1(\lambda) = \hat{\mathcal{S}}_2(\lambda) \) where \( \hat{\mathcal{S}}_i \) is the \( \Omega \)-observable subspace of \( \Sigma_i \).

We finally turn to the proof of claim (1) in (ii)(c). Suppose that \( \begin{bmatrix} x_1^\delta \\ x_2^\delta \end{bmatrix} \in \Omega_{\Sigma_{i/s/o}}^\delta \), where \( \Omega_{\Sigma_{i/s/o}}^\delta \) is the \( \Omega \)-unobservable subspace of \( \Sigma_{i/s/o} \), i.e., there exists an \( \Omega \)-unobservable \( \Omega \)-trajectory \( \left( \begin{bmatrix} \frac{z_2}{x_2} \\ \frac{z_1}{x_1} \end{bmatrix}, 0; \begin{bmatrix} x_1^\delta \\ x_2^\delta \end{bmatrix}, 0 \right) \) of \( \Sigma_{i/s/o} \). Let us denote the \( i/s/o \) resolvent matrices of \( \Sigma_{i/s/o}^j \) by \( \hat{\mathcal{E}}_{i/s/o}^j \) and \( \hat{\mathcal{E}}_{i/s/o}^j \) for \( j = 1, 2 \). Then by Lemmas 5.2.35 and 6.1.7 \( \hat{x}_j(\lambda) = \hat{\mathcal{E}}_{i/s/o}^j(\lambda)x_{o}^j, \lambda \in \Omega, j = 1, 2 \). Define \( \hat{y}_j(\lambda) = \hat{\mathcal{E}}_{i/s/o}^j(\lambda)x_{o}^j \),
\[ \lambda \in \Omega, \ j = 1, 2. \] Then \((\hat{x}_j, \hat{y}_j; x_0^j, 0)\) is an \(\Omega\)-trajectory of \(\Sigma_j, \ j = 1, 2, \) and therefore
\[
\left(\begin{array}{c}
\hat{x}_2
\hat{y}_1

\end{array}\right), \ \left(\begin{array}{c}
x_0^2
0
\end{array}\right)
\] is an \(\Omega\)-trajectory of \(\Sigma_{ij/s/o}. \) Since such a trajectory is determined uniquely by its initial state and input function, it follows that \(\hat{y}_2 - \hat{y}_1 = 0, \) i.e., \(\hat{y}_1 = \hat{y}_2. \) Let us denote \(\hat{w} = \hat{y}_1 = \hat{y}_2. \) Then \((\left[\begin{array}{c}
\hat{x}_2
\hat{y}_1

\end{array}\right], \hat{w}; \left[\begin{array}{c}
x_0^2
0
\end{array}\right])\) is an \(\Omega\)-trajectory of \(\Sigma. \) This shows that \(P_{\text{max}} = \Omega_{ij/s/o}^1\) is contained in the set of all possible initial states of all \(\Omega\)-trajectories of \(\Sigma. \)

Suppose next that \(\left[\begin{array}{c}
x_0^2
x_1^2
\end{array}\right]\) is the initial state of some \(\Omega\)-trajectory \((\left[\begin{array}{c}
\hat{x}_2
\hat{y}_1

\end{array}\right], \hat{w}; \left[\begin{array}{c}
x_0^0
x_1^0
\end{array}\right])\) of \(\Sigma. \) Then \((\hat{x}_j, \hat{w}; x_0^j)\) is an \(\Omega\)-trajectory of \(\Sigma_j, \ j = 1, 2, \) and \((\hat{x}_j, \hat{y}_j; x_0^j, \hat{u})\) is an \(\Omega\)-trajectory of \(\Sigma_{ij/s/o}, \ j = 1, 2, \) where \(\hat{u} = P_{ij}^2 \hat{w}\) and \(\hat{y} = P_{ij}^1 \hat{w}. \) By (6.1.2),

\[
\left[\begin{array}{c}
\hat{x}_j(\lambda)
\hat{y}(\lambda)
\end{array}\right] = \left[\begin{array}{c}
\hat{\mathcal{H}}_{ij/s/o}(\lambda) x_0^j
\hat{\mathcal{C}}_{ij/s/o}(\lambda) x_0^j
\end{array}\right] \left[\begin{array}{c}
\hat{\mathcal{D}}_{ij/s/o}(\lambda)\hat{y}(\lambda)
\end{array}\right], \quad \lambda \in \Omega, \ j = 1, 2.
\]

For \(j = 1, 2\) the above trajectory \((\hat{x}_j, \hat{y}_j; x_0^j, \hat{u})\) of \(\Sigma_{ij/s/o}\) can be written as a sum of the two trajectories \((\hat{x}_1^j, \hat{y}_1^j; x_0^j, 0)\) and \((\hat{x}_2^j, \hat{y}_2^j, 0, \hat{u}), \)

\[
\left[\begin{array}{c}
\hat{x}_1^j(\lambda)
\hat{y}_1^j(\lambda)
\end{array}\right] = \left[\begin{array}{c}
\hat{\mathcal{H}}_{ij/s/o}(\lambda) x_0^j
\hat{\mathcal{C}}_{ij/s/o}(\lambda) x_0^j
\end{array}\right] \left[\begin{array}{c}
\hat{\mathcal{D}}_{ij/s/o}(\lambda)\hat{y}(\lambda)
\end{array}\right], \quad \lambda \in \Omega, \ j = 1, 2.
\]

By Lemmas 6.1.11 and 7.1.15, \(\hat{\mathcal{H}}_{ij/s/o}(\lambda) = \hat{\mathcal{D}}_{ij/s/o}(\lambda)\) for all \(\lambda \in \Omega,\) and therefore \(\hat{y}_1^j(\lambda) = \hat{y}_2^j(\lambda)\) for all \(\lambda \in \Omega.\) This implies that also

\[
\hat{y}_1^j(\lambda) = \hat{y}(\lambda) - \hat{y}_2^j(\lambda) = \hat{y}(\lambda) - \hat{y}_1^j(\lambda) = \hat{y}_2^j(\lambda).
\]

Consequently, \((\left[\begin{array}{c}
\hat{x}_2
\hat{y}_1

\end{array}\right], \hat{y}_1^j, -\hat{y}_1^j, 0; \left[\begin{array}{c}
x_0^2
x_1^2
\end{array}\right])\) is an \(\Omega\)-unobservable \(\Omega\)-trajectory of \(\Sigma_{ij/s/o},\) and hence \(\left[\begin{array}{c}
x_0^2
x_1^2
\end{array}\right] \in \Omega_{ij/s/o}^1. \) This proves the claim that \(P_{\text{max}}^\Omega = \Omega_{ij/s/o}^1\) coincides with the set of all possible initial states of all \(\Omega\)-trajectories of \(\Sigma. \)

**7.1.38.** **COROLLARY** (cf. Corollary 6.1.29). Let \(\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}), \) \(i = 1, 2, \) be two resolvable frequency domain s/s systems (with the same signal space), and let \(\Omega\) be a separately frequency i/s/o-admissible domain for \(\Sigma_1\) and \(\Sigma_2.\) Moreover, suppose that both \(\Sigma_1\) and \(\Sigma_2\) are \(\Omega\)-controllable and \(\Omega\)-observable. (According to Theorem 7.1.56 below, this is equivalent to the assumption that both \(\Sigma_1\) and \(\Sigma_2\) are \(\Omega\)-minimal.) Then \(\Sigma_1\) and \(\Sigma_2\) are \(\Omega\)-pseudo-similar if and only if \(\Sigma_1\) and \(\Sigma_2\) are externally \(\Omega\)-equivalent. Among all the \(\Omega\)-pseudo-similarities between \(\Sigma_1\) and \(\Sigma_2\) there is a (unique) minimal one \(P_{\text{min}}^\Omega\) and a (unique) maximal one \(P_{\text{max}}^\Omega,\) namely those defined in Theorem 7.1.37 (both of which in this case are single-valued densely defined injective operators with dense range).

**PROOF.** If \(\Sigma_1\) and \(\Sigma_2\) are \(\Omega\)-pseudo-similar, then it follows from Theorem 7.1.37(i) that \(\Sigma_1\) and \(\Sigma_2\) are externally \(\Omega\)-equivalent.

Conversely, suppose that \(\Sigma_1\) and \(\Sigma_2\) are externally \(\Omega\)-equivalent. By Theorem 7.1.37 \(\Sigma_1\) and \(\Sigma_2\) are \(\Omega\)-intertwined by some \(P \in \mathcal{M}(\mathcal{X}_1; \mathcal{X}_2). \) By Lemma 7.1.32 and the controllability assumption, dom \((P)\) is dense in \(\mathcal{X}_1\) and rng \((P)\) is dense in \(\mathcal{X}_1. \) Furthermore, by Lemma 7.1.32 and the observability assumption, both ker \((P) = \{0\}\) and mul \((P) = \{0\}. \) Thus, \(P\) is both injective and single-valued. \(\square\)
7.1.6. Frequency domain compressions, restrictions, and projections.

7.1.39. **Definition** (cf. Definition 6.1.30). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \), and let \( \Omega \) be an open set in \( \mathbb{C} \). We call \( \Sigma_1 \) a \( \Omega \)-compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \), and we call \( \Sigma_2 \) a \( \Omega \)-dilation of \( \Sigma_1 \) along \( \mathcal{Z}_1 \), if the following two condition holds for all \( \lambda \in \Omega \):

(i) If \((\hat{x}_2, \hat{w}; x_0^0)\) is an \( \Omega \)-trajectory of \( \Sigma_2 \) with \( x_0 \in \mathcal{X}_1 \), then \( P_{\mathcal{X}_1}^{x_0} \hat{x}_2, \hat{w}; x_0^0) \) is an \( \Omega \)-trajectory of \( \Sigma_1 \).

(ii) For each \( \Omega \)-trajectory \((\hat{x}_1, \hat{w}; x_1^0)\) of \( \Sigma_1 \) there exists some \( \Omega \)-trajectory \((\hat{x}_2, \hat{w}; x_1^0)\) of \( \Sigma_2 \) satisfying \( \hat{x}_1 = P_{\mathcal{X}_1}^{x_0} \hat{x}_2 \).

7.1.40. **Remark.** In the above definition we do not require \( \Omega \)-compressions to preserve regularity in the sense that even in the case where the s/s node \( \Sigma_2 \) in Definition 7.1.39 is regular we do not require the node \( \Sigma_1 \) in Definition 7.1.39 to be regular. This is important in the study of \( \Omega \)-minimality of frequency domain s/s systems. See also Remarks 7.1.53 and 7.1.55 below.

7.1.41. **Lemma** (cf. Lemma 6.1.33). If the frequency domain s/s system \( \Sigma_1 \) is the \( \Omega \)-compression of the frequency domain s/s system \( \Sigma_2 \), then \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent.

**Proof.** This follows immediately from Definitions 7.1.14 and 7.1.39.

7.1.42. **Lemma** (cf. Lemma 6.1.34). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \) be three frequency domain s/s systems (with the same signal space), and let \( \Omega \) be an open set in \( \mathbb{C} \). If \( \Sigma_2 \) is the \( \Omega \)-compression of \( \Sigma_3 \) onto \( \mathcal{X}_2 \) along \( \mathcal{Z}_2 \), and if \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_3 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \), then \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_3 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 + \mathcal{Z}_2 \).

**Proof.** This follows from Definition 7.1.39 since \( P_{\mathcal{X}_1}^{x_1} + P_{\mathcal{X}_2}^{x_2} \).

7.1.43. **Lemma** (cf. Lemma 6.1.35). Let the frequency domain s/s system \( \Sigma_1 \) be the \( \Omega \)-compression of the s/s system \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \). For \( i = 1, 2 \) we denote the exactly \( \Omega \)-reachable subspace of \( \Sigma_i \) by \( \mathfrak{R}_i^{\Omega, \text{exact}} \), the \( \Omega \)-reachable subspace of \( \Sigma_i \) by \( \mathfrak{R}_i^{\Omega} \), and the \( \Omega \)-unobservable subspace of \( \Sigma_i \) by \( \mathfrak{U}_i^{\Omega} \). Then

\[
(7.1.17) \quad \mathfrak{R}_1^{\Omega, \text{exact}} = P_{\mathcal{X}_1}^{x_1} \mathfrak{R}_2^{\Omega, \text{exact}}, \quad \mathfrak{R}_1^{\Omega} = P_{\mathcal{X}_1}^{x_1} \mathfrak{R}_2^{\Omega}, \quad \mathfrak{U}_1^{\Omega} = \mathfrak{U}_2^{\Omega} \cap \mathcal{X}_1.
\]

In particular, if \( \Sigma_2 \) is \( \Omega \)-controllable or \( \Omega \)-observable, then so is \( \Sigma_1 \).

**Proof.** The proof is analogous to the proof of Lemmas 1.5.32.

7.1.44. **Definition** (cf. Definition 6.1.36). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), and let \( \Omega \) be an open set in \( \mathbb{C} \). We call \( \Sigma_1 \) a \( \Omega \)-restriction of \( \Sigma_2 \) to \( \mathcal{X}_1 \) if the following two condition holds for all \( \lambda \in \Omega \):

(i) Every \( \Omega \)-trajectory of \( \Sigma_1 \) is also an \( \Omega \)-trajectory of \( \Sigma_2 \).

(ii) For each \( \Omega \)-trajectory \((\hat{x}_2, \hat{w}; x_0^0)\) of \( \Sigma_2 \) with \( x_0 \in \mathcal{X}_1 \), then \( \hat{x}_2(\lambda) \in \mathcal{X}_1 \) for all \( \lambda \in \Omega \), and \((\hat{x}_2, \hat{w}; x_0^0)\) is also an \( \Omega \)-trajectory of \( \Sigma_1 \).

7.1.45. **Lemma** (cf. Lemma 6.1.37). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \). Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X} \). Then the following conditions are equivalent:
(i) \( \Sigma_1 \) is an \( \Omega \)-restriction of \( \Sigma_2 \) to \( \mathcal{X}_1 \).

(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by the embedding operator \( \mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \).

(iii) \( \mathcal{X}_1 \) is a strongly \( \Omega \)-invariant subspace for \( \Sigma_2 \), and \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

**Proof.** The proof is analogous to the proof of Lemma 1.5.36 \( \square \)

7.1.46. Definition (cf. Definition 6.1.38). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \), and let \( \Omega \) be an open set in \( \mathbb{C} \). We call \( \mathcal{X}_1 \) an \( \Omega \)-projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) if the following two conditions hold for all \( \lambda \in \Omega \):

(i) If \( (\hat{x}_2, \hat{w}; x_2^0) \) is an \( \Omega \)-trajectory of \( \Sigma_2 \), then \( (P_{\mathcal{X}_1}^2 \hat{x}_2, \hat{w}; P_{\mathcal{X}_1}^2 x_2^0) \) is an \( \Omega \)-trajectory of \( \Sigma_1 \).

(ii) If \( (\hat{x}_1, \hat{w}; x_1^0) \) is an \( \Omega \)-trajectory of \( \Sigma_1 \), then for each \( x_2^0 \in \mathcal{X}_2 \) satisfying \( P_{\mathcal{X}_1}^2 x_2^0 = x_1^0 \), there exists an \( \Omega \)-trajectory \( (\hat{x}_2, \hat{w}; x_2^0) \) of \( \Sigma_2 \) satisfying \( P_{\mathcal{X}_1}^2 x_2 = x_1 \).

7.1.47. Lemma (cf. Lemma 6.1.39). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two frequency domain s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) is an \( \Omega \)-projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(ii) \( \Sigma_2 \) and \( \Sigma_1 \) are \( \Omega \)-intertwined by the projection operator \( P_{\mathcal{X}_1}^2 \).

(iii) \( \mathcal{Z}_1 \) is an \( \Omega \)-unobservably \( \Omega \)-invariant subspace for \( \Sigma_2 \), and \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

**Proof.** The proof is analogous to the proof of Lemma 1.5.40 \( \square \)

7.1.48. Lemma. Let \( \Sigma_j = (V_j; \mathcal{X}_j, \mathcal{W}) \), \( j = 1, 2 \), be two frequency domain s/s systems (with the same signal space), where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \) with a direct complement \( \mathcal{Z}_1 \) in \( \mathcal{X}_2 \). Let \( \Sigma^j_{i/s/o} = (S_j; \mathcal{X}_j, \mathcal{U}, \mathcal{Y}) \) be i/s/o representations of \( \Sigma_j \), \( j = 1, 2 \) (with the same input and output spaces), and let \( \Omega \) be an open set in \( \mathbb{C} \). Then

(i) \( \Sigma_1 \) is an \( \Omega \)-compression of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) if and only if \( \Sigma^1_{i/s/o} \) is an \( \Omega \)-compression of \( \Sigma^2_{i/s/o} \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(ii) \( \Sigma_1 \) is an \( \Omega \)-restriction of \( \Sigma_2 \) to \( \mathcal{X}_1 \) if and only if \( \Sigma^1_{i/s/o} \) is an \( \Omega \)-restriction of \( \Sigma^2_{i/s/o} \) to \( \mathcal{X}_1 \), and

(iii) \( \Sigma_1 \) is an \( \Omega \)-projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) if and only if \( \Sigma^1_{i/s/o} \) is an \( \Omega \)-projection of \( \Sigma^2_{i/s/o} \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

**Proof.** This follows from Definitions 6.1.30 6.1.36 6.1.38 7.1.39 7.1.44 7.1.46 7.1.49. Lemma (cf. Lemma 6.1.42). Let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \), \( i = 1, 2 \), be two resolvable frequency domain s/s systems (with the same signal space) where \( \mathcal{X}_1 \) is a closed subspace of \( \mathcal{X}_2 \), and let \( \mathcal{Z}_1 \) be a direct complement to \( \mathcal{X}_1 \) in \( \mathcal{X}_2 \). Suppose further that \( \Omega \) is a separately frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \).
(i) If $\Sigma_1$ is an $\Omega$-compression of $\Sigma_2$ onto $X_1$ along $Z_1$, or an $\Omega$-restriction of $\Sigma_2$ to $X_1$, or an $\Omega$-projection of $\Sigma_2$ onto $X_1$ along $Z_1$, then $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent.

(ii) Suppose that $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent. Then
\begin{itemize}
  \item[(a)] conditions (i) and (ii) in Definition 7.1.39 are equivalent to each other,
  \item[(b)] conditions (i) and (ii) in Definition 7.1.44 are equivalent to each other, and
  \item[(c)] conditions (i) and (ii) in Definition 7.1.46 are equivalent to each other.
\end{itemize}

\textbf{Proof.} (i) That (i) is true follows from Lemmas 7.1.41, 7.1.45, and 7.1.47.

(ii) Since $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent we may assume without loss of generality that $(U_1, \mathcal{Y}_1) = (U_2, \mathcal{Y}_2)$. Let $\Sigma_{ij}^i$ be the $i/s/o$ representations of $\Sigma_j$, $j = 1, 2$, corresponding to this common decomposition of the signal space $W$. It follows from Lemma 7.1.7 that condition (i) or (ii) in Definition 7.1.39 holds for $\Sigma_1$ and $\Sigma_2$ if and only if condition (i) respectively (ii) in Definition 6.1.36 holds for $\Sigma_{ij}^i$ respectively $\Sigma_{ij}^2$, and hence (a) follows from part (i) of Lemma 6.1.42. In the same way claims (b) and (c) follow from claims follow from parts (ii) and (iii) of Lemma 6.1.42.

\section{Corollary} (cf. Corollary 6.1.43. Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two resolvable frequency domain $s/s$ systems (with the same signal space) where $X_1$ is a closed subspace of $X_2$, and let $Z_i$ be a direct complement to $X_1$ in $X_2$. Suppose further that $\Omega$ is a separately frequency $i/s/o$-admissible domain for $\Sigma_1$ and $\Sigma_2$, and that $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent. Then
\begin{itemize}
  \item[(i)] $\Sigma_1$ is the $\Omega$-restriction of $\Sigma_2$ to $X_1$ if and only if every $\Omega$-trajectory of $\Sigma_1$ is also an $\Omega$-trajectory of $\Sigma_2$, and
  \item[(ii)] $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ onto $X_1$ along $Z_1$ if and only if $\langle (P_{X_1}^2, \hat{x}_2, \hat{w}; P_{X_1}^0, x_2^0) \rangle$ is an $\Omega$-trajectory of $\Sigma_1$ whenever $(\hat{x}_2, \hat{w}; x_2^0)$ is an $\Omega$-trajectory of $\Sigma_2$.
\end{itemize}

\textbf{Proof.} This follows immediately from Lemma 7.1.49.

\section{Lemma} (cf. Lemma 6.1.44. Let $\Sigma_i = (V_i; X_i, W)$, $i = 1, 2$, be two resolvable frequency domain $s/s$ systems (with the same signal space), and suppose that $X_1$ is a closed subspace of $X_2$ with a direct complement $Z_1$ in $X_2$. Let $\mathcal{E}_i$, respectively $\hat{\mathcal{E}}_i$ be the the characteristic node and signal bundles of $\Sigma_i$, and let $\mathcal{L}_i$, $\mathcal{A}_i$, $\mathcal{B}_i$, and $\hat{\mathcal{E}}_i$ be the state-signal/state, unobservable state/state, signal/state, and state/signal resolvents of $\Sigma$. Finally, let $\Omega$ be a separately frequency $i/s/o$-admissible domain for $\Sigma_1$ and $\Sigma_2$. Then the following claims are true.
\begin{itemize}
  \item[(i)] $\Sigma_1$ is the $\Omega$-restriction of $\Sigma_2$ onto $X_1$ if and only if
\begin{equation}
\hat{\mathcal{E}}_1(\lambda) = \left( \hat{\mathcal{E}}_2(\lambda) \cap \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right)
\end{equation}
for all $\lambda \in \Omega$, or equivalently, if and only if
\begin{equation}
\mathcal{L}_1(\lambda) = \mathcal{L}_2(\lambda)[X_1 \mid W]
\end{equation}
\end{itemize}
for all $\lambda \in \Omega$. These equations imply that
\begin{equation}
(7.1.19) \quad \hat{A}_1(\lambda) = \hat{A}_2(\lambda)|_{X_1}, \quad \hat{B}_1(\lambda) = \hat{B}_2(\lambda),
\end{equation}
\begin{equation}
(7.1.20a) \quad \hat{C}_1(\lambda) = \hat{C}_2(\lambda)|_{X_1}, \quad \hat{F}_1(\lambda) = \hat{F}_2(\lambda),
\end{equation}
for all $\lambda \in \Omega$.

(ii) $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ onto $X_1$ along $Z_1$ if and only if
\begin{equation}
(7.1.20b) \quad \hat{E}_1(\lambda) = \begin{bmatrix} P_{X_1} & 0 & 0 & 0 \\ 0 & P_{X_1} & 0 & P_{X_1} \\ 0 & 0 & 1_{1_{\overline{W}}} \end{bmatrix} \hat{E}_2(\lambda)
\end{equation}
for all $\lambda \in \Omega$, or equivalently, if and only if
\begin{equation}
(7.1.21) \quad \hat{E}_1(\lambda) = \begin{bmatrix} P_{X_1} & 0 & 0 & 0 \\ 0 & P_{X_1} & 0 & P_{X_1} \\ 0 & 0 & 1_{1_{\overline{W}}} \end{bmatrix} \hat{E}_2(\lambda)
\end{equation}
for all $\lambda \in \Omega$. These equations imply that
\begin{equation}
(7.1.22a) \quad \hat{E}_1(\lambda) = \begin{bmatrix} 1_{X_1} & 0 & 0 & 0 \\ 0 & P_{X_1} & 0 & P_{X_1} \\ 0 & 0 & 1_{1_{\overline{W}}} \end{bmatrix} \left( \hat{E}_2(\lambda) \cap \begin{bmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{bmatrix} \right)
\end{equation}
for all $\lambda \in \Omega$, or equivalently, if and only if
\begin{equation}
(7.1.22b) \quad \hat{E}_1(\lambda) = P_{X_1} \hat{E}_2(\lambda)|_{X_1}
\end{equation}
for all $\lambda \in \Omega$. These equations imply that
\begin{equation}
(7.1.23) \quad \hat{E}_1(\lambda) = \hat{E}_2(\lambda)|_{X_1}, \quad \hat{F}_1(\lambda) = \hat{F}_2(\lambda)
\end{equation}
for all $\lambda \in \Omega$.

PROOF. Let us begin by showing that the two different versions of (7.1.18), (7.1.20), and (7.1.22) are equivalent to each other. According to Lemma 5.3.21, for all $\lambda \in \Omega$ and $i = 1, 2$,
\[ \text{gph} \left( \hat{E}_i(\lambda) \right) = \begin{bmatrix} 0 & 1_{X_i} & 0 \\ 1_{X_i} & 0 & 0 \\ 0 & 0 & 1_{1_{\overline{W}}} \end{bmatrix} \hat{E}_i(\lambda), \]
and therefore conditions (7.1.18a), (7.1.20a), and (7.1.22a) are equivalent to the following respective conditions

\[(7.1.24)\]  
\[\text{gph}\left(\tilde{E}_1(\lambda)\right) = \text{gph}\left(\tilde{E}_2(\lambda)\right) \cap \left[\mathcal{X}_2 \atop \mathcal{W}\right],\]

\[(7.1.25)\]  
\[\text{gph}\left(\tilde{E}_1(\lambda)\right) = \begin{bmatrix} P_{\mathcal{X}_1} & 0 & 0 \\ 0 & P_{\mathcal{X}_1} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \text{gph}\left(\tilde{E}_2(\lambda)\right),\]

\[(7.1.26)\]  
\[\text{gph}\left(\tilde{E}_1(\lambda)\right) = \begin{bmatrix} P_{\mathcal{X}_1} & 0 & 0 \\ 0 & 1_{\mathcal{X}_1} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(\text{gph}\left(\tilde{E}_2(\lambda)\right) \cap \left[\mathcal{X}_2 \atop \mathcal{W}\right]\right).\]

These conditions are equivalent to conditions (7.1.18b), (7.1.20b), and (7.1.22b), respectively. Thus, the two different versions of (7.1.18), (7.1.20), and (7.1.22) are equivalent to each other.

Next we show that (7.1.18), (7.1.20), and (7.1.22) imply (7.1.19), (7.1.21), and (7.1.23), respectively. The formulas for \(\tilde{F}_1\), \(\tilde{E}_1\), and \(\tilde{C}_1\) in (7.1.19), (7.1.21), and (7.1.23) follow from Lemma 5.3.21 and (7.1.18b), (7.1.20b), and (7.1.22b). The formulas for \(\tilde{F}_1(\lambda)\) and \(\tilde{C}_1(\lambda)\) in (7.1.21) imply that \(\tilde{F}_1(\lambda) = P_{\mathcal{X}_1} \tilde{F}_2(\lambda)|_{\mathcal{X}_1}\) and \(\tilde{C}_1(\lambda) = \tilde{C}_2(\lambda)|_{\mathcal{X}_1}\). Thus, (7.1.21) implies (7.1.23). Formulas (7.1.19) and (7.1.23) implies that the multi-valued parts of \(\tilde{F}_1(\lambda)\) and \(\tilde{F}_2(\lambda)\) are the same. These multi-valued parts are \(\tilde{S}_1(\lambda)\) respectively \(\tilde{S}_2(\lambda)\), and consequently \(\tilde{S}_1(\lambda) = \tilde{S}_2(\lambda)\).

We finally turn to the proofs of the main claims in (i)–(iii). If \(\Sigma_{i/s/o}^1\) is the \(\Omega\)-restriction of \(\Sigma_{i/s/o}^2\) to \(\mathcal{X}_1\), or the \(\Omega\)-projection or \(\Omega\)-compression of \(\Sigma_{i/s/o}^2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\), then by Lemma 7.1.49 \(\Sigma_1\) and \(\Sigma_2\) are externally \(\Omega\)-equivalent. On the other hand, if (7.1.18), (7.1.20), or (7.1.22), then as we have shown above, \(\tilde{S}_1(\lambda) = \tilde{S}_2(\lambda)\) for all \(\lambda \in \Omega\). Thus, in both cases we can apply Lemma 7.1.19 to conclude that \(\Omega\) is a jointly frequency i/s/o-admissible domain for \(\Sigma_1\) and \(\Sigma_2\), which means that \(\Sigma_j\) has an \(\Omega\)-resolvable i/s/o representation \(\Sigma_{i/s/o}^j = (S_j, \mathcal{X}_j, \mathcal{U}, \mathcal{Y})\), \(j = 1, 2\) (with the same input and output spaces). By Lemma 7.1.48, the conditions

(a) \(\Sigma_1\) is the \(\Omega\)-restriction of \(\Sigma_2\) to \(\mathcal{X}_1\),
(b) \(\Sigma_1\) is the \(\Omega\)-projection of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\),
(c) \(\Sigma_1\) is the \(\Omega\)-compression of \(\Sigma_2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\),

for the two frequency domain s/s systems is equivalent to the respective conditions

(a') \(\Sigma_{i/s/o}^1\) is the \(\Omega\)-restriction of \(\Sigma_{i/s/o}^2\) to \(\mathcal{X}_1\),
(b') \(\Sigma_{i/s/o}^1\) is the \(\Omega\)-projection of \(\Sigma_{i/s/o}^2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\),
(c') \(\Sigma_{i/s/o}^1\) is the \(\Omega\)-compression of \(\Sigma_{i/s/o}^2\) onto \(\mathcal{X}_1\) along \(\mathcal{Z}_1\),

for the corresponding frequency domain i/s/o systems. However, by Lemmas 6.1.41 and 5.3.28, the conditions (a'), (b'), and (c') are equivalent to (7.1.18b), (7.1.20b), or (7.1.22b), respectively.

7.1.52. Theorem (cf Theorem 6.1.44). Let \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) be a resolvable frequency domain s/s system, let \(\Omega\) be a frequency i/s/o-admissible domain for \(\Sigma\), and let \(\mathcal{X} = \mathcal{X}_1 \uplus \mathcal{Z}_1\) be a direct sum decomposition of \(\mathcal{X}\).
(i) If $\Sigma$ has an $\Omega$-restriction $\Sigma_1$ to $\mathcal{X}_1$, then $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma$. Conversely, if $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma$, then $\Sigma$ has a resolvable $\Omega$-restriction $\Sigma_1$ to $\mathcal{X}_1$ with the property that $\Omega$ is a jointly frequency i/s/o-admissible domain for $\Sigma_1$ and $\Sigma$. The above $\Omega$-restriction $\Sigma_1$ is unique within the class of all $\Omega$-resolvable $\Omega$-restrictions of $\Sigma$ to $\mathcal{X}_1$.

(ii) If $\Sigma$ has an $\Omega$-projection $\Sigma_1$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, then $\mathcal{Z}_1$ is unobservably $\Omega$-invariant for $\Sigma$. Conversely, if $\mathcal{Z}_1$ is unobservably $\Omega$-invariant for $\Sigma$, then $\Sigma$ has a resolvable $\Omega$-projection $\Sigma_1$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ with the property that $\Omega$ is a jointly frequency i/s/o-admissible domain for $\Sigma_1$ and $\Sigma$. The above $\Omega$-projection $\Sigma_1$ is unique within the class of all $\Omega$-resolvable $\Omega$-projections of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$.

Proof. By Lemmas 7.1.45 and 7.1.47, if $\Sigma$ has an $\Omega$-restriction to $\mathcal{X}_1$, then $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma$, and if $\Sigma$ has an $\Omega$-projection onto $\mathcal{X}_1$ along $\mathcal{Z}_1$, then $\mathcal{Z}_1$ is unobservably $\Omega$-invariant for $\Sigma$.

Suppose next that $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma$. Let $\Sigma_{i/s/o}^2 = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of $\Sigma$ satisfying $\Omega \subset \rho(\Sigma_{i/s/o}^2)$. By Lemma 7.1.26, $\mathcal{X}_1$ is strongly $\Omega$-invariant for $\Sigma_{i/s/o}^2$, and hence by Theorem 6.1.44, $\Sigma_{i/s/o}^2$ has a restriction $\Sigma_{i/s/o}^1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ to $\mathcal{X}_1$ satisfying $\Omega \subset \rho(\Sigma_{i/s/o}^0)$. By Lemma 6.1.41, the i/s/o resolvent matrices of $\Sigma_{i/s/o}^1$ and $\Sigma_{i/s/o}^2$ satisfy the conditions (6.1.12b). Let $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ be the s/s node whose generating subspace $V_1$ is given by (2.2.46) with $S$ and $V$ replaced by $S_1$ and $V_1$, so that $\Sigma_{i/s/o}^1$ is the i/s/o representation of $\Sigma_1$ corresponding to the i/o decomposition $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$. Then it follows from (6.1.12b) and (5.3.8a) that (7.1.18a) holds. Consequently, by Lemma 7.1.51, $\Sigma_1$ is the $\Omega$-restriction of $\Sigma$. The final uniqueness claim in (i) follows from Lemma 7.1.51.

The proof of the remaining part of (ii) is analogous to the proof given above, with (6.1.12b) replaced by (6.1.13b) and (7.1.18a) replaced by (7.1.20a).

7.1.53. Remark. In the above theorem we do not claim that the $\Omega$-restriction $\Sigma_1$ in (i) and $\Omega$-projection $\Sigma_1$ in (ii) are regular, even in the case where the s/s node $\Sigma$ in Theorem 7.1.52 is regular.

7.1.54. Definition (cf. Definition 6.2.5). Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be a resolvable frequency domain s/s system, and let $\Omega$ be an open subset $\rho(\Sigma)$.

(i) $\Sigma$ is called $\Omega$-minimal if $\Sigma$ does now have any (non-trivial) resolvable $\Omega$-comression $\Sigma_1$ satisfying $\Omega \subset \rho(\Sigma_1)$.

(ii) By a $\Omega$-minimal compression of $\Sigma$ we mean an $\Omega$-comression $\Sigma_1$ of $\Sigma$ satisfying $\Omega \subset \rho(\Sigma_1)$ which is $\Omega$-minimal (i.e., $\Sigma_1$ does not have any further non-trivial $\Omega$-comression $\Sigma_2$ which satisfies $\Omega \subset \rho(\Sigma_2)$).

7.1.55. Remark. Observe that in the above definition, even in the case where the s/s node $\Sigma_1$ is regular, we do not require the possible nontrivial $\Omega$-compressions to be regular, i.e., in order for $\Sigma$ to be minimal it must not have any (regular or non-regular) nontrivial $\Omega$-compression. The reason for this that it is not known to what extent Theorem 7.1.56 below is valid for regular s/s nodes if we require an $\Omega$-compression of a regular s/s node to be regular.

7.1.56. Theorem (cf. Theorem 6.2.7). Let $\Sigma = (V, \mathcal{X}, \mathcal{W})$ be a resolvable frequency domain s/s system, and let $\Omega$ be a frequency i/s/o-admissible domain for $\Sigma$. Then $\Sigma$ is $\Omega$-minimal if and only if $\Sigma$ is both $\Omega$-controllable and $\Omega$-observable.
Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o representation of \( \Sigma \) satisfying \( \Omega \subset \rho(\Sigma_{i/s/o}) \), and apply Theorem 6.2.7 and Lemmas 7.1.26 and 7.1.48.

7.1.7. The general structure of a frequency domain compression.

7.1.57. Lemma (cf. Lemma 6.1.46). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable frequency domain s/s system with state-signal/state resolvent \( \hat{\mathcal{A}} \), unobservable state/state resolvent \( \hat{\mathcal{B}} \), and state/signal resolvent \( \hat{\mathcal{C}} \), let \( \mathcal{X} = \mathcal{X}_1 + \mathcal{Z}_1 \), and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \). Let \( \Sigma_{\text{ext}} = (V_{\text{ext}}; \mathcal{X}, \begin{bmatrix} \mathcal{W} \\ \mathcal{X}_1 \end{bmatrix}) \) be the i/o extension of \( \Sigma \) with control operator equal to the embedding operator \( \mathcal{I}_{\mathcal{X}_1} : \mathcal{X}_1 \hookrightarrow \mathcal{X} \), observation operator \( P_{\mathcal{Z}_1}^{\mathcal{X}_1} \), and feedthrough operator zero, i.e.,

\[
V_{\text{ext}} = \left\{ \begin{bmatrix} z + u_1 \\ x \\ w \\ z_1 \\ P_{\mathcal{Z}_1}^{\mathcal{X}_1} x \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \\ \mathcal{X}_1 \\ \mathcal{X}_1 \end{bmatrix} \mid \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, u_1 \in \mathcal{X}_1 \right\}.
\]

(i) There exists a (unique) minimal closed strongly \( \Omega \)-invariant subspace \( \mathcal{X}^\Omega_{\text{min}} \) for \( \Sigma \) which contains \( \mathcal{X}_1 \) (i.e., \( \mathcal{X}^\Omega_{\text{min}} \) is closed and strongly \( \Omega \)-invariant for \( \Sigma \), and \( \mathcal{X}^\Omega_{\text{min}} \) is contained in every other closed strongly \( \Omega \)-invariant subspace of \( \Sigma \) which contains \( \mathcal{X}_1 \)). This subspace has the following alternative descriptions:

(a) \( \mathcal{X}^\Omega_{\text{min}} = \mathcal{X}_1 \vee \mathcal{M}^\Omega_{\text{ext}} \) where \( \mathcal{M}^\Omega_{\text{ext}} \) is the \( \Omega \)-reachable subspace of \( \Sigma_{\text{ext}} \);

(b) \( \mathcal{X}^\Omega_{\text{min}} \) is equal to the subspace \( \mathcal{X}^\Omega_{\text{min}} \) in Lemma 6.1.46 with \( \Sigma \) replaced by an arbitrary i/s/o representation \( \Sigma_{i/s/o} \) of \( \Sigma \) satisfying \( \Omega \subset \rho(\Sigma_{i/s/o}) \);

(c) \( \mathcal{X}^\Omega_{\text{min}} \) is given by

\[
\mathcal{X}^\Omega_{\text{min}} = \mathcal{X}_1 \bigvee_{\lambda \in \Omega} \left[ 0 \ 1_{\mathcal{X}} \ 0 \right] \left( \hat{\mathcal{C}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \right);
\]

(d) \( \mathcal{X}^\Omega_{\text{min}} \) is given by

\[
\mathcal{X}^\Omega_{\text{min}} = \mathcal{X}_1 \bigvee_{\lambda \in \Omega} \text{rng} \left( \hat{\mathcal{C}}(\lambda) \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{W} \end{bmatrix} \right).
\]

(ii) The space \( \mathcal{X}^\Omega_{\text{min}} \) has a direct sum decomposition \( \mathcal{X}^\Omega_{\text{min}} = \mathcal{X}_1 + \mathcal{Z}^\Omega_{\text{min}} \), where

\[
\mathcal{Z}^\Omega_{\text{min}} = \mathcal{X}^\Omega_{\text{min}} \cap \mathcal{Z}_1 = P_{\mathcal{Z}_1}^{\mathcal{X}_1} \mathcal{X}^\Omega_{\text{min}} = P_{\mathcal{Z}_1}^{\mathcal{X}_1} \mathcal{M}^\Omega_{\text{ext}}.
\]

The subspace \( \mathcal{Z}^\Omega_{\text{min}} \) has the following alternative descriptions:

(a) \( \mathcal{Z}^\Omega_{\text{min}} \) is given by

\[
\mathcal{Z}^\Omega_{\text{min}} = \bigvee_{\lambda \in \Omega} \left[ 0 \ P_{\mathcal{Z}_1}^{\mathcal{X}_1} \ 0 \right] \left( \hat{\mathcal{C}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \right);
\]

(b) \( \mathcal{Z}^\Omega_{\text{min}} \) is given by

\[
\mathcal{Z}^\Omega_{\text{min}} = \bigvee_{\lambda \in \Omega} \text{rng} \left( P_{\mathcal{Z}_1}^{\mathcal{X}_1} \hat{\mathcal{C}}(\lambda) \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{W} \end{bmatrix} \right).
\]
(iii) There exists a (unique) maximal unobservably $\Omega$-invariant subspace $Z_{\Omega}^{\text{max}}$ for $\Sigma$ which is contained in $Z_1$ (i.e., $Z_{\Omega}^{\text{max}}$ is observably $\Omega$-invariant for $\Sigma$, and $Z_{\Omega}^{\text{max}}$ contains every other unobservably $\Omega$-invariant subspace for $\Sigma$ which is contained in $Z_1$). This subspace has the following alternative descriptions:

(a) $Z_{\Omega}^{\text{max}}$ is the $\Omega$-unobservable subspace of $\Sigma_{\text{ext}}$;
(b) $Z_{\Omega}^{\text{max}}$ is equal to the subspace $Z_{\Omega}^{\text{max}}$ in Lemma 7.1.22 with $\Sigma$ replaced by an arbitrary i/s/o representation $\Sigma_{i/s/o}$ of $\Sigma$ satisfying $\Omega \subset \rho(\Sigma_{i/s/o})$.
(c) $Z_{\Omega}^{\text{max}}$ is given by

\[(7.1.31a) \quad Z_{\Omega}^{\text{max}} = \bigcap_{\lambda \in \Omega} \left\{ Z_1 \middle| \mathbf{1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{E}(\lambda) & \hat{Z}_1 \mathbf{1} \end{bmatrix} \right\}; \]

(d) $Z_{\Omega}^{\text{max}}$ is given by

\[(7.1.31b) \quad Z_{\Omega}^{\text{max}} = \bigcap_{\lambda \in \Omega} \ker \left( \left[ \begin{array}{c} \hat{p}_{\lambda} \hat{Z}_1 \hat{A}(\lambda) \\ \hat{\mathcal{E}}(\lambda) \end{array} \right] \right) \bigcap Z_1. \]

In particular, $Z_{\Omega}^{\text{max}}$ is closed.

**Proof.** This follows from Definition 7.1.9 and Lemmas 5.3.21, 6.1.46, and 7.1.22. \(\square\)

7.1.58. **Theorem** (cf. Theorem 6.1.48). Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a resolvable frequency domain s/s system, let $\mathcal{X} = X_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}$, and let $\Omega$ be an frequency i/s/o-admissible domain for $\Sigma$. Let $\Lambda_{\text{min}}^{\Omega}$ be the minimal strongly $\Omega$-invariant subspace of $\Sigma$ which contains $X_1$, let $Z_{\Omega}^{\text{max}}$ be the maximal unobservably $\Omega$-invariant subspace of $\Sigma$ which is contained in $Z_1$, and let $Z_{\Omega}^{\text{min}} = \Lambda_{\text{min}}^{\Omega} \cap Z_1$ (cf. Lemma 7.1.57). Then the following conditions are equivalent:

(i) There exists a (unique) $\Omega$-resolvable $\Omega$-compression $\Sigma_1$ of $\Sigma$ onto $X_1$ along $Z_1$.
(ii) $Z_1$ contains some closed subspace $Z_{\Omega}^{\text{max}}$ such that $Z_{\Omega}^{\text{max}}$ is unobservably $\Omega$-invariant for $\Sigma$ and $X_1 + Z_{\Omega}^{\text{max}}$ is strongly $\Omega$-invariant for $\Sigma$.
(iii) $Z_{\Omega}^{\text{min}}$ is unobservably $\Omega$-invariant for $\Sigma$.
(iv) $X + Z_{\Omega}^{\text{max}}$ is strongly $\Omega$-invariant for $\Sigma$.
(v) $Z_{\Omega}^{\text{min}} \subset Z_{\Omega}^{\text{max}}$.

The set $\Omega$ is a jointly frequency domain i/s/o-admissible domain for $\Sigma$ and $\Sigma_1$. Two possible choices of the subspace $Z_{\Omega}^{\text{max}}$ in (ii) are $Z_{\Omega}^{\text{min}} = Z_{\Omega}^{\text{min}}$ and $Z_{\Omega}^{\text{max}} = Z_{\Omega}^{\text{max}}$, and every possible subspace $Z_{\Omega}^{\text{max}}$ in (ii) satisfies $Z_{\Omega}^{\text{min}} \subset Z_{\Omega}^{\text{min}} \subset Z_{\Omega}^{\text{max}}$.

**Proof.** Let $\Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o representation of $\Sigma$ satisfying $\Omega \subset \rho(\Sigma_{i/s/o})$, and denote the i/s/o resolvent matrix of $\Sigma_{i/s/o}$ by $\hat{\Sigma}_{i/s/o} = \begin{bmatrix} \hat{A}_{i/s/o} & \hat{B}_{i/s/o} \\ \hat{C}_{i/s/o} & \hat{D}_{i/s/o} \end{bmatrix}$. It follows from the respective assertions in Theorem 6.1.48 combined with Lemmas 7.1.26 and 7.1.48 that all the claims in Theorem 7.1.58 are true if we define $Z_{\Omega}^{\text{min}}$ and $Z_{\Omega}^{\text{max}}$ as in (6.1.17) and (6.1.18) with $\hat{\Sigma}$ replaced by $\hat{\Sigma}_{i/s/o}$. As we saw in the proof of Lemma 7.1.57 above, the formulas (6.1.17) and (6.1.18) (applied to $\Sigma_{i/s/o}$) are equivalent to (7.1.30) respectively (7.1.31). \(\square\)
7.1.59. **Corollary** (cf. Corollary 6.1.49). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable frequency domain s/s system with signal-state/state resolvent \( \mathcal{K} \) and unobservable state/state resolvent \( \hat{\mathcal{K}} \), let \( \mathcal{X} = X_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \), and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \). Then \( \Sigma \) has an \( \Omega \)-resolvable \( \Omega \)-compression \( \Sigma_1 = (V_1; X_1, W) \) onto \( X_1 \) along \( Z_1 \) with signal-state/state resolvent \( \hat{\mathcal{K}}_1 \) if and only if \( \Sigma_1 \) has a direct sum decomposition \( Z_1 = Z_1^\Omega + Z_c \) such that the following conditions hold for all \( \lambda \in \Omega \):

\[
\text{rng} \left( \mathcal{H}(\lambda)|_{[X_1 + Z_1^\Omega]} \right) \subseteq X_1 + Z_1^\Omega,
\]

\[
Z_1^\Omega \subseteq \ker (\mathcal{H}(\lambda)) \text{ and } \mathcal{K}(\lambda) Z_1^\Omega \subseteq Z_1^\Omega,
\]

\[
\mathcal{K}_1(\lambda) = P_{X_1^\Omega} \mathcal{H}(\lambda)|_{[X_1^\Omega]}.
\]

**Proof.** This follows from Lemma 7.1.27 and Theorem 7.1.58. \( \square \)

7.1.60. **Theorem** (cf. Theorem 6.1.50). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable frequency domain s/s system, and let \( \mathcal{X} = X_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \), and suppose that \( \Sigma_1 = (V_1; X_1, W) \) is an \( \Omega \)-resolvable \( \Omega \)-compression of \( \Sigma \) onto \( X_1 \) along \( Z_1 \) where \( \Omega \) is some frequency i/s/o-admissible domain for \( \Sigma \). Let \( Z_1^\Omega \) satisfy the conditions listed in (ii) in Theorem 7.1.58, and let \( Z_c \) be an arbitrary direct complement to \( Z_1^\Omega \) in \( Z_1 \).

(i) Let \( \Sigma_2 \) be the \( \Omega \)-resolvable \( \Omega \)-restriction of \( \Sigma \) to the strongly \( \Omega \)-invariant subspace \( X_1 + Z_1^\Omega \) for \( \Sigma \) given by Theorem 7.1.52(i). Then \( Z_1^\Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma_2 \), and \( \Sigma_1 \) is the \( \Omega \)-projection onto \( X_1 \) along \( Z_1^\Omega \) of \( \Sigma_2 \).

(ii) Let \( \Sigma_3 \) be the \( \Omega \)-resolvable \( \Omega \)-projection of \( \Sigma \) onto \( X_1 + Z_c \) along \( Z_1^\Omega \) given by Theorem 7.1.52(ii). Then \( X_1 \) is strongly \( \Omega \)-invariant for \( \Sigma_3 \), and \( \Sigma_1 \) is the \( \Omega \)-restriction to \( X_1 \) of \( \Sigma_3 \).

**Proof.** Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) and \( \Sigma_{i/s/o}^1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) be i/s/o representations of \( \Sigma \) respectively \( \Sigma_1 \) satisfying \( \Omega \subset \rho(\Sigma_{i/s/o}) \cap \rho(\Sigma_{i/s/o}^1) \), and apply Theorem 6.1.50 and Lemmas 7.1.26 and 7.1.48. \( \square \)

7.1.61. **Lemma** (cf. Lemma 6.1.51). Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) and \( \Sigma_1 = (V_1; X_1, W) \) be two resolvable frequency domain i/s/o systems (with the same signal space) with \( X_2 = X_1 + Z_1 \), and suppose that \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \). Then the following two conditions are equivalent.

(i) \( \Sigma_1 \) is the \( \Omega \)-compression of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \).

(ii) \( Z_1 \) contains some closed subspace \( Z_1^\Omega \) such that \( \Sigma_2 \) and \( \Sigma_1 \) are \( \Omega \)-intertwined by the operator \( P_{X_1^\Omega}|_{X_1 + Z_1^\Omega} \).

Condition (ii) above holds for some particular subspace \( Z_1^\Omega \) if and only condition (ii) in Theorem 7.1.58 holds for the same subspace \( Z_1^\Omega \). Thus, in particular, two possible choices of the subspace \( Z_1^\Omega \) in (ii) are the subspaces \( Z_1^\Omega = Z_1^{\min} \) and \( Z_1^\Omega = Z_1^{\max} \), and every possible subspace \( Z_1^\Omega \) satisfies \( Z_1^{\min} \subset Z_1^\Omega \subset Z_1^{\max} \).

**Proof.** Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) and \( \Sigma_{i/s/o}^1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) be i/s/o representations of \( \Sigma \) respectively \( \Sigma_1 \) satisfying \( \Omega \subset \rho(\Sigma_{i/s/o}) \cap \rho(\Sigma_{i/s/o}^1) \), and apply Lemmas 6.1.51, 7.1.26, and 7.1.33. \( \square \)
7.1.8. Results for connected frequency domains. Earlier in this chapter we have given various results related to \( \Omega \)-trajectories of frequency domain s/s systems. As in the i/s/o case discussed in Chapter 6, it is possible to sharpen some of these results by assuming some connectivity properties of the set \( \Omega \).

7.1.62. Lemma (cf. Lemma 7.1.18). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), and let \( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}) \) be two \( \Omega^o \)-resolvable frequency domain s/s systems with characteristic signal bundles \( \hat{\mathcal{F}}_{\lambda_i} \), \( i = 1, 2 \). Let \( \Omega' \) be an arbitrary subset of \( \Omega^o \) which has a cluster point in \( \Omega^o \), and let \( \Omega \) be a separately frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \) which is contained in \( \Omega^o \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent;
(ii) \( \hat{\mathcal{F}}_{\lambda_1}(\lambda) = \hat{\mathcal{F}}_{\lambda_2}(\lambda) \) for all \( \lambda \in \Omega \);
(iii) \( \hat{\mathcal{F}}_{\lambda_1}(\lambda) = \hat{\mathcal{F}}_{\lambda_2}(\lambda) \) for all \( \lambda \in \Omega^o \);
(iv) \( \hat{\mathcal{F}}_{\lambda_1}(\lambda) = \hat{\mathcal{F}}_{\lambda_2}(\lambda) \) for all \( \lambda \in \Omega' \).

Thus, the external \( \Omega \)-equivalence of \( \Sigma_1 \) and \( \Sigma_2 \) does not depend on the choice of \( \Omega \), as long as \( \Omega \) is a separately frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \) which is contained in \( \Omega^o \).

Proof. (i) \( \Leftrightarrow \) (ii): See Lemma 7.1.18
(ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv): This follows from Lemmas 5.3.25 and A.3.9 \( \square \)

7.1.63. Remark. It is not known if conditions (i)–(iii) above are equivalent to the condition

(iv) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega^o \)-equivalent,

Of course, if \( \Omega^o \) is a separately frequency i/s/o-admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \), then it follows from Lemma 7.1.18 that (iv) is equivalent to (i)–(iii), but it is also not known under what conditions the components of \( \rho(\Sigma_1) \cap \rho(\Sigma_2) \) are separately frequency i/s/o-admissible domains for \( \Sigma_1 \) and \( \Sigma_2 \). Analogous remarks apply to Lemmas 7.1.64, 7.1.65, 7.1.66, 7.1.68, and 7.1.69 and Theorem 7.1.67 below.

7.1.64. Lemma (cf. Lemma 7.1.24). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), and let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be an \( \Omega^o \)-resolvable frequency domain s/s system with signal/state and state/signal resolvents \( \hat{\mathcal{B}} \) respectively \( \hat{\mathcal{C}} \). Let \( \Omega' \) be an arbitrary subset of \( \Omega^o \) which has a cluster point in \( \Omega^o \), and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \) which is contained in \( \Omega^o \). Denote the \( \Omega \)-reachable and \( \Omega \)-observable subspaces of \( \Sigma \) by \( \mathcal{R}_{\Sigma}^o \) respectively \( \mathcal{U}_{\Sigma}^o \). Then

\[
\mathcal{R}_{\Sigma}^o = \bigvee_{\lambda \in \Omega} \text{rng} (\hat{\mathcal{B}}(\lambda)) = \bigvee_{\lambda \in \Omega^o} \text{rng} (\hat{\mathcal{B}}(\lambda)) = \bigvee_{\lambda \in \Omega'} \text{rng} (\hat{\mathcal{B}}(\lambda)),
\]

\[
\mathcal{U}_{\Sigma}^o = \bigcap_{\lambda \in \Omega} \text{ker} (\hat{\mathcal{C}}(\lambda)) = \bigcap_{\lambda \in \Omega^o} \text{ker} (\hat{\mathcal{C}}(\lambda)) = \bigcap_{\lambda \in \Omega'} \text{ker} (\hat{\mathcal{C}}(\lambda)).
\]

Thus, \( \mathcal{R}_{\Sigma}^o \) and \( \mathcal{U}_{\Sigma}^o \) do not depend on the choice of \( \Omega \), as long as \( \Omega \) is a frequency i/s/o-admissible domain for \( \Sigma \) which is contained in \( \Omega^o \).

Proof. That \( \mathcal{R}_{\Sigma}^o = \bigvee_{\lambda \in \Omega} \text{rng} (\hat{\mathcal{B}}(\lambda)) \) and \( \mathcal{U}_{\Sigma}^o = \bigcap_{\lambda \in \Omega} \text{ker} (\hat{\mathcal{C}}(\lambda)) \) follows from Lemma 7.1.24. The second and third identities in (7.1.33) and (7.1.34) follow from Corollary 5.3.26 and Lemma A.3.10 \( \square \)

7.1.65. Lemma (cf. Lemma 7.1.27). Let \( \Omega^o \) be an open connected set in \( \mathbb{C} \), let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be an \( \Omega^o \)-resolvable frequency domain s/s system, let \( \mathcal{Z} \) be a closed
subspace of \( \mathcal{X} \), and let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \) which is contained in \( \Omega^\circ \).

(i) The following conditions are equivalent:
(a) \( \mathcal{Z} \) of \( \mathcal{X} \) is strongly \( \Omega \)-invariant for \( \Sigma \);
(b) the three equivalent conditions \( (7.1.7) \) hold for all \( \lambda \in \Omega \);
(c) the three equivalent conditions \( (7.1.7) \) hold for all \( \lambda \in \Omega^\circ \);
(d) the three equivalent conditions \( (7.1.7) \) hold for some \( \lambda \in \Omega^\circ \).

Thus, if \( \mathcal{Z} \) is strongly \( \Omega \)-invariant for \( \Sigma \) for some frequency i/s/o-admissible domain \( \Omega \) for \( \Sigma \) which is contained in \( \Omega^\circ \), then \( \mathcal{Z} \) is strongly \( \Omega \)-invariant for \( \Sigma \) for every frequency i/s/o-admissible domain \( \Omega \) for \( \Sigma \) which is contained in \( \Omega^\circ \).

(ii) The following conditions are equivalent:
(a) \( \mathcal{Z} \) of \( \mathcal{X} \) is unobservably \( \Omega \)-invariant for \( \Sigma \);
(b) the three equivalent conditions \( (7.1.8) \) hold for all \( \lambda \in \Omega \);
(c) the three equivalent conditions \( (7.1.8) \) hold for all \( \lambda \in \Omega^\circ \);
(d) the three equivalent conditions \( (7.1.8) \) hold for some \( \lambda \in \Omega^\circ \).

Thus, if \( \mathcal{Z} \) is unobservably \( \Omega \)-invariant for \( \Sigma \) for some frequency i/s/o-admissible domain \( \Omega \) for \( \Sigma \) which is contained in \( \Omega^\circ \), then \( \mathcal{Z} \) is unobservably \( \Omega \)-invariant for \( \Sigma \) for every frequency i/s/o-admissible domain \( \Omega \) for \( \Sigma \) which is contained in \( \Omega^\circ \).

Proof. That (a) and (b) are equivalent both in (i) and in (ii) follows from Lemma 7.1.27. Obviously (c) \( \Rightarrow \) (b) \( \Rightarrow \) (d) both in (i) and (ii). Thus, it remains to prove the implications (d) \( \Rightarrow \) (c) in (i) and (ii).

(i)(d) \( \Rightarrow \) (i)(c): Let \( \lambda_0 \in \Omega^\circ \). By Theorem 5.3.9 there exists some i/s/o representation \( \Sigma_{i/s/o} = (S; \mathcal{X}; \mathcal{U}; \mathcal{Y}) \) satisfying \( \lambda_0 \in \rho(\Sigma_{i/s/o}) \). If (i)(d) holds, then it follows from Lemma 5.3.28 that condition (ii)(e) in Lemma 6.1.153 holds. Therefore by Lemmas 5.3.28 and 6.1.153 condition \( (7.1.7) \) holds for all \( \lambda \) in the connected component of \( \rho(\Sigma_{i/s/o}) \) which contains \( \lambda_0 \). By Lemma 5.3.25 the vector bundle \( \hat{\mathcal{E}}(\lambda) \cap \left[ \begin{array}{c} z \\ \mathcal{X} \\ \mathcal{Y} \end{array} \right] \) is analytic in \( \Omega^\circ \), and it follows from Lemma A.3.9 that \( (7.1.7) \) holds for all \( \lambda \in \Omega^\circ \).

(ii)(d) \( \Rightarrow \) (ii)(c): This proof is analogous to the proof of the implication (vii) \( \Rightarrow \) (vi) in Lemma 3.4.8 with references to Proposition 2.5.49 and Lemmas 3.2.3 and 3.4.1 replaced by references to Theorems 5.3.6 and Lemmas 5.3.28, 6.1.16, and 7.1.26.

7.1.66. Lemma (cf. Lemma 7.1.34). Let \( \Omega^\circ \) be an open connected set in \( \mathbb{C} \), let \( \Sigma_i = (V_i; \mathcal{X}_i; \mathcal{W}) \) be two \( \Omega^\circ \)-resolvable frequency domain s/s systems (with the same signal space), let \( P \in M\mathcal{L}(\mathcal{X}_1; \mathcal{X}_2) \) be closed, and let \( \Omega \) be a separately frequency i/s/o-admissible domain \( \Omega \) for \( \Sigma_1 \) and \( \Sigma_2 \) which is contained in \( \Omega^\circ \). Then the following conditions are equivalent:

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \);
(ii) The equivalent conditions (a) and (b) in parts (ii) and (iii) of Lemma 7.1.34 hold for all \( \lambda \in \Omega \);
(iii) The equivalent conditions (a) and (b) in parts (ii) and (iii) of Lemma 7.1.34 hold for all \( \lambda \in \Omega^\circ \);
(iv) The equivalent conditions (a) and (b) in parts (ii) and (iii) of Lemma 7.1.34 hold for some \( \lambda \in \Omega^\circ \).
Thus, if $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P$ for some separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$, then $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P$ for every separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$.

**PROOF.** The proof is analogous to the proof of Lemma 7.1.65 with Lemma 6.1.53 replaced by Lemma 6.1.55. \qed

7.1.67. **Theorem (cf. Theorem 7.1.37).** Let $\Omega^\circ$ be an open connected set in $\mathbb{C}$, and let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W})$, $i = 1, 2$, be two $\Omega^\circ$-resolvable s/s nodes (with the same signal space). Denote the signal/state and state/signal resolvents of $\Sigma_i$ by $\tilde{\mathcal{B}}_i$, respectively $\mathcal{E}_i$, $i = 1, 2$. Let $\Omega'$ be an arbitrary subset of $\Omega^\circ$ which has a cluster point in $\Omega^\circ$, and let $\Omega$ be a separately frequency i/s/o-admissible domain for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$. Furthermore, suppose that $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-invariant, and define $P_{\text{min}}^\Omega$ and $P_{\text{max}}^\Omega$ by (7.1.13) and (7.1.14). Then

\[
\text{gph}(P_{\text{min}}^\Omega) = \bigvee_{\lambda \in \Omega} \text{rng} \left( \begin{bmatrix} \tilde{\mathcal{B}}_2(\lambda) \\ \tilde{\mathcal{B}}_1(\lambda) \end{bmatrix} \right) = \bigvee_{\lambda \in \Omega^\circ} \text{rng} \left( \begin{bmatrix} \tilde{\mathcal{B}}_2(\lambda) \\ \tilde{\mathcal{B}}_1(\lambda) \end{bmatrix} \right)
\]

(7.1.35)

\[
\text{gph}(P_{\text{max}}^\Omega) = \bigcap_{\lambda \in \Omega^\circ} \ker \left( [\tilde{\mathcal{E}}_2(\lambda) - \tilde{\mathcal{E}}_1(\lambda)] \right) = \bigcap_{\lambda \in \Omega^\circ} \ker \left( [\tilde{\mathcal{E}}_2(\lambda) - \tilde{\mathcal{E}}_1(\lambda)] \right)
\]

(7.1.36)

Thus, $P_{\text{min}}^\Omega$ and $P_{\text{max}}^\Omega$ do not depend on the choice of $\Omega$, as long as $\Omega$ is a separately frequency i/s/o-admissible domain for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$ and $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-invariant.

**PROOF.** The proof is analogous to the proof of Lemma 7.1.65 with Lemma 6.1.53 replaced by Theorem 6.1.56. \qed

7.1.68. **Lemma (cf. Lemma 7.1.51).** Let $\Omega^\circ$ be an open connected set in $\mathbb{C}$, let $\Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W})$ be two $\Omega^\circ$-resolvable frequency domain s/s systems, and suppose that $\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_2$ with a direct complement $\mathcal{Z}_1$ in $\mathcal{X}_2$. Let $\Omega'$ be an arbitrary subset of $\Omega^\circ$ which contains a cluster point in $\Omega^\circ$, and let $\Omega$ be a separately frequency i/s/o-admissible domain for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$.

(i) The following conditions are equivalent:
   (a) $\Sigma_1$ is the $\Omega$-restriction of $\Sigma_2$ to $\mathcal{X}_1$;
   (b) the two equivalent conditions (7.1.18) hold for all $\lambda \in \Omega^\circ$;
   (c) the two equivalent condition (7.1.18) hold for some $\lambda \in \Omega^\circ$.

Thus, if $\Sigma_1$ is the $\Omega$-restriction of $\Sigma_2$ to $\mathcal{X}_1$ for some separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$, then $\Sigma_1$ is the $\Omega$-restriction of $\Sigma_2$ to $\mathcal{X}_1$ for every separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$.

(ii) The following conditions are equivalent:
   (a) $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$;
   (b) the two equivalent conditions (7.1.20) hold for all $\lambda \in \Omega^\circ$;
   (c) the two equivalent conditions (7.1.20) hold for some $\lambda \in \Omega^\circ$. 

Thus, if $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ to $X_1$ for some separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$, then $\Sigma_1$ is the $\Omega$-projection of $\Sigma_2$ to $X_1$ for every separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$.

(iii) The following conditions are equivalent:
(a) $\Sigma_1$ is the $\Omega$-compression of $\Sigma_2$ onto $X_1$ along $Z_1$;
(b) condition (7.1.22) holds for all $\lambda \in \Omega^\circ$;
(c) condition (7.1.22) holds for all $\lambda \in \Omega'$.

Thus, if $\Sigma_1$ is the $\Omega$-compression of $\Sigma_2$ to $X_1$ for some separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$, then $\Sigma_1$ is the $\Omega$-compression of $\Sigma_2$ to $X_1$ for every separately frequency i/s/o-admissible domain $\Omega$ for $\Sigma_1$ and $\Sigma_2$ which is contained in $\Omega^\circ$.

Proof. The proof is analogous to the proof of Lemma 7.1.65 with Lemma 6.1.53 replaced by Lemma 6.1.59.

7.1.69. Lemma (cf. Lemma 7.1.57). Let $\Omega^\circ$ be an open connected set in $\mathbb{C}$, let $\Sigma = (V; \mathcal{X}, \mathcal{W})$, $\Omega^\circ$-resolvable frequency domain s/s system with characteristic node bundle $\mathcal{E}$, signal-state/state resolvent $\hat{\mathcal{L}}$, state/signal resolvent $\hat{\mathcal{C}}$, and unobservable state/state resolvent $\hat{\mathcal{A}}$, and let $\mathcal{X} = X_1 + Z_1$. Let $\Omega^\circ$ be an arbitrary subset of $\Omega^\circ$ which has a cluster point in $\Omega^\circ$, let $\lambda_0 \in \Omega^\circ$, and let $\Omega$ be a frequency i/s/o-admissible domain for $\Sigma$. Then the following claims are true:

(i) The subspace $X_1^{y}_{\text{min}}$ in Lemma 7.1.57 is given by

$$
X_1^{y}_{\text{min}} := X_1 \bigvee_{\lambda \in \Omega} [0 \ 1_{Z_1} \ 0] \left( \mathcal{E}(\lambda) \cap \left[ \begin{array}{c}
X_1 \\
X_1 \\
\mathcal{W}
\end{array} \right] \right)
$$

or equivalently, by

$$
X_1^{y}_{\text{min}} = X_1 \bigvee_{\lambda \in \Omega} \text{rng} \left( \hat{\mathcal{E}}(\lambda)[_{X_1}]_{_{\mathcal{W}}} \right)
$$

(7.1.37a)

(ii) The subset $X_1^{y}_{\text{min}}$ in Lemma 7.1.57 is given by

$$
X_1^{y}_{\text{min}} := X_1 \bigvee_{\lambda \in \Omega^\circ} [0 \ 1_{Z_1} \ 0] \left( \mathcal{E}(\lambda) \cap \left[ \begin{array}{c}
X_1 \\
X_1 \\
\mathcal{W}
\end{array} \right] \right)
$$

or equivalently, by

$$
X_1^{y}_{\text{min}} = X_1 \bigvee_{\lambda \in \Omega^\circ} \text{rng} \left( \hat{\mathcal{E}}(\lambda)[_{X_1}]_{_{\mathcal{W}}} \right)
$$

(7.1.37b)
(ii) The subspace \( \mathcal{Z}_\Omega^{\text{min}} \) in Lemma 7.1.57 is given by

\[
\mathcal{Z}_\Omega^{\text{min}} := \bigvee_{\lambda \in \Omega} \begin{bmatrix} 0 & P_{\mathcal{X}_1} \end{bmatrix} \left( \tilde{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{X}_1 \\\ \mathcal{X}_2 \\\ \mathcal{W} \end{bmatrix} \right) \]

or equivalently, by

\[
\mathcal{Z}_\Omega^{\text{min}} = \bigvee_{\lambda \in \Omega} \text{rng} \left( P_{\mathcal{X}_1} \tilde{\mathcal{E}}(\lambda) \right) \bigg| \begin{bmatrix} \mathcal{X}_1 \\\ \mathcal{W} \end{bmatrix} \bigg).
\]

(iii) The subspace \( \mathcal{Z}_\Omega^{\text{max}} \) in Lemma 7.1.57 is given by

\[
\mathcal{Z}_\Omega^{\text{max}} = \bigwedge_{\lambda \in \Omega} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left( \tilde{\mathcal{E}}(\lambda) \cap \begin{bmatrix} \mathcal{Z}_1 \\
\mathcal{Z}_1 \{0\} \end{bmatrix} \right) \]

or equivalently, by

\[
\mathcal{Z}_\Omega^{\text{max}} = \bigwedge_{\lambda \in \Omega} \ker \left( \begin{bmatrix} P_{\mathcal{X}_1} \tilde{\mathcal{A}}(\lambda) \\
\tilde{\mathcal{E}}(\lambda) \end{bmatrix} \bigg| \begin{bmatrix} \mathcal{Z}_1 \{0\} \end{bmatrix} \right).
\]

**Proof.** The proof is analogous to the proof of Lemma 7.1.65 with Lemma 6.1.53 replaced by Lemma 6.1.60.

7.1.9. Frequency domain notions for \( \Omega \)-resolvable s/s nodes. In Definition 3.5.10 we transferred a number of dynamical notions for the continuous time and discrete time s/s systems induced by bounded s/s nodes into notions which applies to the node itself. The same idea can also be used to transfer notions defined
in terms of the frequency domain s/s system induced by a resolvable s/s node into
a notion which applies to the node itself.

7.1.70. **Definition.** Let \( \Omega \) be an open set in \( \mathbb{C} \), and let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) and
\( \Sigma_i = (V_i; \mathcal{X}_i, \mathcal{W}_i) \) be \( \Omega \)-resolvable s/s nodes with characteristic node bundles \( \hat{E} \)
respectively \( \hat{E}_i \), \( i = 1, 2 \) and signal bundles \( \hat{F} \) respectively \( \hat{F}_i \), \( i = 1, 2 \)

(i) A closed subspace \( Z \) of \( \mathcal{X} \) is strongly \( \Omega \)-invariant for \( \Sigma \) if the equivalent
conditions (7.1.7) hold (if \( \Omega \) is a frequency i/s/o admissible domain for \( \Sigma \), then this is equivalent to the condition that \( Z \) is strongly \( \Omega \)-invariant
for the frequency domain s/s system induced by \( \Sigma \)).

(ii) A closed subspace \( Z \) of \( \mathcal{X} \) is unobservably \( \Omega \)-invariant for \( \Sigma \) if the equivalent
conditions (7.1.8) hold (if \( \Omega \) is a frequency i/s/o admissible domain for \( \Sigma \), then this is equivalent to the condition that \( Z \) is unobservably
\( \Omega \)-invariant for the frequency domain s/s system induced by \( \Sigma \)).

(iii) The subspace \( \mathcal{R}_\Sigma \) defined in (7.1.5) is called the \( \Omega \)-reachable subspace
of \( \Sigma \) (if \( \Omega \) is a frequency i/s/o admissible domain for \( \Sigma \), then this is the
\( \Omega \)-reachable subspace of the frequency domain s/s system induced by \( \Sigma \)).

(iv) \( \Sigma \) is \( \Omega \)-controllable if the \( \Omega \)-reachable subspace of \( \Sigma \) is the full state space
\( \mathcal{X} \).

(v) The subspace \( \mathcal{U}_\Sigma \) defined in (7.1.6) is called the \( \Omega \)-unobservable subspace
of \( \Sigma \) (if \( \Omega \) is a frequency i/s/o admissible domain for \( \Sigma \), then this is the
\( \Omega \)-unobservable subspace of the frequency domain s/s system induced by \( \Sigma \)).

(vi) \( \Sigma \) is \( \Omega \)-observable if the \( \Omega \)-unobservable subspace of \( \Sigma \) is \{0\}.

(vii) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent if \( \hat{F}_1(\lambda) = \hat{F}_2(\lambda) \) for all \( \lambda \in \Omega \)
and this is equivalent to the condition that the frequency domain s/s systems
induced by \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent: see Lemma 7.1.18).

(viii) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) if the conditions (ii)
and (iii) in Lemma 7.1.34 hold (if \( \Omega \) is a jointly frequency i/s/o admissible domain for \( \Sigma_1 \) and \( \Sigma_2 \), then this is equivalent to the condition that the
frequency domain s/s systems induced by \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by
\( P \)).

(ix) \( \Sigma_1 \) is the \( \Omega \)-restriction to \( \mathcal{X}_1 \), or \( \Omega \)-projection or \( \Omega \)-compression onto
\( \mathcal{X}_1 \) along \( Z_1 \) of \( \Sigma_2 \) if if conditions (7.1.18), (7.1.20), or (7.1.22) hold,
respectively (if \( \Omega \) is a jointly frequency i/s/o admissible domain for \( \Sigma_1 \)
and \( \Sigma_2 \), then this is equivalent to the condition that the frequency domain
s/s system induced by \( \Sigma_1 \) is the \( \Omega \)-restriction to \( \mathcal{X}_1 \), or \( \Omega \)-projection or
\( \Omega \)-compression onto \( \mathcal{X}_1 \), respectively, along \( Z_1 \) of the frequency domain
s/s system induced by \( \Sigma_2 \)).

(x) \( \Sigma \) is \( \Omega \)-minimal if the \( \Sigma \) does not have any non-trivial \( \Omega \)-resolvable \( \Omega \)-compression
(if \( \Omega \) is a frequency i/s/o admissible domain for \( \Sigma \), then this is equivalent to the condition that the frequency domain s/s system
induced by \( \Sigma \) is \( \Omega \)-minimal).

If \( \Sigma \) is a bounded s/s node, then we can apply both Definition 3.5.10 and
Definition 7.1.70 to \( \Sigma \). The following lemma describes the connection between the
notions introduced in these two definitions.
7.1.71. **Lemma.** Let $\Sigma = (V; X, W)$ and $\Sigma_i = (V_i; X_i, W_i)$ be bounded s/s nodes. Then all the dynamical notions listed in Definition 3.5.10 are equivalent to the corresponding $\Omega$-dynamical notions listed in Definition 7.1.70 with $\Omega = \rho_\infty(\Sigma)$ (in those notions which refer only to $\Sigma$) or $\Omega$ equal to the unbounded component of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$ (in those notions that refer to $\Sigma_1$ and $\Sigma_2$).

**Proof.** The proof is analogous to the proof of Lemma 3.5.9. \qed
7.2. The Generating Subspaces of Ω-Compressed S/S Systems (Jan 02, 2016)

In the preceding section we investigated a number of relationships between s/s systems in the frequency domain, such as restrictions, projections, compressions, and intertwinements. It is also possible to describe the same relationships in terms of the generating subspaces of the involved systems.

7.2.1. The generating subspace of an s/s Ω-restriction.

7.2.1. Theorem. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable s/s node, let \( \mathcal{X}_1 \) be a closed subspace of \( \mathcal{X} \), and let \( \Sigma_{\text{part}} = (V_{\text{part}}; \mathcal{X}_1, \mathcal{W}) \) be the part of \( \Sigma \) in \( \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{W} \end{array} \right] \). Finally, let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \). Then the following conditions are equivalent:

(i) \( \mathcal{X}_1 \) is strongly \( \Omega \)-invariant for \( \Sigma \);
(ii) \( \Sigma \) has a (unique) \( \Omega \)-resolvable \( \Omega \)-restriction to \( \mathcal{X}_1 \);
(iii) \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma_{\text{part}} \) and \( \Sigma \).

If these equivalent conditions hold, then \( \Sigma_{\text{part}} \) is the unique \( \Omega \)-resolvable \( \Omega \)-restriction of \( \Sigma \) to \( \mathcal{X}_1 \).

Proof. (i) \( \iff \) (ii): See Theorem 7.1.52.(i).

(ii) \( \Rightarrow \) (iii): If (ii) holds, then so does (i), and it follows from Lemmas 7.1.27 and 7.1.51 that the characteristic node bundle of the restriction \( \Sigma_1 \) in (ii) satisfies \( \widehat{\mathcal{E}}_1(\lambda) = \mathcal{E}(\lambda) \cap \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{W} \end{array} \right] \) for all \( \lambda \in \Omega \). This combined with Lemma 1.6.15 implies that \( V_1 = V \cap \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{W} \end{array} \right] \), i.e., \( \Sigma_1 = \Sigma_{\text{part}} \). By Lemmas 7.1.41 and 7.1.45 \( \Sigma_{\text{part}} \) and \( \Sigma \) are externally equivalent. Thus (iii) holds.

(iii) \( \Rightarrow \) (ii): Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o representation of \( \Sigma \) satisfying \( \Omega \in \rho(\Sigma_{i/s/o}) \). Then by Lemma 7.1.19 \( \Sigma_{\text{part}} \) has an i/s/o representation \( \Sigma_{i/s/o}^{\text{part}} = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) (with the same input and output spaces) satisfying \( \Omega \in \rho(\Sigma_{i/s/o}^{\text{part}}) \). By Lemma 1.6.15 \( S_1 \) is the part of \( S \) in \( \mathcal{X}_1 \). By Theorem 6.2.1 \( \Sigma_{i/s/o}^{\text{part}} \) is the \( \Omega \)-restriction of \( \Sigma_{i/s/o} \) to \( \mathcal{X}_1 \), and hence by Lemma 7.1.48 \( \Sigma_{\text{part}} \) is the \( \Omega \)-restriction of \( \Sigma \) to \( \mathcal{X}_1 \). Thus (ii) holds. \( \Box \)

7.2.2. The generating subspace of an s/s Ω-projection.

7.2.2. Theorem. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a resolvable s/s node, let \( \mathcal{X}_1 \) be a closed subspace of \( \mathcal{X} \) with a direct complement \( \mathcal{X}_2 \), and let \( \Sigma_{\text{proj}} = (V_{\text{proj}}; \mathcal{X}_1, \mathcal{W}) \) be the static projection of \( \Sigma \) onto \( \left[ \begin{array}{c} \mathcal{X}_1 \\ \mathcal{W} \end{array} \right] \). Finally, let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \). Then the following conditions are equivalent:

(i) \( \mathcal{X}_1 \) is unobservably \( \Omega \)-invariant for \( \Sigma \);
(ii) \( \Sigma \) has a (unique) \( \Omega \)-resolvable \( \Omega \)-projection onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \);
(iii) \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma_{\text{proj}} \) and \( \Sigma \).

If these equivalent conditions hold, then \( \Sigma_{\text{proj}} \) is the unique \( \Omega \)-resolvable \( \Omega \)-projection of \( \Sigma \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

Proof. (i) \( \iff \) (ii): See Theorem 7.1.52.(ii).

(ii) \( \Rightarrow \) (iii): If (ii) holds, then so does (i), and it follows from Lemmas 7.1.27 and 7.1.51 that the characteristic node bundle of the restriction \( \Sigma_1 \) in (ii) satisfies
\[ \mathcal{E}_1(\lambda) = \begin{bmatrix} \rho_{\lambda_1}^S & 0 & 0 \\ 0 & \rho_{\lambda_1}^S & 0 \\ 0 & 0 & 1_w \end{bmatrix} \mathcal{E}(\lambda) \] for all \( \lambda \in \Omega \). This combined with Lemma 1.6.15 implies that \( V_1 = \begin{bmatrix} \rho_{\lambda_1}^S & 0 & 0 \\ 0 & \rho_{\lambda_1}^S & 0 \\ 0 & 0 & 1_w \end{bmatrix} V \), i.e., \( \Sigma_1 = \Sigma_{\text{proj}} \). By Lemmas 7.1.41 and 7.1.47 \( \Sigma_{\text{proj}} \) and \( \Sigma \) are externally equivalent. Thus (iii) holds.

(iii) \( \Rightarrow \) (ii): Let \( \Sigma_{i/s/o} = (S;X,U,Y) \) be an i/s/o representation of \( \Sigma \) satisfying \( \Omega \in \rho(\Sigma_{i/s/o}) \). Then by Lemma 7.1.19 \( \Sigma_{\text{proj}} \) has an i/s/o representation \( \Sigma_{1{i/s/o}} = (S_1;X_1,U,Y) \) (with the same input and output spaces) satisfying \( \Omega \in \rho(\Sigma_{1{i/s/o}}) \).

By Theorem 6.2.2 \( \Sigma_{1{i/s/o}} \) is the \( \Omega \)-projection of \( \Sigma_{i/s/o} \) to \( \Sigma_1 \), and hence by Lemma 7.1.48 \( \Sigma_{\text{proj}} \) is the \( \Omega \)-projection restriction of \( \Sigma \) to \( \Sigma_1 \). Thus (ii) holds. \( \square \)

7.2.3. The generating subspace of an s/s \( \Omega \)-compression.

7.2.3. Theorem. Let \( \Sigma = (V;X,W) \) be a resolvable s/s node, let \( \Omega \) be a frequency i/s/o-admissible domain for \( \Sigma \), and let \( X = X_1 + Z \Omega + Z_c \) be a direct sum decomposition of \( X \). Then the following conditions are equivalent:

(i) \( Z \Omega \) is unobservably \( \Omega \)-invariant for \( \Sigma \) and \( X_1 + Z \Omega \) is strongly \( \Omega \)-invariant for \( \Sigma \).

(ii) If we denote the part of \( \Sigma \) in \( X_1 + Z \Omega \) by \( \Sigma_{\text{part}} = (V_{\text{part}};X_1 + Z \Omega,W) \) and the projection of \( \Sigma_{\text{part}} \) onto \( X_1 \) along \( Z \Omega \) by \( (\Sigma_{\text{part}})_{\text{proj}} = ((V_{\text{part}})_{\text{proj}};X_1,W) \), then \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma \), \( \Sigma_{\text{part}} \), and \( (\Sigma_{\text{part}})_{\text{proj}} \).

(iii) If we denote the projection of \( \Sigma \) onto \( X_1 + Z_c \) by \( \Sigma_{\text{proj}} = (V_{\text{proj}};X_1 + Z_c,W) \) and the part of \( \Sigma_{\text{proj}} \) in \( X_1 \) by \( (\Sigma_{\text{proj}})_{\text{part}} = ((V_{\text{proj}})_{\text{part}};X_1,W) \), then \( \Omega \) is a jointly frequency i/s/o-admissible domain for \( \Sigma \), \( \Sigma_{\text{proj}} \), and \( (\Sigma_{\text{proj}})_{\text{part}} \).

If these equivalent conditions hold, then according to Theorem 7.1.55 \( \Sigma \) has a unique \( \Omega \)-resolvable \( \Omega \)-compression onto \( X_1 \) along \( Z_1 := Z \Omega + Z_c \), and conversely, if \( Z_1 \) is a direct complement to \( X_1 \) in \( \Sigma \) and \( \Sigma \) has an \( \Omega \)-resolvable \( \Omega \)-compression onto \( X_1 \) along \( Z_1 \), then \( \Sigma \) can be decomposed into \( \Sigma = X_1 + Z \Omega + Z_c \) in such a way that the above equivalent conditions (i)–(iii) hold. The unique \( \Omega \)-resolvable \( \Omega \)-compression \( \Sigma_1 \) of \( \Sigma \) onto \( X_1 \) along \( Z_1 \) is given by

\[ \Sigma_1 = (\Sigma_{\text{part}})_{\text{proj}} = (\Sigma_{\text{proj}})_{\text{part}}. \]

Proof. The proof is analogous to the proofs of Theorems 7.2.1 and 7.2.2 with Theorems 6.2.1 and 6.2.2 replaced by Theorem 6.2.3. \( \square \)

7.2.4. Compressions into minimal s/s systems.

7.2.4. Theorem (cf. Theorem 6.2.8). Let \( \Sigma = (V;X,W) \) be a resolvable s/s node, and let \( \Omega \) be a frequency domain i/s/o-admissible domain for \( \Sigma \). Then \( \Sigma \) has an \( \Omega \)-minimal \( \Omega \)-compression. Two families of such \( \Omega \)-compressions are described below, where we have denoted the \( \Omega \)-reachable and \( \Omega \)-unobservable subspaces of \( \Sigma \) by \( \Omega_{\text{reach}} \) and \( \Omega_{\text{unobs}} \) respectively:

(i) Let \( X_1 \) be a direct complement to \( \Omega_{\text{unobs}} \) in \( X \), and let \( X_0 = \frac{1}{\rho_{\lambda_1}^s} \Omega_{\text{reach}} \), and let \( \Sigma_o = (V_o;X_o,W) \), where \( V_o \) is the part in \( X_o \) of the projection of \( V \) onto
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\[ X_1 \text{ along } \Omega_{11}, \text{ i.e., } \]

\[
(7.2.2) \quad V_o = \left( \begin{array}{ccc}
P_{X_1} \Omega & 0 & 0 \\
0 & P_{X_1} \Omega & 0 \\
0 & 0 & 1_W
\end{array} \right) V \bigg \vert \begin{bmatrix}
X_c \\
\Omega_{11} \\
W
\end{bmatrix}.
\]

Then \( \Sigma_c \) is an \( \Omega \)-minimal \( \Omega \)-compression of \( \Sigma \). One gets this minimal \( \Omega \)-compression by first \( \Omega \)-projecting \( \Sigma \) onto \( X_1 \) along its \( \Omega \)-unobservable subspace \( \Omega_{11} \), and then \( \Omega \)-restricting the resulting system to its \( \Omega \)-reachable subspace \( X_c \).

(ii) Let \( X_* \) be a direct complement to \( \Omega_{12}^{\Omega} \cap \Omega_{11}^{\Omega} \) in \( \Omega_{12}^{\Omega} \), and let \( \Sigma_* = (V_*; X_*, W) \), where \( V_* \) is the projection onto \( X_* \) along \( \Omega_{12}^{\Omega} \cap \Omega_{11}^{\Omega} \) of the part of \( V \) in \( \Omega_{12}^{\Omega} \), i.e,

\[
(7.2.3) \quad V_* = \left( \begin{array}{ccc}
P_{X_*} \Omega & 0 & 0 \\
0 & P_{X_*} \Omega & 0 \\
0 & 0 & 1_W
\end{array} \right) \left( V \bigg \vert \begin{bmatrix}
\Omega_{12}^{\Omega} \\
\Omega_{11}^{\Omega} \\
W
\end{bmatrix} \right).
\]

Then \( \Sigma_* \) is an \( \Omega \)-minimal \( \Omega \)-compression of \( \Sigma \). One gets this \( \Omega \)-compression by first \( \Omega \)-restricting \( \Sigma \) to its \( \Omega \)-reachable subspace \( \Omega_{12}^{\Omega} \), and then \( \Omega \)-projecting the resulting system onto \( X_* \) along its \( \Omega \)-unobservable subspace \( \Omega_{11}^{\Omega} \).

**Proof.** Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an \( \Omega \)-resolvable i/o representation of \( \Sigma \), and apply Theorem 6.2.8 and Lemmas 7.1.26 and 7.1.48.

7.2.5. **Lemma.** Let \( \Sigma \) be a resolvable s/s node, and let \( \Omega \) be a frequency domain i/o-admissible domain for \( \Sigma \). Then the \( \Omega \)-minimal \( \Omega \)-compression of \( \Sigma \) is unique if and only at least one of conditions (i) and (ii) below holds:

(i) \( \Sigma \) is \( \Omega \)-observable, i.e., \( \Omega_{11}^{\Omega} = \{0\} \), where \( \Omega_{11}^{\Omega} \) is the \( \Omega \)-unobservable subspace of \( \Sigma \),

(ii) the following equivalent conditions hold:

(a) \( \Sigma \) has an \( \Omega \)-minimal \( \Omega \)-compression with state space \( \{0\} \),

(b) the characteristic signal bundle of \( \Sigma \) is a constant in \( \Omega \),

(c) \( \Omega_{12}^{\Omega} \subset \Omega_{11}^{\Omega} \), where \( \Omega_{12}^{\Omega} \) is the \( \Omega \)-reachable subspace of \( \Sigma \).

In case (i) the unique \( \Omega \)-minimal \( \Omega \)-compression \( \Sigma_{\text{min}} \) is the part of \( \Sigma \) in \( \Omega_{12}^{\Omega} \), i.e., \( \Sigma_{\text{min}} = (V_{\text{min}}; \Omega_{12}^{\Omega}, W) \) where

\[
(7.2.4) \quad V_{\text{min}} = V \bigg \vert \begin{bmatrix}
\Omega_{12}^{\Omega} \\
\Omega_{11}^{\Omega} \\
\mathcal{U}
\end{bmatrix}.
\]

In case (ii) the unique minimal compression is \( \Sigma_{\text{min}} = \begin{bmatrix} \{0\} \\ \{0\} \\ W_0 \end{bmatrix} \), where \( W_0 \) is the constant value of the characteristic signal bundle of \( \Sigma \) in \( \Omega \). If neither (i) nor (ii) holds, then \( \Sigma \) has an infinite number of minimal compressions.

**Proof.** Let \( \Sigma_{i/s/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an \( \Omega \)-resolvable i/o representation of \( \Sigma \), and apply Lemmas 6.2.9, 7.1.26, and 7.1.48.
7.2.5. Dynamical properties of the resolvent family of bounded \( s/s \) nodes. At this point the reader may want to recall Definition 5.3.31 which explains what we mean by the resolvent family of bounded \( s/s \) nodes induced by a resolvable \( s/s \) node \( \Sigma \).

7.2.6. Lemma (cf. Lemma 6.2.12). Let \( \Omega \) be an open set in \( C \), let \( \Sigma = (V; X, W) \) and \( \Sigma_i = (V; X_i, W) \), \( i = 1, 2 \), be \( \Omega \)-resolvable \( s/s \) systems, and let \( \Sigma^\lambda = (\mathcal{E}(\lambda); X, W) \), \( \lambda \in \rho(\Sigma) \), and \( \Sigma^\lambda_i = (\mathcal{E}_i(\lambda); X_i, W) \), \( \lambda \in \rho(\Sigma_i) \), \( i = 1, 2 \), be the resolvent families of bounded \( s/s \) nodes induced by \( \Sigma \) respectively \( \Sigma_i \), \( i = 1, 2 \). Then the following claims are true:

(i) A closed subspace \( Z \) of \( X \) is strongly or unobservably \( \Omega \)-invariant for \( \Sigma \) if and only if \( Z \) is strongly respectively unobservably invariant for \( \Sigma^\lambda \) for all \( \lambda \in \Omega \).

(ii) \( \mathfrak{M}^\lambda_\Sigma = \bigvee_{\lambda \in \Omega} \mathfrak{M}^\lambda \), where \( \mathfrak{M}^\lambda_\Sigma \) is the \( \Omega \)-reachable subspace of \( \Sigma \) and \( \mathfrak{M}^\lambda \) is the reachable subspace of \( \Sigma^\lambda \).

(iii) \( \mathfrak{M}^\lambda_\Sigma = \bigwedge_{\lambda \in \Omega} \mathfrak{M}^\lambda \), where \( \mathfrak{M}^\lambda \) is the \( \Omega \)-unobservable subspace of \( \Sigma \) and \( \mathfrak{M}^\lambda \) is the unobservable subspace of \( \Sigma^\lambda \).

(iv) \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent if and only if \( \Sigma^\lambda_1 \) and \( \Sigma^\lambda_2 \) are externally equivalent for all \( \lambda \in \Omega \).

(v) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by the closed multi-valued operator \( P \in \mathcal{MC}(X_1; X_2) \) if and only if \( \Sigma^\lambda_1 \) and \( \Sigma^\lambda_2 \) are intertwined by \( P \) for all \( \lambda \in \Omega \).

(vi) \( \Sigma_1 \) is the \( \Omega \)-restriction to \( X_1 \), or the \( \Omega \)-projection or \( \Omega \)-compression onto \( X_1 \) along \( Z_1 \) of \( \Sigma_2 \) if and only if \( \Sigma^\lambda_1 \) is the restriction to \( X_1 \), or the projection or compression onto \( X_1 \) along \( Z_1 \) of \( \Sigma^\lambda_2 \), respectively, for all \( \lambda \in \Omega \).

Proof. This follows from Theorem 5.3.9, Lemma 6.2.12 and Propositions 2.5.49 and 2.5.50 applied to the bounded \( i/s/o \) representations \( \Sigma^\lambda_{i/s/o} \) of the bounded \( s/s \) systems \( \Sigma^\lambda \).

There are certain claims that are missing from Lemma 7.2.6, namely claims about controllability, observability, and minimality. The reason for this omission is that in the setting of Lemma 7.2.6

(i) \( \Omega \)-controllability of \( \Sigma \) does not imply controllability of \( \Sigma^\lambda \) for all \( \lambda \in \Omega \),

(ii) \( \Omega \)-observability of \( \Sigma \) does not imply observability of \( \Sigma^\lambda \) for all \( \lambda \in \Omega \),

(iii) \( \Omega \)-minimality of \( \Sigma \) does not imply minimality of \( \Sigma^\lambda \) for all \( \lambda \in \Omega \).

However, as we shall see below, these additional claims are true whenever \( \Omega \) is connected, or more generally, \( \Omega \) is contained in some connected component of \( \rho(\Sigma) \).

7.2.7. Lemma (cf. Lemma 6.2.13). Let \( \Omega^0 \) be an open connected set in \( C \), let \( \Sigma = (V; X, W) \) and \( \Sigma_i = (V_i; X_i, W) \), \( i = 1, 2 \), be three \( \Omega^0 \)-resolvable \( s/s \) systems, and let \( \Sigma^\lambda = (\mathcal{E}(\lambda); X, W) \), \( \lambda \in \rho(\Sigma) \), and \( \Sigma^\lambda_i = (\mathcal{E}_i(\lambda); X_i, W) \), \( \lambda \in \rho(\Sigma_i) \), \( i = 1, 2 \), be the resolvent families of bounded \( s/s \) nodes induced by \( \Sigma \) respectively \( \Sigma_i \), \( i = 1, 2 \). Let \( \Omega \) be an open subset of \( \Omega^0 \). Then the following claims are true.

(i) For each closed subspace \( Z \) of \( X \) the following conditions are equivalent:

(a) \( Z \) is strongly or unobservably \( \Omega \)-invariant for \( \Sigma \);

(b) \( Z \) is strongly respectively unobservably invariant for \( \Sigma^\lambda \) for all \( \lambda \in \Omega^0 \);

(c) \( Z \) is strongly respectively unobservably invariant for \( \Sigma^\lambda_{0} \) for some \( \lambda_0 \in \Omega^0 \).
(ii) $\mathcal{R}_{\Sigma^\lambda}^1 = \mathcal{R}_{\Sigma^\lambda}$ for all $\lambda \in \Omega^0$, where $\mathcal{R}_{\Sigma^\lambda}^1$ is the $\Omega$-reachable subspace of $\Sigma$ and $\mathcal{R}_{\Sigma^\lambda}$ is the reachable subspace of $\Sigma^\lambda$.

(iii) $\mathcal{U}_{\Sigma}^1 = \mathcal{U}_{\Sigma^\lambda}$ for all $\lambda \in \Omega^0$, where $\mathcal{U}_{\Sigma}^1$ is the $\Omega$-observable subspace of $\Sigma$ and $\mathcal{U}_{\Sigma^\lambda}$ is the unobservable subspace of $\Sigma^\lambda$.

(iv) The following conditions are equivalent:
(a) $\Sigma$ is $\Omega$-controllable or $\Omega$-observable;
(b) $\Sigma^\lambda$ is controllable respectively observable for all $\lambda \in \Omega^0$;
(c) $\Sigma^\lambda_{\Omega}$ is controllable respectively observable for some $\lambda_0 \in \Omega^0$.

(v) The following conditions are equivalent:
(a) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent;
(b) $\Sigma^\lambda_1$ and $\Sigma^\lambda_2$ are externally equivalent for all $\lambda \in \Omega^0$;
(c) $\Sigma^\lambda_{\Omega_1}$ and $\Sigma^\lambda_{\Omega_2}$ are externally equivalent for some $\lambda_0 \in \Omega^0$.

(vi) For each closed $P \in \mathcal{MC}(\mathcal{X}_1; \mathcal{X}_2)$ the following conditions are equivalent:
(a) $\Sigma_1$ and $\Sigma_2$ are $\Omega$-intertwined by $P$;
(b) $\Sigma^\lambda_1$ and $\Sigma^\lambda_2$ are intertwined by $P$ for all $\lambda \in \Omega^0$;
(c) $\Sigma_{\Omega_1}$ and $\Sigma_{\Omega_2}$ are intertwined by $P$ for some $\lambda_0 \in \Omega^0$.

(vii) The following conditions are equivalent:
(a) $\Sigma_1$ is the $\Omega$-restriction to $\mathcal{X}_1$, or the $\Omega$-projection or $\Omega$-compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ of $\Sigma_2$;
(b) $\Sigma^\lambda_1$ is the restriction to $\mathcal{X}_1$, or the projection or compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ of $\Sigma^\lambda_2$, respectively, for all $\lambda \in \Omega^0$;
(c) $\Sigma^\lambda_{\Omega_1}$ is the restriction to $\mathcal{X}_1$, or the projection or compression onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ of $\Sigma^\lambda_{\Omega_2}$, respectively, for some $\lambda_0 \in \Omega^0$.

(viii) For each $\lambda \in \Omega^0$ the subspaces $\mathcal{X}^\Omega_{\min}, Z^\Omega_{\min}$, and $Z^\Omega_{\max}$ in Lemma 7.1.57 coincide with the corresponding subspaces $\mathcal{X}_{\min}, Z_{\min}$, and $Z_{\max}$ in Lemma 3.4.23 applied to $\Sigma^\lambda$.

(ix) For each $\lambda \in \Omega^0$ the $\Omega$-compressions of $\Sigma$ described in Theorem 7.1.58 have the same structure as the compressions of $\Sigma^\lambda$ described in Theorem 3.4.23.

(x) The following conditions are equivalent:
(a) $\Sigma$ is $\Omega$-minimal;
(b) $\Sigma^\lambda$ is minimal for all $\lambda \in \Omega^0$;
(c) $\Sigma^\lambda_{\Omega}$ is minimal for some $\lambda_0 \in \Omega^0$.

(xi) For each $\lambda \in \Omega^0$ there is a one-to-one correspondence between the families of minimal $\Omega$-compressions of $\Sigma$ described in Theorem 7.2.4 and the families of minimal compressions of $\Sigma^\lambda$ described in Theorem 3.4.22.

(xii) The following conditions are equivalent:
(a) The minimal $\Omega$-compression of $\Sigma$ is unique;
(b) The minimal compression of $\Sigma^\lambda$ is unique for all $\lambda \in \Omega^0$;
(c) The minimal compression of $\Sigma^\lambda_{\Omega}$ is unique for some $\lambda_0 \in \Omega^0$.

Proof. This follows from Theorem 5.3.9, Lemma 6.2.13, and Propositions 2.5.49 and 2.5.50 applied to the bounded i/s/o representations $\Sigma^\lambda_{\i/o}$ of the bounded s/s systems $\Sigma^\lambda$.

$\square$
7.3. Notes and Comments (Jan 02, 2016)
CHAPTER 8

Well-Posed Input/State/Output Systems (Jan 02, 2016)

In Chapters 6 and 7 we studied i/s/o systems and s/s systems in the frequency domain, and developed several frequency domain notions. We now go back to the time domain, and introduce the classes of well-posed s/s systems and well-posed i/s/o systems. They correspond to each other in the sense that a s/s system is well-posed if and only if it has a well-posed i/s/o representations. In order to prepare the ground for the theory of passive s/s and i/s/o systems we now switch from using signals in $L^1_{loc}$ to using signals in $L^2_{loc}$ instead. In this chapter we look at well-posed i/s/o systems, and return to the class of well-posed s/s systems in Chapter 9.
8.1. Basic Properties of Well-Posed Input/State/Output Systems (Jan 02, 2016)

8.1.1. The definition of a well-posed i/s/o system. In this chapter we shall use a slightly different setting from the one used in Chapters \[1\] and change the definition of a generalized trajectory by replacing the space \(L^1_{\text{loc}}\) by \(L^p_{\text{loc}}\) for some \(p\). As the following remark says, this is, indeed, possible.

8.1.1. REMARK. All the results in Chapters \[1\] remain true if we throughout replace \(L^1_{\text{loc}}\) by \(L^p_{\text{loc}}\) in the definition of a generalized trajectory on some interval \(I\), where \(1 \leq p < \infty\). That is,

(i) \([\begin{bmatrix} x \\ w \end{bmatrix}]\) is a generalized trajectory of a s/s system \(\Sigma = (V; \mathcal{X}, \mathcal{W})\) on the interval \(I\) if \([\begin{bmatrix} x \\ w \end{bmatrix}] \in \left[C(I; \mathcal{X}) \atop L^p_{\text{loc}}(I; \mathcal{W})\right]\) and \([\begin{bmatrix} x \\ w \end{bmatrix}]\) is the limit in \(\left[C(I; \mathcal{X}) \atop L^p_{\text{loc}}(I; \mathcal{W})\right]\) of a sequence of classical trajectories of \(\Sigma\) in \(I\), and

(ii) \([\begin{bmatrix} x \\ u \\ y \end{bmatrix}]\) is a generalized trajectory of an i/s/o system \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) on the interval \(I\) if \([\begin{bmatrix} x \\ u \\ y \end{bmatrix}] \in \left[C(I; \mathcal{X}) \atop L^p_{\text{loc}}(I; \mathcal{U})\right] \times \left[C(I; \mathcal{X}) \atop L^p_{\text{loc}}(I; \mathcal{Y})\right]\) and \([\begin{bmatrix} x \\ u \\ y \end{bmatrix}]\) is the limit in \(\left[C(I; \mathcal{X}) \atop L^p_{\text{loc}}(I; \mathcal{U})\right] \times \left[C(I; \mathcal{X}) \atop L^p_{\text{loc}}(I; \mathcal{Y})\right]\) of a sequence of classical trajectories of \(\Sigma\) in \(I\).

Note that the convergence with respect to the state component \(x\) remains the same as before (i.e., convergence in \(C(I; \mathcal{X})\)). In the sequel, when we refer to results in Chapters \[1\] we throughout mean references to the corresponding results which are valid for generalized trajectories of the \(L^p\)-type described above. (All notions related to classical trajectories remain the same as before.)

Remark 8.1.1 makes it possible to study i/s/o and s/s systems which are well-posed in a setting where inputs and outputs belong locally to \(L^p\) for any \(p \in [1, \infty)\). However, we shall not do so in this text, but instead choose the specific value \(p = 2\). This is the value which is by far the most important one in the study of passive i/s/o and s/s systems. Thus,

throughout the rest of this monograph we shall throughout assume that all inputs and outputs of generalized i/s/o trajectories and all signal parts of generalized s/s trajectories belong locally to \(L^2\), and that we use convergence in \(L^2_{\text{loc}}\) in the input, output, and signal components when generalized trajectories are approximated by classical trajectories.

See, e.g., Staffans \[2005\] for a treatment which covers all values of \(p \in [1, \infty)\), as well as some results for the case \(p = \infty\).

8.1.2. Definition. An i/s/o system \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) is well-posed if it has the following properties:

(i) \(S\) is single-valued and closed;

(ii) \(\Sigma\) is solvable (i.e., for every \([\begin{bmatrix} x^0 \\ u_0 \end{bmatrix}] \in \text{dom} (S)\) there exists some classical future trajectory \([\begin{bmatrix} x \\ u \end{bmatrix}]\) with \([\begin{bmatrix} x(0) \\ u(0) \end{bmatrix}] = [\begin{bmatrix} x^0 \\ u_0 \end{bmatrix}]\);

(iii) for each \(x^0 \in \mathcal{X}\) and each \(u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U})\) there exists a generalized future trajectory \([\begin{bmatrix} x \\ u \end{bmatrix}]\) of \(\Sigma\) with initial state \(x(0) = x^0\) (and input function \(u\));

(iv) for some triple of admissible norms \(\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{U}},\) and \(\|\cdot\|_{\mathcal{Y}}\) in \(\mathcal{X}, \mathcal{U},\) and \(\mathcal{Y},\) respectively, there exists a nonnegative locally bounded function \(\eta\) on
\[ \mathbb{R}^+ \text{ such that all generalized future trajectories } \left[ \begin{array}{c} x \\ u \end{array} \right] \text{ of } \Sigma \text{ satisfy} \]

\[
\|x(t)\|_X^2 + \int_0^t \|y(s)\|_Y^2 \, ds
\leq \eta(t) \left( \|x(0)\|_X^2 + \int_0^t \|u(s)\|_U^2 \, ds \right), \quad t \in \mathbb{R}^+.
\]

(8.1.1)

Here "locally bounded" means that \( \eta \) is bounded on each finite interval \([0, T]\). Since all admissible norms in \( X, U, \) and \( Y \) are equivalent to each other, if condition (8.1.1) holds for some admissible norms \( \| \cdot \|_X, \| \cdot \|_U, \) and \( \| \cdot \|_Y \) in \( X, U, \) and \( Y \), then it holds for all admissible norms \( \| \cdot \|_X, \| \cdot \|_U, \) and \( \| \cdot \|_Y \) in \( X, U, \) and \( Y \) (for some different function \( \eta \), which may depend on the chosen norms).

8.1.3. Lemma. Every semi-bounded \( i/s/o \) system is well-posed.

Proof. Let \( \Sigma = (\left[ \begin{array}{c} A \\ B \\ C \\ D \end{array} \right]; X, U, Y) \) be semi-bounded, and denote the evolution semigroup of \( \Sigma \) by \( \mathcal{S} \). Then condition (i) in Definition 8.1.2 holds. That also conditions (ii)–(iv) hold follows from Lemma 4.2.7 and Theorem 4.2.6 together with the fact that \( B, C, \) and \( D \) are bounded and \( \sup_{t \in [0, T]} \mathcal{S}^t \| x \|_X < \infty \) for each finite \( T \).

8.1.4. Lemma. Every well-posed \( i/s/o \) system \( \Sigma = (S; X, U, Y) \) has the following properties:

(i) The domain of the main operator \( A \) of \( \Sigma \) is dense in \( X \).
(ii) For every \( u^0 \in U \) there exists some \( x^0 \in X \) such that \( \left[ \begin{array}{c} x^0 \\ u^0 \end{array} \right] \in \text{dom} \, (S) \).
(iii) \( \text{dom} \, (S) \) is dense in \( \left[ X \atop U \right] \).
(iv) \( \Sigma \) is regular.
(v) \( \Sigma \) has the uniqueness property.
(vi) \( \Sigma \) has the continuation property.

Proof. Proof of (i): For every \( x^0 \in X \) there exists a generalized future trajectory \( \left[ \begin{array}{c} x \\ u \end{array} \right] \) of \( \Sigma \) with initial state \( x(0) = x^0 \). It follows from Lemma 4.2.5 that there exists a sequence of classical future trajectories \( \left[ \begin{array}{c} x_n \\ y_n \end{array} \right] \) of \( \Sigma \) such that \( \left[ \begin{array}{c} x_n \\ y_n \end{array} \right] \rightarrow \left[ \begin{array}{c} x \\ y \end{array} \right] \) in \( \left[ X \atop Y \right] \) as \( n \rightarrow \infty \). Here each \( x_n(0) \in \text{dom} \, (A) \) and \( x_n(0) \rightarrow x^0 \) as \( n \rightarrow \infty \), and hence \( \text{dom} \, (A) \) is dense in \( X \).

Proof of (ii): Let \( u^0 \in U \). Then by condition (iii) in Definition 8.1.2 there exists a generalized future trajectory \( \left[ \begin{array}{c} x \\ u \end{array} \right] \) of \( \Sigma \) with initial state zero and input function \( u(t) = u^0 \), \, t \in \mathbb{R}^+ \). From Lemma 4.2.5 follows that there exists a sequence of classical future trajectories \( \left[ \begin{array}{c} x_n(t) \\ y_n(t) \end{array} \right] \) (with the same constant input function). These trajectories satisfy \( \left[ \begin{array}{c} x_n(t) \\ u_n(t) \end{array} \right] = \left[ \begin{array}{c} x_n(0) \\ u^0(t) \end{array} \right] \in \text{dom} \, (S) \) for all \( t \in \mathbb{R}^+ \). This proves (ii).

Proof of (iii): It follows from (ii) that given any \( \left[ \begin{array}{c} x \\ u \end{array} \right] \in \left[ X \atop U \right] \) we may first choose some \( x^1 \in X \) such that \( \left[ \begin{array}{c} x^1 \\ u^0 \end{array} \right] \in \text{dom} \, (S) \). We may then use (i) to approximate \( x^0 - x^1 \) by a sequence \( x_n \in \text{dom} \, (A) \) which converges to \( x^0 - x^1 \) in \( X \) as \( n \rightarrow \infty \). Then \( \left[ \begin{array}{c} x^1 + x_n \\ u^0 \end{array} \right] = \left[ \begin{array}{c} x^1 \\ u^0 \end{array} \right] + \left[ \begin{array}{c} x_n \\ u^0 \end{array} \right] \in \text{dom} \, (S) \) and \( \left[ \begin{array}{c} x^1 + x_n \\ u^0 \end{array} \right] \rightarrow \left[ \begin{array}{c} x^0 \\ u^0 \end{array} \right] \) in \( \left[ X \atop U \right] \) as \( n \rightarrow \infty \).
8.1. BASIC PROPERTIES OF WELL-POSED INPUT/STATE/OUTPUT SYSTEMS (Jan 02, 2016) 445

Proof of (iv): That (iv) holds follows from (iii) and condition (i) in Definition 8.1.2.

Proof of (v): Suppose that \( \begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix} \) is a classical trajectory of \( \Sigma \) on the interval \([0, T]\) with initial state \( x(0) = 0 \). Since \( \Sigma \) is solvable this trajectory can be extended to a classical future trajectory of \( \Sigma \). This trajectory is also a generalized future trajectory of \( \Sigma \), and therefore it follows from condition (iv) in Definition 8.1.2 that \( x = 0 \). Thus \( \Sigma \) has the uniqueness property.

Proof of (vi): Let \( \begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix} \) be a generalized trajectory of \( \Sigma \) on the interval \([0, T]\). By condition (iii) in Definition 8.1.2 and the time-invariance of \( \Sigma \) (see Lemma 2.4.1) \( \Sigma \) has a generalized trajectory on \([T, \infty)\) with initial state \( x_1(T) \) and input function zero. Define \( \begin{bmatrix} u \\ x \\ y \end{bmatrix} \) by (2.4.1). Then by Lemma 2.4.29 \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a generalized future trajectory of \( \Sigma \). This shows that \( \Sigma \) has the continuation property.

8.1.5. LEMMA. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system, and let \( I \) be the interval \( I = [t_0, t_1] \) or \( I = [t_0, \infty) \) where \( t_0 \) and \( t_1 \) are finite. Then the following claims are true:

(i) For each \( x^0 \in X \) and each \( u \in L^2_{\text{loc}}(I; U) \) there exists a unique generalized trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) satisfying \( x(t_0) = x^0 \).

(ii) For every triple of admissible norms \( \| \cdot \|_X, \| \cdot \|_U, \) and \( \| \cdot \|_Y \) in \( X, U, \) and \( Y, \) respectively, there exists a nonnegative locally bounded function \( \eta \) on \( \mathbb{R}^+ \) such that all generalized trajectories \( \begin{bmatrix} u \\ x \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) satisfy

\[
\|x(t)\|_X^2 + \int_{t_0}^{t} \|y(s)\|_Y^2 \, ds \\
\leq \eta(t - t_0) \left( \|x(t_0)\|_X^2 + \int_{t_0}^{t} \|u(s)\|_U^2 \, ds \right), \quad t \in I.
\]

(8.1.2)

The function \( \eta \) in [8.1.2] can be taken to be equal to the function \( \eta \) in [8.1.1] if the admissible norms in \( X, U, \) and \( Y \) are chosen to be the same. If \( \eta \) is bounded, then the function \( \eta(t - t_0) \) in [8.1.2] may be replaced by the finite constant \( \sup_{s \in I} \eta(s - t_0) \) (this supremum is finite since \( \eta \) is locally bounded).

Proof. If \( I = [t_0, \infty) \) then this follows from condition (iv) in Definition 8.1.2 and the time-invariance of \( \Sigma \) (see Lemma 2.4.1). If \( I = [t_0, t_1] \) then every trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) on \([t_0, t_1]\) can be extended to a trajectory on \([t_0, \infty)\) since \( \Sigma \) has the continuation property, and consequently (8.1.2) holds in this case, too.

8.1.6. LEMMA. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system.

(i) Let \( I \) be the interval \( I = \mathbb{R} \) or \( I = (-\infty, t_1] \) for some \( t_1 \in \mathbb{R} \). For every \( u \in L^2_{\text{loc}}(I; U) \) whose support is bounded to the left there exists a unique generalized trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) whose support is bounded to the left.

More precisely, if \( u \) vanishes on some interval \((-\infty, T) \subset I \), then also \( x \) and \( y \) vanish on \((-\infty, T) \).

(ii) For every triple of admissible norms \( \| \cdot \|_X, \| \cdot \|_U, \) and \( \| \cdot \|_Y \) in \( X, U, \) and \( Y, \) respectively, there exists a nonnegative locally bounded function \( \eta \) on \( \mathbb{R}^+ \) such that all generalized trajectories \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \( I \) which vanish on
the interval \([-∞, T]\) ⊂ I satisfy
\[
\|x(t)\|_X^2 + \int_T^t \|y(s)\|_Y^2 \, ds
\]
(8.1.3)
\[
\leq \eta(t - T) \int_T^t \|u(s)\|_U^2 \, ds, \quad t \in I.
\]

The function \(\eta\) in (8.1.3) can be taken to be equal to the function \(\eta\) in (8.1.1) if the admissible norms in \(X, U, Y\) are chosen to be the same.

**Proof.** Let \(u \in L^2_{loc}(I; U)\), and suppose that \(u(t) = 0\) for \(t \in (-∞, T]\). By Lemma 8.1.5, there exists a unique generalized trajectory \(\begin{bmatrix} x \\ y \end{bmatrix}\) of \(Σ\) on the interval \(I \cap \{T, ∞\}\) with initial state \(x(T) = 0\). By Lemma 2.4.30 (see also Lemma 1.3.29), this trajectory can be extended to a trajectory on the full interval \(I\) with input \(u\) by defining \(x(t) = 0\) and \(y(t) = 0\) for \(t \in (-∞, T]\). This proves the existence part of (i).

If \(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\) is another generalized trajectory of \(Σ\) on \(I\) which vanishes on the interval \((-∞, T_1]\), then \(\begin{bmatrix} x-x_1 \\ 0 \end{bmatrix}\) is a generalized trajectory of \(Σ\) on \(I\) which vanishes on \((-∞, T_2]\) where \(T_2 = \min\{T, T_1\}\), and it follows from Lemma 8.1.5 applied to the interval \(I \cap \{T_1, ∞\}\) that \(x_1 = x\) and \(y_1 = y\).

8.1.7. **Remark.** It is possible to assume, without loss of generality, that the function \(\eta\) in condition (iv) of (8.1.1), (8.1.2), and (8.1.3) is nondecreasing, because we may always replace \(\eta(t)\) by \(\sup_{0 \leq s \leq t} \eta(s)\). This supremum is finite for each \(t \in \mathbb{R}^+\) since \(\eta\) is assumed to be locally bounded. Since \(\|x(t)\|_X \to \|x(0)\|_X\) in (8.1.1) as \(t \downarrow 0\) the function \(\eta\) always satisfies \(\liminf_{t\downarrow 0} \eta(t) \geq 1\), so if we assume, in addition, that \(\eta\) is nondecreasing, then \(\eta(t) \geq 1\) for all \(t \in \mathbb{R}^+\).

Conditions (iii) and (iv) in Definition 8.1.2 involves the notion of generalized future trajectories of \(Σ\). It is possible to replace these conditions by two other conditions which involves only classical trajectories future of \(Σ\).

8.1.8. **Lemma.** An i/s/o system \(Σ = (S; X, U, Y)\) is well-posed if and only if the following conditions hold:

(i') \(S\) is single-valued and closed, and the set
\[
X_0 = \left\{ x^0 \in X \mid \begin{bmatrix} x^0 \\ u^0 \end{bmatrix} \in \text{dom} (S) \text{ for some } u^0 \in U \right\}
\]
is dense in \(X\);

(ii') \(Σ\) is solvable (i.e., condition (ii) in Definition 8.1.2 holds);

(iii') the set of all input components \(u\) of all classical future trajectories \(\begin{bmatrix} x \\ y \end{bmatrix}\) of \(Σ\) satisfying \(\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\) is dense in \(L^2_{loc}(\mathbb{R}^+; U)\);

(iv') all classical future trajectories of \(Σ\) satisfy (8.1.1).

**Proof.** Suppose first that \(Σ\) is well-posed, i.e., conditions (i')–(iv') in Definition 8.1.2 hold. That (i') holds follows from condition (i) in Definition 8.1.2 and Lemma 8.1.4. That also (iii') holds follows from condition (iii) in Definition 8.1.2 and Lemma 2.4.25. Finally, condition (iv') in Definition 8.1.2 implies (iv') since every classical trajectories is also a generalized trajectory.
Conversely, suppose that conditions (i'), (ii), (iii'), and (iv') hold. Then conditions (i) and (ii) in Definition 8.1.2 hold. That also (iv) holds follows from the definition of a generalized trajectory and the fact that integrals in (8.1.1) depend continuously on \( u \) and \( y \) in the appropriate local \( L^2 \)-norm.

It remains to prove that condition (iii) in Definition 8.1.2 holds. Fix some arbitrary \( x^0 \in X \) and \( u \in L^2_{\text{loc}}(\mathbb{R}^+;U) \). It follows from condition (i') that we can find some sequence \( \{ x^0_n \} \in \text{dom}(S) \) such that \( x^0_n \to x^0 \) as \( n \to \infty \). Since \( \Sigma \) is solvable, there exists a sequence of classical future trajectories \( \begin{bmatrix} x_n^1 \\ u_n^1 \\ y_n^1 \end{bmatrix} \) of \( \Sigma \) satisfying

\[
\begin{bmatrix} x_n^1(0) \\ u_n^1(0) \end{bmatrix} = \begin{bmatrix} x^0_n \\ u^0_n \end{bmatrix}.
\]

By condition (iii') in Lemma 8.1.8, there exists another sequence of classical future trajectories \( \begin{bmatrix} x_n^2 \\ u_n^2 \\ y_n^2 \end{bmatrix} \) of \( \Sigma \) with \( x_n^2(0) = 0 \) such that \( u_n^1 + u_n^2 \to u \) in \( L^2_{\text{loc}}(\mathbb{R}^+;U) \) as \( n \to \infty \). Define \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_n^1 \\ u_n^1 \\ y_n^1 \end{bmatrix} + \begin{bmatrix} x_n^2 \\ u_n^2 \\ y_n^2 \end{bmatrix} \). Then each \( \begin{bmatrix} x_n \\ u_n \\ y_n \end{bmatrix} \) is a classical future trajectory of \( \Sigma \), and \( x_n(0) \to x^0 \) in \( X \) and \( u_n \to u \) in \( L^2_{\text{loc}}(\mathbb{R}^+;U) \) as \( n \to \infty \). It follows from (8.1.1) applied to the difference of two such trajectories that \( x_n \) is a Cauchy sequence in \( C(\mathbb{R}^+;X) \) and that \( y_n \) is a Cauchy sequence in \( L^2_{\text{loc}}(\mathbb{R}^+;Y) \). Thus there exist functions \( x \in C(\mathbb{R}^+;X) \) and \( y \in L^2_{\text{loc}}(\mathbb{R}^+;Y) \) such that \( x_n \to x \) in \( C(\mathbb{R}^+;X) \) and \( L^2(\mathbb{R}^+;Y) \) as \( n \to \infty \). By Definition 2.1.7, \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) is a generalized future trajectory of \( \Sigma \).

Below we give a third set of equivalent conditions for an i/s/o system to be well-posed. In this set of conditions we concentrate the attention to the set of generalized trajectories of \( \Sigma \) on some finite time interval \([0,T]\). (It is also possible to replaced generalized trajectories by classical trajectories in the same way as we did in Lemma 8.1.8.)

8.1.9. Lemma. An i/s/o system \( \Sigma = (S;X,U,Y) \) is well-posed if and only if it satisfies the following conditions:

(i) \( S \) is single-valued and closed (i.e., condition (i) in Definition 8.1.2 holds);
(ii) \( \Sigma \) is solvable (i.e., condition (ii) in Definition 8.1.2 holds);
(iii') For some constant \( T > 0 \), all \( x^0 \in X \), and all \( u \in L^2([0,T];U) \) there exists a generalized trajectory \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) of \( \Sigma \) on \([0,T]\) with initial state \( x(0) = x^0 \) (and input function \( u \));
(iv") for some some triple of admissible norms \( \| \cdot \|_X, \| \cdot \|_U, \) and \( \| \cdot \|_Y \) in \( X, U, \) and \( Y, \) respectively, there exists a constant \( K \geq 1 \) such that the generalized trajectories \( \begin{bmatrix} x \\ u \\ y \end{bmatrix} \) in (iii") satisfy

\[
\| x(t) \|^2_X + \int_0^t \| y(s) \|^2_Y ds 
\leq K \left( \| x(0) \|^2_X + \int_0^t \| u(s) \|^2_U ds \right), \quad t \in [0,T].
\]

Moreover, it is possible to choose the function \( \eta \) in condition (iv) in Definition 8.1.2 to be equal to \( \eta(t) = Me^{\alpha t} \) for some \( M \geq 1 \) and \( \alpha \geq 0 \).
Assume first that \( \Sigma \) is well-posed. Clearly condition (iii) in Definition 8.1.2 implies (iii\textsuperscript{ii}). Since \( \Sigma \) has the continuation property (see Lemma 8.1.4), each generalized trajectory \( \left[ \begin{array}{c} x \\ u \\ y \end{array} \right] \) of \( \Sigma \) on \([0, T]\) can be extended to a generalized future trajectory of \( \Sigma \), and together with condition (iv) in Definition 8.1.2 implies (iv\textsuperscript{ii}).

In the rest of the proof we assume that (i), (ii), (iii\textsuperscript{ii}) and (iv\textsuperscript{ii}) hold, and prove that this implies conditions (iii) and (iv) in Definition 8.1.2.

Proof of (iii): Let \( x^0 \in \mathcal{X} \) and \( u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \). By (iii\textsuperscript{ii}) there exists a generalized trajectory \( \left[ \begin{array}{c} x_n \\ u_n \\ y_n \end{array} \right] \) of \( \Sigma \) on \([0, T]\) with \( x_0(0) = x^0 \). It follows from condition (iii\textsuperscript{ii}) and the time invariance of \( \Sigma \) that for each \( n \in \mathbb{N} \) there exists a generalized trajectory \( \left[ \begin{array}{c} x_n \\ u_n \\ y_n \end{array} \right] \) of \( \Sigma \) on \([nT, (n + 1)T]\) satisfying \( x_n(nT) = x_{n-1}(nT) \) and \( u_n(t) = u(t) \) for \( t \in [nT, (n + 1)T] \). Define \( x(t) = x_n(t) \) and \( y(t) = y_n(t) \) for \( t \in [nT, (n + 1)T], n \in \mathbb{Z}^+ \). Then \( x \in C(\mathbb{R}^+; \mathcal{X}) \) and \( y \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \). By applying Lemma 2.4.24 first to the intervals \( I_1 = [0, T] \) and \( I_2 = [T, 2T] \), then to the intervals \( I_1 = [0, 2T] \) and \( I_2 = [2T, 3T] \), etc., we find that the restriction of \( \left[ \begin{array}{c} x_n \\ u_n \end{array} \right] \) to any finite interval of the type \([0, kT]\) for some \( k \in \mathbb{N} \) is a mild trajectory of \( \Sigma \) on this interval. Therefore by Lemma 2.4.28 it is also a generalized trajectory of \( \Sigma \) on any finite closed interval with left end-point zero. By Definition 2.1.7 this means that \( \left[ \begin{array}{c} x \\ u \end{array} \right] \) is a generalized future trajectory of \( \Sigma \). This trajectory has the given initial state \( x(0) = x^0 \), and the input function is the given input function \( u \). Thus condition (iii) in Definition 8.1.2 holds.

Proof of (iv): Let \( x^0 \in \mathcal{X} \) and \( u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \). As we saw above, condition (iii) in Definition 8.1.2 holds. Let \( \left[ \begin{array}{c} x \\ u \end{array} \right] \) be a generalized future trajectory of \( \Sigma \) with initial state \( x(0) = x^0 \) (and input function \( u \)). By the time invariance of \( \Sigma \) the condition (8.1.4) holds on any closed interval \([t_0, t_0 + T]\) with \( x(0) \) replaced by \( x(t_0) \) and \( \int_0^t \) replaced by \( \int_{t_0}^{t+T} \) for all \( t \in [t_0, t_0 + T] \). Thus, for all \( n \in \mathbb{Z}^+ \) and all \( t \in [nT, (n + 1)T] \) we have

\[
\|x(t)\|^2_{\mathcal{X}} + \int_{nT}^{t} \|y(s)\|^2_{\mathcal{Y}} ds \leq K \left( \|x(nT)\|^2_{\mathcal{X}} + \int_{nT}^{t} \|u(s)\|^2_{\mathcal{U}} ds \right).
\]

The same inequality with \( n \) replaced by \( k \) and \( t \) replaced by \( k + 1 \) gives

\[
\|x((k + 1)T)\|^2_{\mathcal{X}} + \int_{kT}^{(k + 1)T} \|y(s)\|^2_{\mathcal{Y}} ds \leq K \left( \|x(kT)\|^2_{\mathcal{X}} + \int_{kT}^{(k + 1)T} \|u(s)\|^2_{\mathcal{U}} ds \right).
\]

By adding (8.1.5) to all the inequalities (8.1.6) with \( k = 0, 1, \ldots, n-1 \) (and dropping some nonnegative terms on the left-hand side) we get

\[
\|x(t)\|^2_{\mathcal{X}} + \int_{0}^{t} \|y(s)\|^2_{\mathcal{Y}} ds \leq K \left( \sum_{k=0}^{n} \|x(kT)\|^2_{\mathcal{X}} + \int_{0}^{t} \|u(s)\|^2_{\mathcal{U}} ds \right).
\]

Thus, to complete the proof it suffices to show that there exists a finite constant \( M(n) \) such that

\[
\sum_{k=0}^{n} \|x(kT)\|^2_{\mathcal{X}} \leq M(n) \left( \|x(0)\|^2_{\mathcal{X}} + \int_{0}^{nT} \|u(s)\|^2_{\mathcal{U}} ds \right).
\]
To do this we observe that (8.1.6) implies that
\[
\|x((k+1)T)\|_X^2 \leq K \left( \|x((k)T)\|_X^2 + \int_0^{nT} \|u(s)\|_{U}^2 \, ds \right), \quad 0 \leq k \leq n.
\]
This inequality can be solved recursively, and by using the fact that \( K \geq 1 \) (see Remark 8.1.7) the result can be simplified into
\[
\|x((k)T)\|_X^2 \leq (n+1)K^n \left( \|x((0)T)\|_X^2 + \int_0^{nT} \|u(s)\|_{U}^2 \, ds \right), \quad 0 \leq k \leq n.
\]
By adding over \( k = 0, 1, \ldots, n \) we get
\[
\sum_{k=0}^{n} \|x((k)T)\|_X^2 \leq (n+1)^2K^n \left( \|x((0)T)\|_X^2 + \int_0^{nT} \|u(s)\|_{U}^2 \, ds \right).
\]
Substituting this into (8.1.7) and again using the fact that \( K \geq 1 \) and \( n \leq t/T \) we get
(8.1.8)
\[
\|x(t)\|_X^2 + \int_0^t \|y(s)\|_{Z}^2 \, ds \leq (t/T + 2)^2K^{t/T+1} \left( \|x((0)T)\|_X^2 + \int_0^{t} \|u(s)\|_{U}^2 \, ds \right).
\]
Thus we may take \( \eta(t) = (t/T + 2)^2K^{t/T+1} \) in (8.1.1). If we define \( \alpha' = \log K^{1/T} \), then \( K^{t/T} = e^{\alpha't} \), and \( \eta(t) = (t/T + 2)^2K^{t/T+1} = (t/T + 2)^2Ke^{\alpha't} \). For each \( \alpha > \alpha' \) we have \( \eta(t) \leq Me^{\alpha t} \), where
\[
M = \max_{t \in R^+} (t/T + 2)^2Ke^{-(\alpha-\alpha')t}.
\]
Thus, (8.1.1) holds if we take \( \eta(t) = Me^{\alpha t} \) where \( \alpha \) and \( M \) are defined as above. \( \square \)

8.1.2. The fundamental i/s/o solution of a well-posed i/s/o system.
Below we shall present formulas for the state and output components of generalized future trajectories for any input \( u \in L^2_{\text{loc}}(R^+; U) \) and any initial state \( x^0 \in X \), as well as formulas for trajectories defined on arbitrary closed intervals (if these intervals are unbounded to the left, we shall assume that the support of the trajectories is bounded to the left). This formulas involve a \( C_0 \) semigroup, the so called evolution semigroup of the system, and three additional maps, which we call the input map, the output map, and the input/output map of the system.

Before we introduce the fundamental i/s/o solution of a well-posed i/s/o system we need to define some additional function spaces. (For later reference we formulate this definition so that it can also be applied when \( L^2 \) is replaced by \( L^p \) for some \( p \in [1, \infty) \).)

8.1.10. Notation. Let \( Z \) be a \( B \)-space, and let \( p \in [1, \infty) \).

(i) For each closed unbounded interval \( I = (-\infty, t_0] \) with finite right endpoint \( t_0 \) the space \( L^p_{0}(I; Z) \) is the space of all functions \( z \in L^p(I; Z) \) with bounded support. A sequence \( z_n \in L^p_{0}(I; Z) \) tends to a function \( z \) in \( L^p_{0}(I; Z) \) as \( n \to \infty \) if \( z_n \to z \) in \( L^p(I; Z) \) and, in addition, all the functions \( z_n \) vanish on some common interval \( (-\infty, T) \subset I \).

(ii) The space \( L^p_{0,\text{loc}}(R; Z) \) is the space of all functions \( z \in L^p_{0,\text{loc}}(R; Z) \) with the property that the support of \( z \) is bounded to the left (i.e., \( z \) vanishes on some interval \( (-\infty, T) \)). A sequence \( z_n \in L^p_{0,\text{loc}}(R; Z) \) tends to a
function $z$ in $L^p_{\text{loc}}(\mathbb{R}; \mathcal{Z})$ as $n \to \infty$ if $z_n \to z$ in $L^p_{\text{loc}}(\mathbb{R}; \mathcal{Z})$ and, in addition, all the functions $z_n$ vanish on some common interval $(-\infty, T]$.

8.1.11. **Lemma.** Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a well-posed i/s/o system.

(i) For each $x^0 \in \mathcal{X}$ and $t \in \mathbb{R}^+$, define $\mathfrak{A}^t x^0$ by

$$\mathfrak{A}^t x^0 = x(t), \quad x^0 \in \mathcal{X}, \quad t \in \mathbb{R}^+,$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the generalized future trajectory of $\Sigma$ with initial state $x^0$ (and input zero) given by Lemma 8.1.5. Then $t \mapsto \mathfrak{A}^t$ is a $C_0$ semigroup in $\mathcal{X}$.

(ii) For each $x^0 \in \mathcal{X}$, define $\mathfrak{C} x^0$ by

$$\mathfrak{C} x^0 = y, \quad x^0 \in \mathcal{X},$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the generalized future trajectory of $\Sigma$ with initial state $x^0$ (and input zero) given by Lemma 8.1.5. Then $\mathfrak{C}$ is a continuous linear operator from $\mathcal{X}$ into $L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y})$.

(iii) For each $u \in L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{U})$ (i.e., $u \in L^2(\mathbb{R}^-; \mathcal{U})$ and the whose support of $u$ is bounded to the left), define $\mathfrak{B} u$ by the formula

$$\mathfrak{B} u = x(0), \quad u \in L^2_{\text{loc}}(I; \mathcal{U}),$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the unique generalized trajectory of $\Sigma$ on $\mathbb{R}^-$ given by Lemma 8.1.6. Then $\mathfrak{B}$ is a continuous linear operator from $L^2_{\text{loc}}(I; \mathcal{U})$ into $\mathcal{X}$.

(iv) For each $u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U})$ (i.e., $u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U})$ and the whose support of $u$ is bounded to the left), define $\mathfrak{D} u$ by the formula

$$\mathfrak{D} u = y, \quad u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}),$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the unique generalized trajectory of $\Sigma$ on $\mathbb{R}$ given by Lemma 8.1.6. Then $\mathfrak{D}$ is a continuous linear operator from $L^2_{\text{loc}}(\mathbb{R}; \mathcal{U})$ into $L^2_{\text{loc}}(\mathbb{R}; \mathcal{U})$.

**Proof.** **Proof of (i):** By Lemma 8.1.5 the operator $\mathfrak{C}$ is well-defined, and it follows from the linearity of $\Sigma$ that $\mathfrak{A}^t \in \mathcal{B}(\mathcal{X})$. That $\mathfrak{A}$ is a $C_0$ semigroup is proved in the same way as in the proof of Theorem 4.1.5.

**Proof of (ii):** By Lemma 8.1.5 the operator $\mathfrak{C}$ is well-defined, and it follows from the linearity of $\Sigma$ that $\mathfrak{C}$ is linear. The continuity of $\mathfrak{C}$ follows from (8.1.2).

**Proof of (iii):** By Lemma 8.1.6 the operator $\mathfrak{B}$ is well-defined, and it follows from the linearity of $\Sigma$ that $\mathfrak{B}$ is linear. To show that $\mathfrak{B}$ is continuous we let $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{X})$. Then all the functions $u_n$ vanish on some interval $(-\infty, T]$, and it follows from Lemma 8.1.6 that $\mathfrak{B} u_n \to \{0\}$ as $n \to \infty$. Thus $\mathfrak{B}$ is continuous.

**Proof of (iv):** By Lemma 8.1.6 the operator $\mathfrak{D}$ is well-defined, and it follows from the linearity of $\Sigma$ that $\mathfrak{D}$ is linear. To show that $\mathfrak{D}$ is continuous we let $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{X})$, and denote the corresponding generalized two-sided trajectories of $\Sigma$ by $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$. That $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{X})$ means that all the functions $u_n$ vanish on some interval $(-\infty, 0]$, and for each $t_0 > 0$ their restrictions to the interval $[t_0, t_1]$ tend to zero in $L^2([t_0, t_1]; \mathcal{U})$ as $n \to \infty$. By Lemma 8.1.6 all the functions $y_n$
vanish on \((-\infty, t_0]\) and the restrictions of \(y_n\) to \([t_0, t_1]\) tend to zero in \(L^2([t_0, t_1]; \mathcal{Y})\) as \(n \to \infty\). Thus \(\mathcal{D}\) is continuous. \(\square\)

8.1.12. DEFINITION. Let \(\Sigma\) be a well-posed i/s/o system. Then the \(C_0\) semigroup \(\mathbb{A}\) and the maps \(\mathbb{B}: L^2_0(\mathbb{R}^-; \mathcal{U}) \to \mathcal{X}\), \(\mathbb{C}: \mathcal{X} \to L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{X})\), and \(\mathbb{D}: L^2_{\text{loc}}(\mathbb{R}; \mathcal{X}) \to L^2_{\text{loc}}(\mathbb{R}; \mathcal{X})\) introduced in Lemma 8.1.11 are named as follows:

(i) \(\mathbb{A}\) is the evolution semigroup of \(\Sigma\).
(ii) \(\mathbb{B}\) is the input map of \(\Sigma\).
(iii) \(\mathbb{C}\) is the output map of \(\Sigma\).
(iv) \(\mathbb{D}\) is the (two-sided) input/output map (i/o map) of \(\Sigma\).
(v) The quadruple \(\left[\frac{\mathbb{A}}{\mathbb{B}\mathbb{C}\mathbb{D}}\right]\) is called the fundamental input/state/output solution (fundamental i/s/o solution) of \(\Sigma\).

8.1.13. LEMMA. Let \(\Sigma = \left(\left[\begin{array}{c} \mathcal{X} \\ \mathcal{U} \end{array}\right]\right)\) be a semi-bounded i/s/o system with evolution semigroup \(\mathbb{A}\). Then the input map \(\mathbb{B}\), the output map \(\mathbb{C}\), and the i/o map \(\mathbb{D}\) of \(\Sigma\) are given by the following formulas:

\[
\begin{align*}
(8.1.13a) & \quad \mathbb{B}u = \int_{-\infty}^{0} \mathbb{A}^{-s}Bu(s) \, ds, \quad u \in L^2_0(\mathbb{R}^-; \mathcal{U}), \\
(8.1.13b) & \quad (\mathbb{C}x^0)(t) = C\mathbb{A}^t x^0, \quad x^0 \in \mathcal{X}, \\
(8.1.13c) & \quad (\mathbb{D}u)(t) = Du(t) + \int_{-\infty}^{t} C\mathbb{A}^{t-s}Bu(s) \, ds, \quad u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}), \\
(8.1.13d) & \quad (\mathbb{D}u)(t) = Du(t) + \int_{-\infty}^{t} C\mathbb{A}^{t-s}Bu(s) \, ds, \quad u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}).
\end{align*}
\]

PROOF. This follows from Theorem 4.2.6 and Lemma 8.1.11. \(\square\)

The generalized trajectories of \(\Sigma\) given by Lemmas 8.1.5 and 8.1.6 can be expressed with the help of the fundamental i/s/o solution \(\left[\frac{\mathbb{A}}{\mathbb{B}\mathbb{C}\mathbb{D}}\right]\) as follows:

8.1.14. LEMMA. Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a well-posed i/s/o system fundamental i/s/o solution \(\left[\frac{\mathbb{A}}{\mathbb{B}\mathbb{C}\mathbb{D}}\right]\).

(i) The state and output components \(x\) respectively \(y\) of the generalized trajectory \(\left[\begin{array}{c} x \\ u \\ y \end{array}\right]\) in Lemma 8.1.5 can be written in the form

\[
\begin{align*}
(8.1.14a) & \quad x(t) = \mathbb{A}^{t-t_0}x^0 + \mathbb{B}\rho^{-t}i_1u, \quad t \in I, \\
(8.1.14b) & \quad y = \rho_1\rho^{-t_0}i_1\mathbb{C}x^0 + \rho_1\mathbb{D}t_1u.
\end{align*}
\]

Thus, in 8.1.14a \(u\) is first extended to the function \(i_1u\) defined on \(\mathbb{R}\) (by defining \(u(t) = 0\) for \(t \notin I\)), then \(i_1u\) is translated into \(\tau^t i_1u\) (so that \(t\) is mapped into zero), and finally \(\tau^t i_1u\) is restricted to the function \(\rho^{-t}i_1u\) defined on \(\mathbb{R}^-\) to which \(\mathbb{B}\) can be applied. In 8.1.14b \(\mathbb{C}x^0\) is first extended to the function \(i_1\mathbb{C}x^0\) (by taking the values on \(\mathbb{R}^-\) to be zero), then the new function is translated into \(\tau^{-t_0}i_1\mathbb{C}x^0\) (so that zero is mapped into \(t_0\)) and restricted to the function \(\rho_1\tau^{-t_0}i_1\mathbb{C}x^0\) defined on \(I\). The operator \(\mathbb{D}\) is applied to the extension of \(u\) to the function \(i_1u\) defined on \(\mathbb{R}\), and the result is again restricted to \(I\).
(ii) The state and output components $x$ respectively $y$ of the generalized trajectory $\begin{bmatrix} x \\
 y \end{bmatrix}$ in Lemma 8.1.6 can be written in the form

\begin{align}
(8.1.15a) \quad x(t) &= \mathfrak{B}_t \tau_t u, \quad t \in I, \\
(8.1.15b) \quad y &= \rho_{t} \mathfrak{D}_t u.
\end{align}

The interpretation of (8.1.15) is analogous to the interpretation of (8.1.14).

**Proof.** This follows from the time-invariance of $\Sigma$ and Lemmas 8.1.5 and 8.1.11.

8.1.15. Theorem. The fundamental i/s/o solution $\begin{bmatrix} x \\
 y \end{bmatrix}$ of a well-posed linear system $\Sigma = (\mathcal{S}, \mathcal{X}, \mathcal{U}, \mathcal{Y})$ satisfies the following algebraic conditions:

\begin{align}
(8.1.16a) \quad \mathfrak{A}^{s+t} &= \mathfrak{A}^s \mathfrak{A}^t, \quad s, t \in \mathbb{R}^+, \quad \mathfrak{A}^0 = 1_{\mathcal{X}}, \\
(8.1.16b) \quad \mathfrak{A}^t \mathfrak{B} &= \mathfrak{B} \tau^{-t}_t, \quad t \in \mathbb{R}^+, \\
(8.1.16c) \quad \mathfrak{C} \mathfrak{A}^t &= \tau^+_t \mathfrak{C}, \quad t \in \mathbb{R}^+, \\
(8.1.16d) \quad \tau^+ \mathfrak{D} &= \mathfrak{D} \tau^+_t, \quad t \in \mathbb{R}, \\
(8.1.16e) \quad \pi_- \mathfrak{D}_{\pi_+} &= 0, \quad \rho_{\tau} \mathfrak{D} = \mathfrak{B} \mathfrak{D}.
\end{align}

**Proof.** (i) Condition (8.1.16a) holds since $\mathfrak{A}$ is a $C_0$ semigroup.

(ii) Let $u \in L_0^2(\mathbb{R}^+; \mathcal{U})$ and $t \in \mathbb{R}^+$, and let $\begin{bmatrix} \tau^+_t u \\
 \tau^{-t}_t u \end{bmatrix}$ be the past generalized trajectory given by Lemma 8.1.6 with input function $\tau^+_t u$. From Lemma 8.1.11 we get $x(0) = \mathfrak{B} \tau^+_t u$ and $x(-t) = \mathfrak{B} u$. Since $\tau^+_t u$ vanishes in the interval $[-t, 0]$ it follows from Lemma 8.1.11 that $x(0) = \mathfrak{A}^t x(-t) = \mathfrak{A}^t \mathfrak{B} u$. Thus (8.1.16b) holds.

(iii) Let $x^0 \in \mathcal{X}$ and $t \in \mathbb{R}^+$, and let $\begin{bmatrix} x \\
 y \end{bmatrix}$ be the generalized future trajectory of $\Sigma$ given by Lemma 8.1.5. From Lemma 8.1.11 we get $x(t) = \mathfrak{A}^t x^0$, $y = \mathfrak{C} x^0$, and $\tau^{-t}_t y = \mathfrak{C}(t) = \mathfrak{A}^t x^0$. This proves (8.1.16c).

(iv) Let $u \in L_0^2(\mathbb{R}^+; \mathcal{U})$ and $t \in \mathbb{R}$, and let $\begin{bmatrix} \tau^+_t u \\
 \tau^{-t}_t u \end{bmatrix}$ be the past generalized two-sided trajectories of $\Sigma$ given by Lemma 8.1.6 with input functions $u$ respectively $\tau^t u$. Then $y = \mathfrak{D} u$ and $y_t = \mathfrak{D} \tau^+_t u$. On the other hand, by the time-invariance of $\Sigma$ also $\begin{bmatrix} \tau^+_t x \\
 \tau^{-t}_t y \end{bmatrix}$ is a generalized two-sided trajectories of $\Sigma$ with input function $\tau^+_t u$. It therefore follows from the uniqueness part of Lemma 8.1.6 that $\mathfrak{D} \tau^+_t y = y_t = \mathfrak{D} \tau^+_t y$. This proves (8.1.16d).

(v) That $\pi_+ \mathfrak{D}_{\pi_+} = 0$ follows from (8.1.12) and Lemma 8.1.6. Let $u \in L_0^2(\mathbb{R}^+; \mathcal{U})$ and let $\begin{bmatrix} \tau^+_t u \\
 \tau^{-t}_t u \end{bmatrix}$ be the generalized two-sided trajectory of $\Sigma$ given by Lemma 8.1.6 with input function $\tau^t u$. Since $(\tau^t u)(t) = u(t)$ for $t \in \mathbb{R}^+$ and $(\tau_t u)(t) = 0$ for $t > 0$ we get from (8.1.11), (8.1.10), and (8.1.12) $y = \mathfrak{D} \tau^+_t u$, $x(0) = \mathfrak{B} u$, and $\rho_{\tau} y = \mathfrak{C} x(0)$. Thus $\rho_+ \mathfrak{D} \tau^+_t u = \mathfrak{B} \mathfrak{D}$. This proves (8.1.16e).

In the sequel we shall also need to work with quadruples $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, where $\mathfrak{A}$ is a $C_0$ semigroup and $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ are operators satisfying conditions (8.1.16) in Theorem 8.1.15 without knowing in advance that $\begin{bmatrix} \mathfrak{A} \\
 \mathfrak{B} \\
 \mathfrak{C} \\
 \mathfrak{D} \end{bmatrix}$ is the fundamental i/s/o solution of a well-posed i/s/o system. For this reason we introduce the following notion.
8.1.16. Definition. Let \( \mathcal{X}, \mathcal{U}, \mathcal{Y} \) be \( H \)-spaces. By a formal fundamental i/s/o solution in \( (\mathcal{X}, \mathcal{U}, \mathcal{Y}) \) we mean a quadruple \( \left[ \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \right] \); where \( \mathfrak{A} \) is a \( C_0 \) semigroup in \( \mathcal{X} \), and \( \mathfrak{B}: L^2_0(\mathbb{R}^-; \mathcal{U}) \to \mathcal{X}, \mathfrak{C}: \mathcal{X} \to L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}), \) and \( \mathfrak{D}: L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}) \to L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}) \) are continuous linear operators satisfying conditions (8.1.10) in Theorem 8.1.13.

Since we shall make frequent use of the formulas for \( x \) and \( y \) in Lemmas 8.1.14 we introduce the following notation.

8.1.17. Definition. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system with evolution semigroup \( \mathfrak{A} \), input map \( \mathfrak{B} \), output map \( \mathfrak{C} \), and i/o map \( \mathfrak{D} \).

(i) The state transition map \( \mathfrak{A}^t_s = \mathfrak{A}^{t-s} \), the input map \( \mathfrak{B}^t_s = \mathfrak{B} \rho_{-\tau^t} l_{[s,t]} \), the output map \( \mathfrak{C}^t_s = \rho_{[s,t]} \tau^t + \mathfrak{C} \), and the input/output map \( \mathfrak{D}^t_s = \rho_{[s,t]} \mathfrak{D}_l_{[s,t]} \).

\[
\begin{align*}
\mathfrak{A}^t_s &= \mathfrak{A}^{t-s}, \\
\mathfrak{B}^t_s &= \mathfrak{B} \rho_{-\tau^t} l_{[s,t]}, \\
\mathfrak{C}^t_s &= \rho_{[s,t]} \tau^t + \mathfrak{C}, \\
\mathfrak{D}^t_s &= \rho_{[s,t]} \mathfrak{D}_l_{[s,t]},
\end{align*}
\]

(ii) The input map \( \mathfrak{B}^t = \mathfrak{B}^{t \mapsto (-\infty, t]; \mathcal{U}} \) and the input/output map \( \mathfrak{D}^t = \mathfrak{D}^{t \mapsto (-\infty, t]; \mathcal{U}} \) with initial time \( s \in \mathbb{R} \) and final time \( t \geq s \) are defined by

\[
\begin{align*}
\mathfrak{B}^t &= \mathfrak{B}^{t \mapsto (-\infty, t]} := \mathfrak{B} \rho_{-\tau^t} l_{(-\infty, t]}, \\
\mathfrak{D}^t &= \mathfrak{D}^{t \mapsto (-\infty, t]} := \rho_{(-\infty, t]} \mathfrak{D}_l_{(-\infty, t]},
\end{align*}
\]

(iii) The output map \( \mathfrak{C}_s = \mathfrak{C}^\infty_s = \rho_{[s,\infty)} \tau^s l_{+} \mathfrak{C} \), and the input/output map \( \mathfrak{D}_s = \mathfrak{D}^{\infty}_s = \rho_{[s,\infty)} \mathfrak{D}_l_{[s,\infty)} \), are defined by

\[
\begin{align*}
\mathfrak{C}_s &= \mathfrak{C}^\infty_s := \rho_{[s,\infty)} \tau^s l_{+} \mathfrak{C}, \\
\mathfrak{D}_s &= \mathfrak{D}^{\infty}_s := \rho_{[s,\infty)} \mathfrak{D}_l_{[s,\infty)},
\end{align*}
\]

The fundamental i/s/o solution \( \left[ \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \right] \) of a well-posed i/s/o system \( \Sigma \) can be recovered from the family of fundamental i/s/o solutions \( \left[ \mathfrak{A}^t_s, \mathfrak{B}^t_s, \mathfrak{C}^t_s, \mathfrak{D}^t_s \right] \) with initial time \( s \) and final time \( t \), \( -\infty < s \leq t < \infty \), as follows.

8.1.18. Lemma. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system with the family of fundamental i/s/o solutions \( \left[ \mathfrak{A}^t_s, \mathfrak{B}^t_s, \mathfrak{C}^t_s, \mathfrak{D}^t_s \right] \) with initial time \( s \) and final time \( t \), \( -\infty < s \leq t < \infty \).

(i) For every \( s \in \mathbb{R} \) the evolution semigroup \( \mathfrak{A} \) of \( \Sigma \) is given by \( \mathfrak{A}^t_s = \mathfrak{A}^{t+s}, \) \( t \in \mathbb{R}^+ \).

(ii) For every \( t \in \mathbb{R} \) the input map \( \mathfrak{B}^t \) and i/o map \( \mathfrak{D}^t \) of \( \Sigma \) with final time \( t \) is given by

\[
\begin{align*}
\mathfrak{B}^t &= \lim_{s \to -\infty} \mathfrak{B}^t_s \pi_{[s,t] l_{(-\infty, t]}}, \\
\mathfrak{D}^t &= \lim_{s \to -\infty} \mathfrak{D}^t_s \pi_{[s,t]} l_{[s,t]}.
\end{align*}
\]
(iii) For every \( s \in \mathbb{R} \) the output map \( C_s \) and i/o map \( D_s \) of \( \Sigma \) with initial time \( s \) is given by
\[
C_s = \lim_{t \to \infty} \rho_{[s, \infty)}(t, \cdot)\mathcal{C}_s, \quad D_s = \lim_{t \to \infty} t_{[s, \cdot]}\mathcal{D}_s \pi_{[s, \cdot]}.
\]

(iv) The fundamental i/s/o solution \([\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]\) of \( \Sigma \) is given by
\[
\mathcal{A}^t = \mathcal{A}_0^t, \quad t \in \mathbb{R}^+,
\]
\[
\mathcal{B} = \mathcal{B}^0 = \lim_{s \to -\infty} \mathcal{B}_s^0 \pi_{[s, 0]}(t-),
\]
\[
\mathcal{C} = \mathcal{C}_0 = \lim_{t \to \infty} \rho_{[0, \infty)}(t, \cdot)\mathcal{C}_0, \quad \mathcal{D} = \lim_{s \to -\infty, t \to \infty} t_{[s, \cdot]}\mathcal{D}_s^t \pi_{[s, \cdot]}.
\]

**Proof.** This follows from Definition 8.1.17.

With the notations in Definition 8.1.17 the formulas for \( x \) and \( y \) in Lemma 8.1.14 can be rewritten as follows.

**8.1.19. Lemma.** Let \( \Sigma = (S, X, \mathcal{U}, \mathcal{Y}) \) be well-posed i/s/o system with i/s/o quadruple \([\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]\). With the notation introduce in Definition 8.1.17 we have the following formulas for the state and output components of generalized trajectories of \( \Sigma \).

(i) If \([x, u] \) is a generalized trajectory of \( \Sigma \) on \( \mathbb{R} \) whose support is bounded to the left, then \( x(t) = \mathcal{B}^t u, \ t \in \mathbb{R} \), and \( y = \mathcal{Du} \).

(ii) If \([x, u] \) is a generalized trajectory of \( \Sigma \) on the interval \( I = (-\infty, t_1] \) whose support is bounded to the left, then \( x(t) = \mathcal{B}^t u, \ t \in (-\infty, t_1] \), and \( y = \mathcal{D}^t u \).

(iii) If \([x, u] \) is a generalized trajectory of \( \Sigma \) on the interval \( I = [t_0, \infty) \) satisfying \( x(t_0) = x^0 \), then \( x(t) = \mathcal{A}_0^t x^0 + \mathcal{B}^t u, \ t \in [t_0, \infty) \), and \( y = \mathcal{C}_0 x^0 + \mathcal{D}^t u \).

(iv) If \([x, u] \) is a generalized trajectory of \( \Sigma \) on the interval \( I = [t_0, t_1] \) satisfying \( x(t_0) = x^0 \), then \( x(t) = \mathcal{A}^t_0 x^0 + \mathcal{B}^t u, \ t \in [t_0, t_1] \), and \( y = \mathcal{C}^t_0 x^0 + \mathcal{D}^t_0 u \).

**Proof.** This follows immediately from Lemma 8.1.14 and Definition 8.1.17.

**8.1.3. The growth bound of a well-posed i/s/o system.** In Definition 8.1.2 of a well-posed system \( \Sigma \) we did not put any priori growth restrictions on the function \( \eta \) in \([8.1.1] \), but as we saw in Lemma 8.1.9 it is possible to take \( \eta(t) = Me^{\alpha t} \) for some constants \( M \geq 0 \) and \( \alpha \geq 0 \). Here the restriction \( \alpha \geq 0 \) is natural, since the left-hand side of \([8.1.1] \) will not in general tend to zero as \( t \to \infty \) due to the presence of the nondecreasing term \( \int_0^t \|y(s)\|_X^2 \, ds \). However, this restriction on \( \alpha \) can be removed if we replace \([8.1.1] \) by the following related inequality, which does not have the same limitation:
\[
\|e^{-\alpha t} x(t)\|_X^2 + \int_0^t \|e^{-\alpha s} y(s)\|_Y^2 \, ds
\]
\[
\leq M^2 \left( \|x(0)\|_X^2 + \int_0^t \|e^{-\alpha s} u(s)\|_U^2 \, ds \right), \quad t \in \mathbb{R}^+.
\]

In the above estimate \([x, u] \) is supposed to be a generalized future trajectory of \( \Sigma \), i.e., the initial time is zero. By using the time-invariance of \( \Sigma \) it is not difficult to
show that the appropriate version of conditions \(8.1.23\) for generalized trajectories
on a closed unbounded interval of the type \(I = [t_0, \infty)\) is

\[
\|e^{-\alpha t}x(t)\|_X^2 + \int_0^t \|e^{-\alpha s}y(s)\|_Y^2 \, ds \\
\leq M^2 \left( e^{-\alpha t_0} \|x(t_0)\|_X^2 + \int_0^t \|e^{-\alpha s}u(s)\|_{\mathcal{L}U}^2 \, ds \right), \quad t \in [t_0, \infty).
\]

8.1.20. Definition. By an \(\alpha\)-bounded i/s/o system we mean a i/s/o system
\(\Sigma = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) which, in addition to conditions (i)–(iii) in Definition 8.1.2 satisfies the following condition:

(iv) for some some triple of admissible norms \(\|\cdot\|_X, \|\cdot\|_U,\) and \(\|\cdot\|_Y\) in \(\mathcal{X}, \mathcal{U},\) and \(\mathcal{Y}\), respectively, there exists constants \(M \geq 1\) and \(\alpha \in \mathbb{R}\) such that
all generalized future trajectories \([u, y]\) of \(\Sigma\) satisfy (8.1.23).

As we shall see in Theorem 8.1.24 below, an i/s/o system is well-posed if and only if it is \(\alpha\)-bounded for some \(\alpha \in \mathbb{R}\). In one direction this claim is easy to prove.

8.1.21. Lemma. Every i/s/o system which is \(\alpha\)-bounded from some \(\alpha \in \mathbb{R}\) is well-posed.

Proof. Suppose that \(\Sigma\) is \(\alpha\)-bounded. If \(\alpha \geq 0\), then it follows from (8.1.23)
that

\[
\|x(t)\|_X^2 + \int_0^t \|y(s)\|_Y^2 \, ds \leq M^2 e^{2\alpha t} \left( \|x(0)\|_X^2 + \int_0^t \|u(s)\|_{\mathcal{L}U}^2 \, ds \right), \quad t \in \mathbb{R}^+.
\]

Thus \(8.1.1\) holds with \(\eta(t) = Me^{\alpha t}\), and hence \(\Sigma\) is well-posed. If instead \(\alpha < 0\), then it follows from (8.1.23)
that

\[
e^{-2\alpha t} \left( \|x(t)\|_X^2 + \int_0^t \|y(s)\|_Y^2 \, ds \right) \leq M^2 \left( \|x(0)\|_X^2 + \int_0^t \|u(s)\|_{\mathcal{L}U}^2 \, ds \right), \quad t \in \mathbb{R}^+.
\]

Thus, \(8.1.1\) holds with \(\eta(t) = Me^{\alpha |t|}\), and hence \(\Sigma\) is again well-posed. \(\square\)

The proof of the converse of Lemma 8.1.21 is more difficult. In that proof the following function spaces will be useful. (For completeness we formulate the definition below so that it can also be applied when \(L^2\) is replaced by \(L^p\) for some \(p \in [1, \infty)\).)

8.1.22. Notation. Let \(I\) be an interval, and let \(Z\) be an \(B\)-space, and let \(\alpha \in \mathbb{R}\).

(i) For each \(\alpha \in \mathbb{R}\) the space \(L^p_\alpha(I; Z)\) is the space of all \(Z\)-valued functions \(z\) defined on \(I\) with the property that the function \(s \mapsto e^{-\alpha s}z(s)\) belongs to \(L^p(I; Z)\). This is an \(B\)-space (it is an \(H\)-space if \(Z\) is an \(H\)-space and \(p = 2\), and each admissible norm \(\|\cdot\|_Z\) in \(Z\) induces an admissible norm \(\|z\|_{L^p(I; Z)} = \left( \int_I \|e^{-\alpha s}z(s)\|_Z^p \, ds \right)^{1/p}\) in \(L^p_\alpha(I; Z)\). (Note that \(L^p_\alpha(I; Z) = L^p(I; Z)\) if and only if \(I\) is finite.)

(ii) For each \(\alpha \in \mathbb{R}\) the space \(L^p_{\alpha, c}(\mathbb{R}; Z)\) is the space of all functions \(z \in L^p_\alpha(\mathbb{R}; Z)\) with the property that the support of \(z\) is bounded to the left (i.e., \(z\) vanishes on some interval \((-\infty, T]\)). A sequence \(z_n \in L^p_{\alpha, c}(\mathbb{R}; Z)\) tends to a function \(z\) in \(L^p_{\alpha, c}(\mathbb{R}; Z)\) as \(n \to \infty\) if \(z_n \to z\) in \(L^p_\alpha(\mathbb{R}; Z)\).
and, in addition, all the functions \( z_n \) vanish on some common interval \((-\infty, T]\).

(iii) For each \( \alpha \in \mathbb{R} \) the space \( L^p_{\alpha, \text{loc}}(\mathbb{R}; Z) \) is the space of all functions \( z \in L^p_{\text{loc}}(\mathbb{R}; Z) \) with the property that the restriction of \( z \) to \( \mathbb{R}^+ \) belongs to \( L^p_{\alpha}(\mathbb{R}^+; Z) \). A sequence \( z_n \in L^p_{\alpha, \text{loc}}(\mathbb{R}; Z) \) tends to a function \( z \) in \( L^p_{\alpha, \text{loc}}(\mathbb{R}; Z) \) as \( n \to \infty \) if it is true for every \( T \in \mathbb{R} \) that the restriction of \( z_n \) to \((-\infty, T]\) tends to the restriction of \( z \) to \((-\infty, T] \) in \( L^p_{\alpha}((-\infty, T]; Z) \) as \( n \to \infty \).

8.1.23. **Lemma.** Let \( \left[ \frac{\mathcal{A}x}{\mathcal{B}y} \right] \) be a formal fundamental i/s/o solution in \((\mathcal{X}, \mathcal{U}, \mathcal{Y})\) (for example, \( \left[ \begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array} \right] \) may be the fundamental i/s/o solution of a well-posed i/s/o system \( \Sigma \)). Then the following claims are true for each \( \alpha > \omega(\mathfrak{A}) \), where \( \omega(\mathfrak{A}) \) is the growth bound of \( \mathfrak{A} \).

(i) \( \mathcal{B} \) can be extended to a continuous linear operator \( L^2_{\alpha}(\mathbb{R}^-; \mathcal{U}) \to \mathcal{X} \).

(ii) The range of the map \( \mathcal{E} \) lies in \( L^2_{\alpha}(\mathbb{R}^+; \mathcal{Y}) \), and \( \mathcal{E} \) is continuous as an operator \( \mathcal{X} \to L^2_{\alpha}(\mathbb{R}^+; \mathcal{Y}) \).

(iii) \( \mathfrak{D} \) can be extended to a continuous linear operator \( L^2_{\alpha, \text{loc}}(\mathbb{R}; \mathcal{U}) \to L^2_{\alpha, \text{loc}}(\mathbb{R}; \mathcal{Y}) \), and this extended operator maps \( L^2_{\alpha}(\mathbb{R}; \mathcal{U}) \) continuously into \( L^2_{\alpha}(\mathbb{R}; \mathcal{Y}) \).

(iv) The extended operators defined above still satisfy the conditions listed in parts (i)–(iv) of Theorem [8.1.15]

**Proof.** Claims (i)–(iii) are found in, e.g., [Staffans 2005], Theorem 2.5.4(ii)–(iv)]. That the extended maps must satisfy the conditions listed in (ii)–(iv) of Theorem [8.1.15] follows from the fact that \( L^2_{\alpha, \text{loc}}(\mathbb{R}; \mathcal{U}) \) is dense in \( L^2_{\alpha, \text{loc}}(\mathbb{R}; \mathcal{U}) \) and the continuity of the extended maps.

8.1.24. **Theorem.** Let \( \Sigma = (S, \mathcal{X}, \mathcal{U}, \mathcal{Y}) \).

(i) \( \Sigma \) is well-posed if and only if \( \Sigma \) is \( \alpha \)-bounded for some \( \alpha \in \mathbb{R} \).

(ii) If \( \Sigma \) is \( \alpha \)-bounded for some \( \alpha \in \mathbb{R} \), then \( \Sigma \) is \( \beta \)-bounded for all \( \beta > \alpha \).

(iii) If \( \Sigma \) is well-posed, then \( \Sigma \) is \( \alpha \)-bounded for all \( \alpha > \omega(\mathfrak{A}) \) and \( \Sigma \) is not \( \alpha \)-bounded for any \( \alpha < \omega(\mathfrak{A}) \), where \( \omega(\mathfrak{A}) \) is the growth bound for the evolution semigroup \( \mathfrak{A} \) of \( \Sigma \).

**Proof.** **Proof of (i):** If \( \Sigma \) is \( \alpha \)-bounded for some \( \alpha \in \mathbb{R} \), then it follows from Lemma [8.1.21] that \( \Sigma \) is well-posed. That the converse is also true follows from (iii).

**Proof of (ii):** Suppose that \( \Sigma \) is \( \alpha \)-bounded. Then by (iii) \( \alpha \geq \omega(\mathfrak{A}) \), and consequently \( \beta > \omega(\mathfrak{A}) \). It therefore follows from (iii) that \( \Sigma \) is \( \beta \)-bounded.

**Proof of (iii):** Suppose first that \( \Sigma \) is \( \alpha \)-bounded. By Lemma [8.1.21] is well-posed. Therefore, for each \( x^0 \in \mathcal{X} \) the state component \( x \) of the unique generalized future trajectory \( \left[ \begin{array}{c} \tilde{x} \\ \tilde{y} \end{array} \right] \) of \( \Sigma \) with initial state \( x^0 \) (and zero input) is given by \( x(t) = \mathfrak{A}^t x^0, \ t \in \mathbb{R}^+ \). By [8.1.23] \( \| \mathfrak{A}^t x^0 \|_{\mathcal{X}} \leq M e^{\alpha t} \| x^0 \|_{\mathcal{X}} \). Since this is true for all \( x^0 \in \mathcal{X} \) we get \( \| \mathfrak{A}^t \|_{\mathcal{B}(\mathcal{X})} \leq M e^{\alpha t} \). From this and [4.1.19] follows that \( \omega(\mathfrak{A}) \leq \alpha \). This proves that \( \Sigma \) cannot be \( \alpha \)-bounded for any \( \alpha < \omega(\mathfrak{A}) \).

To complete the proof of (iii) we still need to show that if \( \Sigma \) is is well-posed, then \( \Sigma \) is \( \alpha \)-bounded for every \( \alpha > \omega(\mathfrak{A}) \). Let \( \alpha > \omega(\mathfrak{A}) \), and let \( \left[ \begin{array}{c} \tilde{x} \\ \tilde{y} \end{array} \right] \) be a generalized future trajectory of \( \Sigma \) with initial state \( x^0 \). Then by [8.1.14]

\[
x(t) = \mathfrak{A}^t x^0 + \mathfrak{B} \rho_+ \tau^t t u, \quad t \in \mathbb{R}^+,
\]
\[
y = \mathcal{C} x^0 + \rho_+ \mathfrak{D} t u,
\]
By Lemma 4.1 there exists a constant \( M_1 \geq 1 \) such that
\[
\| e^{-\alpha t} x^0 \|^2_X \leq M_1^2 \| x^0 \|^2_X.
\]

By Lemma 8.1.23(i) there exists a constant \( M_2 \) such that
\[
\| e^{-\alpha t} A t x^0 \|^2_X \leq M_2^2 \| x^0 \|^2_X.
\]

By Lemma 8.1.23(ii)–(iii) there exist constants \( M_3 \) and \( M_4 \) such that for all \( t \in \mathbb{R}^+ \),
\[
\int_0^t \| e^{-\alpha s} (\rho + D \iota) u(s) \|^2_Y ds \leq M_4^2 \int_0^t \| e^{-\alpha s} u(s) \|^2_Y ds.
\]

By adding these estimates we get (8.1.23) with
\[
M_2 = \sum_{i=1}^4 M_i^2. \quad \square
\]

8.1.25. Definition. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system with fundamental i/s/o solution \[
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix}.
\]

(i) \( \mathcal{A} \) is \( \alpha \)-bounded if \( \mathcal{A} \) satisfies \( \sup_{t \geq 0} \| e^{-\alpha t} \mathcal{A} t \| < \infty \);

(ii) \( \mathcal{B} \) is \( \alpha \)-bounded if \( \mathcal{B} \) can be extended to a bounded linear operator \( L_2^2(\mathbb{R}; \mathcal{U}) \rightarrow \mathcal{X} \) (which satisfies \( \mathcal{B} \pi_+ = 0 \)).

(iii) \( \mathcal{C} \) is \( \alpha \)-bounded if \( \mathcal{C} \) is a continuous linear operator \( \mathcal{X} \rightarrow L_2^2(\mathbb{R}^+; \mathcal{Y}) \).

(iv) \( \mathcal{D} \) is \( \alpha \)-bounded if \( \mathcal{D} \) can be extended to a continuous linear operator \( L_2^2(\mathbb{R}; \mathcal{U}) \rightarrow L_2^2(\mathbb{R}; \mathcal{Y}) \), and this extended operator maps \( L_2^2(\mathbb{R}; \mathcal{U}) \)
continuously into \( L_2^2(\mathbb{R}; \mathcal{Y}) \).

(v) \( \Sigma \) is \( \alpha \)-bounded if conditions (i)–(iv) above hold.

(vi) The infimum of all \( \alpha \in \mathbb{C} \) for which \( \Sigma \) is \( \alpha \)-bounded is called the growth bound of \( \Sigma \), and it is denoted by \( \omega(\Sigma) \).

(vii) \( \Sigma \) is stable if \( \Sigma \) is \( \alpha \)-bounded with \( \alpha = 0 \).

(viii) \( \Sigma \) is exponentially stable if \( \omega(\Sigma) < 0 \).

If \( \Sigma \) is \( \alpha \)-bounded then we use same notations \( \mathcal{B}, \mathcal{C}, \) and \( \mathcal{D} \) for the extensions/restrictions listed in (ii)–(iv) as for the original operators.

Note that an i/s/o system \( \Sigma \) may have a growth bound zero, and still be unstable.

8.1.26. Lemma. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system with evolution semigroup \( \mathcal{A} \).

(i) The growth bound of \( \Sigma \) is equal to the growth bound of \( \mathcal{A} \).

(ii) Exponential stability of \( \Sigma \) implies stability.

Proof. This follows from Lemma 8.1.23 and Definition 8.1.25. \( \square \)

Lemma 8.1.23 makes it possible to extend part (i) of Lemmas 8.1.5 and 8.1.6. In the formulation of this extension and its proof we make use of the following function spaces.
8.1.27. NOTATION. Let $I$ be a closed interval, and let $Z$ be an $H$-space (i.e., $Z$ is isomorphic to a Hilbert space).

(i) The space $BC(I; Z)$ is the space of all bounded continuous $Z$-valued functions defined on the interval $I$. This is a $B$-space, and each admissible norm $\|\cdot\|_Z$ in $Z$ induces an admissible norm $\|z\|_{BC(I; Z)} = \sup_{s \in I} \|z(s)\|_Z$ in $BC(I; Z)$. (If $I$ is bounded, then $BC(I; Z) = C(I; Z)$.)

(ii) For each $\alpha \in \mathbb{R}$ the space $BC_\alpha(I; Z)$ is the space of all continuous $Z$-valued functions $z$ defined on $I$ with the property that the function $s \mapsto e^{-\alpha s}z(s)$ belongs to $BC(I; Z)$. This is an $B$-space, and each admissible norm $\|\cdot\|_Z$ in $Z$ induces an admissible norm $\|z\|_{BC(I; Z)} = \sup_{s \in I} \|e^{-\alpha s}z(s)\|_Z$ in $L^2_\alpha(I; Z)$. (If $I$ is bounded, then $BC_\alpha(I; Z) = BC(I; Z) = C(I; Z)$.

(iii) The space $BUC(I; Z)$ is the space of all uniformly continuous $Z$-valued functions defined on the interval $I$. This is a closed subspace of $BC(I; Z)$. (If $I$ is finite, then $BUC(I; Z) = BC(I; Z) = C(I; Z)$.)

(iv) The space $BC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ is the space of all functions $z \in C(\mathbb{R}; Z)$ with the property that the support of $z$ is bounded to the left (i.e., $z$ vanishes on some interval $(-\infty, T]$). A sequence $z_n \in BC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ tends to a function $z$ in $BC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ as $n \to \infty$ if $z_n \to z$ in $C(\mathbb{R}; Z)$ and, in addition, all the functions $z_n$ vanish on some common interval $(-\infty, T]$.

(v) For each $\alpha \in \mathbb{R}$ the space $BUC_{\alpha}(I; Z)$ is the space of all continuous $Z$-valued functions defined on the interval $I$ with the property that the function $s \mapsto e^{-\alpha s}z(s)$ belongs to $BUC(I; Z)$. This is a closed subspace of $BC_{\alpha}(I; Z)$. (If $I$ is finite, then $BUC_{\alpha}(I; Z) = BC(I; Z) = C(I; Z)$.)

(vi) For each $\alpha \in \mathbb{R}$ the space $BUC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ is the space of all functions $z \in BUC_{\alpha}(\mathbb{R}; Z)$ with the property that the support of $z$ is bounded to the left (i.e., $z$ vanishes on some interval $(-\infty, T]$). A sequence $z_n \in BUC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ tends to a function $z$ in $BUC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ as $n \to \infty$ if $z_n \to z$ in $BUC_{\alpha}(\mathbb{R}; Z)$ and, in addition, all the functions $z_n$ vanish on some common interval $(-\infty, T]$.

(vii) For each $\alpha \in \mathbb{R}$ the space $BUC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ is the space of all functions $z \in C(\mathbb{R}; Z)$ with the property that the restriction of $z$ to $\mathbb{R}^-$ belongs to $BUC_{\alpha}(\mathbb{R}^-; Z)$. A sequence $z_n \in BUC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ tends to a function $z$ in $BUC_{\alpha, \text{loc}}(\mathbb{R}; Z)$ as $n \to \infty$ if it is true for every $T \in \mathbb{R}$ that the restriction of $z_n$ to $(-\infty, T]$ tends to the restriction of $z$ to $(-\infty, T]$ in $BUC_{\alpha}((-\infty, T]; Z)$ as $n \to \infty$.

8.1.28. LEMMA. Let $\alpha \in \mathbb{R}$, and let $\Sigma = (S; X, U, \mathcal{Y})$ be an $\alpha$-bounded i/s/o system with fundamental i/s/o solution $\left[\begin{array}{c} u \\ y \end{array}\right]$, and let $t_0, t_1 \in \mathbb{R}$.

(i) For every $x^0 \in X$ and $u \in L^2_\alpha(\mathbb{R}^+; U)$ the unique generalized trajectory $\left[\begin{array}{c} x \\ y \end{array}\right]$ of $\Sigma$ on $[t_0, \infty)$ with initial state $x(t_0) = x^0$ (and input function $u$) satisfies $\left[\begin{array}{c} x \\ y \end{array}\right] \in \left[\begin{array}{c} BUC_{\alpha, \text{loc}}([t_0, \infty); X) \\ L^2_{\alpha, \text{loc}}([t_0, \infty); \mathcal{Y}) \end{array}\right]$.

(ii) For every $u \in L^2_{\alpha}((-\infty, t_1]; U)$ there exists a unique generalized trajectory $\left[\begin{array}{c} x \\ y \end{array}\right]$ of $\Sigma$ on $(-\infty, t_1]$ satisfying $\lim_{t \to -\infty} e^{-\alpha t}x(t) = 0$. The state component $x$ and the output component $y$ satisfy $\left[\begin{array}{c} x \\ y \end{array}\right] \in \left[\begin{array}{c} BUC_{\alpha}((-\infty, t_1]; X) \\ L^2_{\alpha}((-\infty, t_1]; \mathcal{Y}) \end{array}\right]$. and they are given by (8.1.15) with $I = (-\infty, t_1]$ where again $\mathfrak{X}$ and $\mathfrak{D}$.
stand for the extensions of the original input map and input/output map described in Definition \[8.1.22\].

(iii) For every \(u \in L^2_{\alpha, loc}(\mathbb{R}; U)\) there exists a unique generalized two-sided trajectory \([\begin{array}{c} x \\ u \\ y \end{array}]\) of \(\Sigma\) satisfying \(\lim_{t \to -\infty} e^{-\alpha t} x(t) = 0\). The state component \(x\) and the output component \(y\) satisfy \([\begin{array}{c} x \\ \psi \\ y \end{array}] \in \left[ BUC_{\alpha, loc}(\mathbb{R}; X) \right] L^2_{\alpha, loc}(\mathbb{R}; \mathcal{Y})\), and they are given by \([8.1.15]\) with \(I = \mathbb{R}\), where \(\mathcal{B}\) and \(\mathcal{Y}\) stand for the extensions of the original input map and input/output map described in Definition \[8.1.25\]. If \(u \in L^2_{\alpha}(\mathbb{R}; U)\), then \([\begin{array}{c} x \\ \psi \\ y \end{array}] \in \left[ BUC_{\alpha}(\mathbb{R}; X) \right] L^2_{\alpha}(\mathbb{R}; \mathcal{Y})\].

Proof. The proofs of (i), (ii), and (iii) are similar to each other, so below we only prove (iii) and leave the proofs of (i) and (ii) to the reader.

Proof of (iii): Let \(u_n = \pi_{[−n, n]} u\), and let \([\begin{array}{c} x_n \\ u_n \\ y_n \end{array}]\) be the generalized trajectory of \(\Sigma\) given by Lemma \[8.1.11\](i). Then \(x_n \in BUC_0(\mathbb{R}; X)\) and \(u_n \in L^2_{\alpha, loc}(\mathbb{R}; U)\). As \(n \to \infty\) we have \(u_n \to u\) in \(L^2_{\alpha, loc}(\mathbb{R}; U)\), and it follows from Lemma \[8.1.23\] that \(x_n\) and \(y_n\) converge to some functions \(x\) and \(y\) in \(BUC_0(\mathbb{R}; X)\) and \(L^2_{\alpha, loc}(\mathbb{R}; \mathcal{Y})\) respectively. Since the set of generalized trajectories of \(\Sigma\) is closed (see Lemma \[2.4.1\](iii)), also \([\begin{array}{c} x \\ u \\ y \end{array}]\) is a generalized trajectory of \(\Sigma\), and \(x(t) = \mathcal{B} \rho_{-\alpha t} u, t \in \mathbb{R}\), and \(y = \mathcal{D} u\). That \(\lim_{t \to -\infty} e^{-\alpha t} x(t) \to 0\) as \(t \to -\infty\) follows from the fact that \(\|e^{-\alpha t} x(t)\| \leq \|\mathcal{B}\| \|\pi(0, \alpha) u\| L^2_{\alpha}(\mathbb{R}; \mathcal{D})\), where \(\|\mathcal{B}\|\) stands for the norm of \(\mathcal{B}\) as an operator \(L^2_{\alpha}(\mathbb{R}; \mathcal{D}) \to X\) and \(\|\pi(0, \alpha) u\| L^2_{\alpha}(\mathbb{R}; \mathcal{D}) \to 0\) as \(t \to -\infty\). The uniqueness of a trajectory with these properties follows from \([8.1.24]\) by letting \(t_0 \to -\infty\). That \(x \in BUC_{\alpha}(\mathbb{R}; X)\) and \(y \in L^2_{\alpha}(\mathbb{R}; \mathcal{Y})\) whenever \(u \in L^2_{\alpha}(\mathbb{R}; U)\) follows from Lemma \[8.1.23\].

8.1.4. Well-posed i/s/o systems have a nonempty resolvent set.

8.1.29. Theorem. Let \(\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})\) be a well-posed i/s/o system with growth bound \(\omega(\Sigma)\). Then the following claims are true:

(i) The open right half-plane \(\mathbb{C}^+(\Sigma)\) is contained in the resolvent set of \(\Sigma\).

(In particular, \(\rho(\Sigma) \neq \emptyset\).)

(ii) Denote the i/s/o resolvent matrix of \(\Sigma\) by \(\hat{\mathcal{S}} = \left[ \begin{array}{cc} \hat{s} & \hat{\mathcal{B}} \\ \hat{\mathcal{D}} & \hat{D} \end{array} \right]\). Let \(x^0 \in \mathcal{X}\), let \(\alpha > \omega(\Sigma)\), and let \([\begin{array}{c} x \\ u \\ y \end{array}]\) be the generalized future trajectory of \(\Sigma\) with initial state \(x(0) = x^0\) (and input function \(u\)). If \(u \in L^2_{\alpha}(\mathbb{R}; \mathcal{U})\), then the Laplace transforms \([\begin{array}{c} \hat{x}(\lambda) \\ \hat{u}(\lambda) \\ \hat{y}(\lambda) \end{array}]\) of \([\begin{array}{c} x \\ u \\ y \end{array}]\) converge absolutely for all \(\lambda \in \mathbb{C}^+(\omega(\Sigma))\), and they satisfy \([5.1.23]\).

Proof. Let \(x^0 \in \mathcal{X}\), let \(\alpha > \omega(\Sigma)\), let \(u \in L^2_{\alpha}(\mathbb{R}; \mathcal{U})\), and let \([\begin{array}{c} x \\ u \\ y \end{array}]\) be the generalized future trajectory of \(\Sigma\) with initial state \(x^0\) (and input function \(u\)). Then by Lemma \[8.1.28\] \([\begin{array}{c} x \\ u \\ y \end{array}] \in \left[ BUC_{\alpha}(\mathbb{R}^+; \mathcal{X}) \right] L^2_{\alpha}(\mathbb{R}^+; \mathcal{Y})\). Let \([\begin{array}{c} u_n \\ y_n \end{array}]\) be the sequence of classical future trajectories of \(\Sigma\) defined in Lemma \[2.4.25\](i). Then \(\hat{x}_n \in BUC_{\alpha}(\mathbb{R}^+; \mathcal{X})\) and \([\begin{array}{c} x_n \\ u_n \\ y_n \end{array}] \to \hat{x} \in \left[ BUC(\mathbb{R}^+; \mathcal{X}) \right] L^2(\mathbb{R}^+; \mathcal{Y})\) as \(n \to \infty\). In particular, the Laplace transforms of \(\hat{x}_n, x_n, u_n,\) and \(y_n\) converge absolutely in the open half-plane \(\mathbb{C}^+(\Sigma)\). By multiplying
with \( \begin{bmatrix} z \\ y \\ u \end{bmatrix} \) replaced by \( \begin{bmatrix} x_n \\ \dot{u}_n \\ u_n \end{bmatrix} \), integrating by parts in the equation for \( \dot{x}_n \), and using the fact that \( S \) is closed we get

\[
\begin{bmatrix} \dot{x}_n(\lambda) - x_n(0) \\ \dot{u}_n(\lambda) \\ y_n(\lambda) \end{bmatrix} = S \begin{bmatrix} \ddot{x}_n(\lambda) \\ \ddot{u}_n(\lambda) \\ \dot{y}_n(\lambda) \end{bmatrix}, \quad \lambda \in \mathbb{C}_+.
\]

Letting \( n \to \infty \) and using once more the closedness of \( S \) we arrive at equation (5.1.7). In this \( \begin{bmatrix} x^0 \\ u(\lambda) \end{bmatrix} \) can be an arbitrary vector in \( \mathcal{X} \) (for example, we may choose \( u(t) = e^{(\alpha-1)t}u^0, t \in \mathbb{R}^+ \), where \( u^0 \) is an arbitrary vector in \( \mathcal{U} \), which gives \( \hat{u}(\lambda) = (\lambda - \alpha + 1)^{-1}u^0 \)). Moreover, \( \begin{bmatrix} \ddot{x}(\lambda) \\ \ddot{u}(\lambda) \end{bmatrix} \) are uniquely determined by \( \begin{bmatrix} \ddot{x}^0 \\ \ddot{u}(\lambda) \end{bmatrix} \) and depend continuously on \( \begin{bmatrix} x^0 \\ u(\lambda) \end{bmatrix} \). Theorem 8.1.29 follows from Lemma 5.1.3 and Definition 5.1.4.

**8.1.30. Notation.** If \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is a well-posed i/s/o system, then we denote the component of \( \rho(\Sigma) \) which contains some right half-plane by \( \rho_+(\Sigma) \) (this set is equal to \( \rho_+(\mathcal{A}) \), where \( \mathcal{A} \) is the main operator of \( \Sigma \)).

**8.1.31. Theorem.** Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system with growth bound \( \omega(\Sigma) \), fundamental i/s/o solution \( \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \end{bmatrix} \), and i/s/o resolvent matrix \( \hat{\mathcal{G}} = \begin{bmatrix} \hat{\mathcal{A}} \\ \hat{\mathcal{B}} \end{bmatrix} \). Then, for each \( \lambda \in \mathbb{C}_+ \omega(\Sigma) \),

\[
\begin{align*}
\hat{\mathcal{A}}(\lambda)x^0 &= \int_0^\infty e^{-\lambda t}\mathcal{A}x^0 dt, \quad x^0 \in \mathcal{X}, \\
\hat{\mathcal{B}}(\lambda)u_0 &= \mathcal{B}(\rho_-e_\lambda u_0), \quad u_0 \in \mathcal{U}, \\
\hat{\mathcal{C}}(\lambda)x^0 &= \int_0^\infty e^{-\lambda t}(\mathcal{C}x^0)(t) dt, \quad x^0 \in \mathcal{X}, \\
e_\lambda \hat{\mathcal{D}}(\lambda)u_0 &= \mathcal{D}(e_\lambda u_0), \quad u_0 \in \mathcal{U},
\end{align*}
\]

where \( e_\lambda \) is the function \( t \mapsto e^{\lambda t}, t \in \mathbb{R} \), and \( \mathcal{B} \) and \( \mathcal{D} \) stand for the the extensions of the original input respectively input/output maps described in Definition 5.1.29.

**Proof.** The formulas for \( \hat{\mathcal{A}}(\lambda) \) and \( \hat{\mathcal{C}}(\lambda) \) follows from Lemma 8.1.11 and Theorem 8.1.29 by taking \( u = 0 \) and observing that in this case \( x(t) = \mathcal{A}x^0 \) for \( t \in \mathbb{R}^+ \) and \( y = \mathcal{C}x^0 \). To prove the formulas for \( \hat{\mathcal{B}} \) and \( \hat{\mathcal{D}} \) we fix some \( u_0 \in \mathcal{U} \) and define \( x := e_\lambda \hat{\mathcal{B}}(\lambda)u_0, u := e_\lambda u_0, \) and \( y := e_\lambda \hat{\mathcal{D}}(\lambda)u_0 \), where \( \hat{\mathcal{B}}(\lambda) \) and \( \hat{\mathcal{D}}(\lambda) \) are the input/state and input/output resolvents of \( \Sigma \). Then \( \dot{x} = \lambda x = \lambda e_\lambda \hat{\mathcal{B}}(\lambda)u_0 \). This implies that

\[
\begin{bmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \lambda \mathcal{B}(\lambda) \\ \mathcal{D}(\lambda) \\ \mathcal{B}(\lambda) \\ 1_U \end{bmatrix} u(t), \quad t \in \mathbb{R}.
\]

By 5.2.12, \( \begin{bmatrix} x(t) \\ y(t) \\ u(t) \end{bmatrix} \in \text{dom}(S) \) and \( \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \) for all \( t \in \mathbb{R} \). Since, in addition, \( x \) is continuously differentiable and \( u \) and \( y \) are continuous, this means that \( \begin{bmatrix} x \\ y \\ u \end{bmatrix} \) is a classical two-sided trajectory of \( \Sigma \). It is therefore also a generalized two-sided
of defined by (8.1.14) are called the state trajectory respectively the output function

\[ \Sigma = (A, B, C) \]

following notion: For each \( x \) in that setting our notion of a generalized future trajectory is replaced by the local boundedness of \( A \) (ii) and (iii’) in Lemma 8.1.8 hold. Finally, condition (iv’) is a consequence of the (i’) in Lemma 8.1.8, and from [Staffans, 2005, Proposition 4.7.8] that conditions in \( B \hat{\lambda} \) and \( \lambda \) as follows: Let \( \lambda \in \mathbb{C}^+ \), where \( \omega(\Sigma) \) is the growth rate of \( \Sigma \), define \( \hat{\Sigma}(\lambda) \), \( \hat{B}(\lambda) \), and \( \hat{C}(\lambda) \) by the first three formulas in (8.1.25), and let \( \hat{D}(\lambda) \) be the unique operator in \( B(U, \mathcal{Y}) \) which satisfies the last formula in (8.1.25). Then \( S \) is given by (5.1.21) and 5.1.22.

**Proof.** This follows from Lemma 5.1.12 and Theorem 8.1.31.

8.1.32. **Corollary.** The generating operator \( S \) of a well-posed i/s/o system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) can be recovered from the fundamental i/s/o solution \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \) of \( \Sigma \) as follows: Let \( \lambda \in \mathbb{C}^+ \), where \( \omega(\Sigma) \) is the growth rate of \( \Sigma \), define \( \hat{\Sigma}(\lambda) \), \( \hat{B}(\lambda) \), and \( \hat{C}(\lambda) \) by the first three formulas in (8.1.25), and let \( \hat{D}(\lambda) \) be the unique operator in \( B(U, \mathcal{Y}) \) which satisfies the last formula in (8.1.25). Then \( S \) is given by (5.1.21) and 5.1.22.

8.1.33. **Theorem.** Every formal fundamental i/s/o solution in \( (\mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is the fundamental i/s/o solution of a unique well-posed i/s/o system \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \).

**Proof.** The proof of existence of \( \Sigma \) is based on a result from Staffans [2005, Definition 4.6.4], starting from the given formal fundamental i/s/o solution \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \). It follows from Staffans [2005, Proposition 4.7.1] that \( S \) satisfies condition (i’) in Lemma 8.1.8 and from Staffans [2005, Proposition 4.7.8] that conditions (ii) and (iii’) in Lemma 8.1.8 hold. Finally, condition (iv’) is a consequence of the local boundedness of \( \hat{\mathcal{A}} \) and the continuity of \( \hat{\mathcal{B}}, \hat{\mathcal{C}}, \) and \( \hat{\mathcal{D}} \). Thus \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is a well-posed i/s/o system. That the fundamental i/s/o solution of \( \Sigma \) is equal to \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \) follows from Staffans [2005, Theorems 4.6.5 and 4.6.11].

Uniqueness of \( \Sigma \) can be proved as follows. Let \( \Sigma \) and \( \Sigma_1 \) be two well-posed i/s/o systems with the fundamental i/s/o solution \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \). By Lemma 8.1.11, the fundamental i/s/o solutions of \( \Sigma \) and \( \Sigma_1 \) determines the set of all generalized future trajectories of \( \Sigma \) and \( \Sigma_1 \) uniquely, and by Corollary 2.4.33, the system operators \( S \) and \( S_1 \) of \( \Sigma \) and \( \Sigma_1 \) are determined uniquely by the set of all generalized future trajectories of \( \Sigma \) and \( \Sigma_1 \). Thus \( \Sigma = \Sigma_1 \).

8.1.34. **Remark.** Although our present notion of a well-posed linear system is equivalent to the notion of an \( L^2 \)-well-posed linear system introduced in Staffans [2005], at a first glance these two notions seem to be quite different from each other. More precisely, in the setting of Staffans [2005] an \( L^2 \)-well-posed linear system consists of a formal fundamental i/s/o solution \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \) in \( (\mathcal{X}, \mathcal{U}, \mathcal{Y}) \), and in that setting our notion of a generalized future trajectory is replaced by the following notion: For each \( x^0 \in \mathcal{X} \) and \( u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \) the functions \( x \) and \( y \) defined by (8.1.14) are called the state trajectory respectively the output function of \( \Sigma \) with initial state \( x^0 \) and input function \( u \). Thus, if follows from Theorem 8.1.15 that if \( \left[ \begin{array}{c} x \\ y \end{array} \right] \) is a generalized future trajectory of a well-posed i/s/o system \( \Sigma \) with fundamental i/s/o solution \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \), then \( x \) and \( y \) are the state trajectory respectively output function of \( \left[ \begin{array}{c} S \\ U \\ \mathcal{Y} \end{array} \right] \) with initial state \( x(0) \) and input function \( u \).
in the sense of [Staffans 2005]. That the converse is also true follows from Lemma 8.1.11 and Theorem 8.1.33. Thus, there is a one-to-one correspondence between the class of well-posed i/s/o systems in Definition 8.1.2 and the class of $L^2$-well-posed i/s/o systems defined in [Staffans 2005]. This means, in particular, that all the results proved in [Staffans 2005] about $L^2$-well-posed i/s/o systems in a Hilbert space setting are also valid in the setting of our Definition 8.1.2.

The last of the formulas in (8.1.25) can be extended to the case where less is assumed of $D$, namely that $D$ is an arbitrary continuous linear operator mapping into $L^2_{\text{loc}}(\mathbb{R};Y)$ which is shift-invariant and causal.

8.1.35. Lemma. Let $X$, $U$, and $Y$ be $H$-spaces, let $\alpha \in \mathbb{R}$, and let $D$ be a continuous linear operator mapping $L^2_{\text{loc}}(\mathbb{R};U)$ into $L^2_{\text{loc}}(\mathbb{R};Y)$ which is shift-invariant and causal in the sense that $\tau^\alpha D = D\tau^\alpha$, $t \in \mathbb{R}$, and $\pi_D\pi_+ = 0$. Then, for each $\lambda \in \mathbb{C}_\alpha^+$, there exists a unique operator $\hat{D}(\lambda) \in \mathcal{B}(U,Y)$ such that

$$ e_\lambda \hat{D}(\lambda) u_0 = D(e_\lambda u_0), \quad u_0 \in U, $$

where $e_\lambda$ is the function $t \mapsto e^{\lambda t}$.

Observe that $e_\lambda \in L^2_{\text{loc}}(\mathbb{R})$ since $\Re \lambda > \alpha$, so that the right-hand side of (8.1.26) is well-defined.

**Proof of Lemma 8.1.35.** Let $u_0 \in U$ and $\lambda \in \mathbb{C}_\alpha^+$, and define $u := e_\lambda u_0$. Then $\tau^\alpha e_\lambda u_0 = e_\lambda(\tau) e_\lambda u_0$, and hence $\tau^\alpha D e_\lambda u_0 = D e_\lambda(\tau) e_\lambda u_0 = e_\lambda(\tau) D e_\lambda u_0$. Thus, if we denote $y := Du = D(e_\lambda u_0)$, then $\tau^\alpha y = e_\lambda(\tau) y = e^{\lambda \tau} y$. This implies that there exists a vector $y_0 \in Y$ such that $y$ is equal to the function $e_\lambda y_0$ in the $L^2$-sense. By the continuity and causality of $D$, there is a finite constant $M$ such that $\|y\|_{L^2_\text{loc}(\mathbb{R};Y)} \leq M\|u\|_{L^2_\text{loc}(\mathbb{R};U)}$. Since $u = e_\lambda u_0$ and $y = e_\lambda y_0$ this means that $\|\hat{e}_\lambda\|_{L^2_\text{loc}(\mathbb{R};Y)} \|y_0\|_Y \leq M\|\hat{e}_\lambda\|_{L^2_\text{loc}(\mathbb{R};U)}\|u_0\|_U$, or equivalently, $\|y_0\|_Y \leq M\|u_0\|_U$. It follows from the linearity of $D$ that the map from $u_0$ to $y_0$ is linear, and the above estimate shows that it is continuous. Thus, there exists a (unique) operator $\hat{D}(\lambda) \in \mathcal{B}(U,Y)$ such that $y_0 = \hat{D}(\lambda) u_0$, and hence $D(e_\lambda u_0) = y_0 = \hat{D}(\lambda) (e_\lambda u_0)$, i.e., (8.1.26) holds.

Clearly $\hat{D}(\lambda)$ is determined uniquely by (8.1.26). \hfill \Box

8.1.36. Lemma. Let $\Sigma$ be a well-posed i/s/o system with growth bound $\omega(\Sigma)$ and i/s/o resolvent matrix $\hat{\Sigma} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$, and let $\|\cdot\|_X$, $\|\cdot\|_U$, and $\|\cdot\|_Y$ be some admissible norms in $X$, $U$, and $Y$, respectively. Then for each $\alpha > \omega(\Sigma)$ there exists a finite constant $M$ such that

$$
\|\hat{A}(\lambda)\|_{\mathcal{B}(X)} \leq \frac{M}{\Re \lambda - \alpha + 1}, \quad \|\hat{B}(\lambda)\|_{\mathcal{B}(U,X)} \leq \frac{M}{(\Re \lambda - \alpha + 1)^{1/2}},
$$

$$
\|\hat{C}(\lambda)\|_{\mathcal{B}(X,Y)} \leq \frac{M}{(\Re \lambda - \alpha + 1)^{1/2}}, \quad \|\hat{D}(\lambda)\|_{\mathcal{B}(U,Y)} \leq M, \quad \lambda \in \mathbb{C}_\alpha^+,
$$

where the norms in (8.1.27) stand for the operator norms of $\hat{A}(\lambda)$, $\hat{B}(\lambda)$, $\hat{C}(\lambda)$, and $\hat{D}(\lambda)$, with respect to the norms let $\|\cdot\|_X$, $\|\cdot\|_U$, and $\|\cdot\|_Y$ in $X$, $U$, and $Y$.

**Proof.** By Corollary 4.1.13 and Lemma 8.1.26 the estimate for $\|\hat{A}(\lambda)\|_{\mathcal{B}(X)}$ in (8.1.27) holds for some $M \geq 1$, so it only remains to prove the three other estimates.
If \( \alpha > \omega(\mathfrak{A}) + 2 \) (this is, in particular true if \( \omega(\mathfrak{A}) = -\infty \)), then we define \( \alpha_0 = \alpha - 1 \), and if \( \alpha \leq \omega(\mathfrak{A}) + 2 \) (and hence \( \omega(\mathfrak{A}) \) is finite) then we define \( \alpha_0 = \frac{1}{2}(\alpha + \omega(\mathfrak{A})) \). Then in both cases \( \omega(\mathfrak{A}) < \alpha_0 < \alpha \) and \( \epsilon := \alpha - \alpha_0 \) satisfies \( 0 < \epsilon \leq 1 \). Since \( \alpha_0 > \omega(\mathfrak{A}) \) the boundedness conditions in Lemma 8.1.23 holds with \( \alpha \) replaced by \( \alpha_0 \), i.e., there exists a constant \( M_0 \) such that for all

\[
\| \mathfrak{B} u \|_\mathcal{X} \leq M_0 \| u \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{M})}, \quad u \in L^2_{\alpha_0}(\mathbb{R}; \mathcal{U}),
\]

(8.1.28)

\[
\| \mathfrak{C} x^0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} \leq M_0 \| x^0 \|_\mathcal{X}, \quad x^0 \in \mathcal{X},
\]

\[
\| \mathfrak{D} u \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} \leq M_0 \| u \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{M})}, \quad u \in L^2_{\alpha_0}(\mathbb{R}; \mathcal{U}).
\]

Let \( \Re \lambda \geq \alpha \) and let \( u^0 \) be an arbitrary vector in \( \mathcal{U} \). Take \( u(t) = e^{\lambda t} u^0, t \in \mathbb{R}^- \), in the first estimate in (8.1.28) and use the second identity in (8.1.25) to get

\[
\| \mathfrak{B}(\lambda) u_0 \|_{\mathcal{X}}^2 \leq M_0^2 \| e_{\lambda} u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{M})}^2 = \frac{M_0^2 \| u_0 \|_{\mathcal{U}}^2}{2(\Re \lambda - \alpha_0)} = \frac{M_0^2 \| u_0 \|_{\mathcal{U}}^2}{2(\Re \lambda - \alpha + \epsilon)}.
\]

Here

\[
\Re \lambda - \alpha + \epsilon = \epsilon((\Re \lambda - \alpha)/(\epsilon + 1) \geq \epsilon(\Re \lambda - \alpha) + 1),
\]

and thus we get the estimate for \( \| \mathfrak{B}(\lambda) \|_{\mathcal{B}(\mathcal{U}; \mathcal{X})} \) in (8.1.27) with \( M^2 = M_0^2/(2\epsilon) \). In the same way we get from the third estimate in (8.1.28) and the lst identity in (8.1.25)

\[
\| e_{\lambda} \mathfrak{D}(\lambda) u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} \leq M_0 \| e_{\lambda} u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{M})}.
\]

Here

\[
\| e_{\lambda} \mathfrak{D}(\lambda) u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} = \| \mathfrak{D}(\lambda) u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} \quad \text{and}
\]

\[
\| e_{\lambda} u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} = \| u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})}.
\]

and dividing both sides of the above inequality by \( \| e_{\lambda} u_0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})} \) we get \( \| \mathfrak{D}(\lambda) u_0 \|_{\mathcal{Y}} \leq M_0 \| x^0 \|_{\mathcal{U}} \). This gives the estimate for \( \| \mathfrak{D}(\lambda) \|_{\mathcal{B}(\mathcal{U}; \mathcal{Y})} \) in (8.1.27) with \( M = M_0 \).

Let \( \Re \lambda \geq \alpha \) and let \( x^0 \) be an arbitrary vector in \( \mathcal{X} \). It follows from the third identity in (8.1.25), Hölder’s inequality, and the second estimate in (8.1.28) that

\[
\| \mathfrak{C}(\lambda) x^0 \|_{\mathcal{Y}}^2 = \left( \int_0^\infty e^{(\alpha_0 - \lambda)t} e^{-\alpha_0 t} (\mathfrak{C} x^0)(t) \, dt \right)^2 \leq \left( \int_0^\infty e^{2(\alpha_0 - \Re \lambda)t} \, dt \right) \| \mathfrak{C} x^0 \|_{L^2_{\alpha_0}(\mathbb{R}^{-}\setminus\mathcal{Y})}^2 = \frac{M_0^2 \| x^0 \|_{\mathcal{X}}^2}{2(\Re \lambda - \alpha_0)} = \frac{M_0^2 \| x^0 \|_{\mathcal{X}}^2}{2(\Re \lambda - \alpha - \epsilon)}.
\]

This gives the estimate for \( \| \mathfrak{C}(\lambda) \|_{\mathcal{B}(\mathcal{X}; \mathcal{Y})} \) in (8.1.27) with constant \( M^2 = M_0^2/(2\epsilon) \).

\[\square\]

8.1.6. Realizations of shift-invariant causal linear operators (Jan 02, 2016).

8.1.37. Definition. Let \( \mathcal{U} \) and \( \mathcal{Y} \) be \( H \)-spaces, and let \( \mathfrak{D} \) be a continuous linear operator \( \mathfrak{D} : L^2_{\alpha_0, \text{loc}}(\mathbb{R}; \mathcal{U}) \to L^2_{\alpha_0, \text{loc}}(\mathbb{R}; \mathcal{Y}) \).

(i) \( \mathfrak{D} \) is called shift-invariant if \( \mathfrak{D} \tau^t = \tau^t \mathfrak{D} \) for all \( t \in \mathbb{R} \) (here the operator \( \tau^t \) on the left-hand side stands for a shift in \( L^2_{\alpha_0, \text{loc}}(\mathbb{R}; \mathcal{U}) \), and the operator \( \tau^t \) on the right-hand side stands for a shift in \( L^2_{\alpha_0, \text{loc}}(\mathbb{R}; \mathcal{Y}) \).

(ii) \( \mathfrak{D} \) is called causal if \( \pi_{(\alpha_0, \epsilon)} \mathfrak{D} \pi_{(t, \epsilon)} = 0 \) for all \( t \in \mathbb{R} \).
(iii) $\mathcal{D}$ is $\alpha$-bounded (where $\alpha \in \mathbb{R}$) if $\mathcal{D}$ can be extended to a continuous linear operator $L^2_{\alpha, \text{loc}}(\mathbb{R}; U) \to L^2_{\alpha, \text{loc}}(\mathbb{R}; Y)$, and this extended operator maps $L^2_{\alpha}(\mathbb{R}; U)$ continuously into $L^2_{\alpha}(\mathbb{R}; Y)$.

(iv) $\mathcal{D}$ is exponentially bounded if $\mathcal{D}$ is $\alpha$-bounded for some $\alpha \in \mathbb{R}$.

(v) If $\mathcal{D}$ is exponentially bounded, then the growth bound $\omega(\mathcal{D})$ of $\mathcal{D}$ is the infimum over all $\alpha \in \mathbb{R}$ for which $\mathcal{D}$ is $\alpha$-bounded. If $\mathcal{D}$ is not exponentially bounded then we write $\omega(\mathcal{D}) = \infty$.

(vi) We denote the set of all continuous linear causal shift-invariant operators $L^2_{\alpha, \text{loc}}(\mathbb{R}; U) \to L^2_{\alpha, \text{loc}}(\mathbb{R}; Y)$ by $\text{TIC}(\mathbb{R}; Y)$, and the set of all $\alpha$-bounded operators in $\text{TIC}(\mathbb{R}; Y)$ by $\text{TIC}_\alpha(\mathbb{R}; Y)$.

It is possible to have $\omega(\mathcal{D}) = -\infty$, e.g., when $\hat{\mathcal{D}}$ is an operator of the type $(\mathcal{D}u)(t) = D\hat{y}(t)$ for some $D \in \mathcal{B}(U; Y)$. (An operator of this type is called static.)

8.1.38. Lemma. The i/o map $\mathcal{D}$ of a well-posed i/s/o system $\Sigma$ is an exponentially bounded shift-invariant causal linear operator whose growth bound $\omega(\mathcal{D})$ satisfies $\omega(\mathcal{D}) \leq \omega(\Sigma)$, where $\omega(\Sigma)$ is the growth bound of $\Sigma$.

Proof. This follows from Theorem 8.1.15, Lemma 8.1.37 and Definition 8.1.37.

8.1.39. Definition. Let $\mathcal{D}$ be an exponentially bounded shift-invariant causal linear operator $L^2_{\alpha, \text{loc}}(\mathbb{R}^+; U) \to L^2_{\alpha, \text{loc}}(\mathbb{R}^+; Y)$. Any well-posed i/s/o system whose i/o map coincides with $\mathcal{D}$ is called a well-posed realization of $\mathcal{D}$.

8.1.40. Theorem. Every exponentially bounded shift-invariant causal linear operator $\mathcal{D} : L^2_{\alpha, \text{loc}}(\mathbb{R}^+; U) \to L^2_{\alpha, \text{loc}}(\mathbb{R}^+; Y)$ has a well-posed i/s/o realization. Moreover, given any $\varepsilon > 0$, it is possible to find a realization whose growth bound is at most $\omega(\mathcal{D}) + \varepsilon$.

Proof. Fix any $\alpha \in \mathbb{R}$ for which $\mathcal{D}$ is $\alpha$-bounded. Four different well-posed $\alpha$-bounded realizations of $\mathcal{D}$ are given in [Staffans 2005, Example 2.6.5]. (The evolution semigroup in these four examples are left-shifts in the spaces $L^2_{\alpha}(\mathbb{R}; U)$, $L^2_{\alpha}(\mathbb{R}; Y)$, $L^2_{\alpha}(\mathbb{R}^+; U)$, and $L^2_{\alpha}(\mathbb{R}^+; Y)$.)

8.1.41. Theorem. Let $\alpha \in \mathbb{R}$.

(i) Let $\mathcal{D} \in \text{TIC}_\alpha(\mathbb{R}; Y)$. Then there exists a unique function $\hat{\mathcal{D}} \in H^\infty(\mathbb{C}_\alpha^+; \mathcal{B}(U; Y))$ with the following property: $u \in L^2_{\alpha}(\mathbb{R}^+; U)$ and $y \in L^2_{\alpha}(\mathbb{R}^+; Y)$ satisfy

\[ y = \mathcal{D}u \text{ if and only if the Laplace transforms } \hat{u} \text{ and } \hat{y} \text{ of } u \text{ respectively } y \text{ satisfy} \]

\[ \hat{y}(\lambda) = \hat{\mathcal{D}}(\lambda)\hat{u}(\lambda), \quad \lambda \in \mathbb{C}_\alpha^+. \]

(ii) Conversely, if $\hat{\mathcal{D}} \in H^\infty(\mathbb{C}_\alpha^+; \mathcal{B}(U; Y))$, then there exists a unique operator $\mathcal{D} \in \text{TIC}_\alpha(\mathbb{R}; Y)$ such that $\mathcal{D}$ and $\hat{\mathcal{D}}$ are related to each other as described in (i) above.

\[ \text{If } \mathcal{D} \text{ and } \hat{\mathcal{D}} \text{ are related as above, then the norm of } \mathcal{D} \text{ as a bounded linear operator } L^2_{\alpha}(\mathbb{R}^+; U) \to L^2_{\alpha}(\mathbb{R}^+; Y) \text{ is equal to the } H^\infty(\mathbb{C}_\alpha^+; \mathcal{B}(U; Y))\text{-norm of } \hat{\mathcal{D}}. \]

Proof. See, e.g., [Staffans 2005, Lemma 10.3.3 and Theorem 10.3.5].

8.1.42. Lemma. If the operator $\mathcal{D} \in \text{TIC}(\mathbb{R}; Y)$ is $\alpha_0$-bounded for some $\alpha_0$, then it is also $\alpha$-bounded for every $\alpha > \alpha_0$, and the norm of $\mathcal{D}$ as an operator in $\mathcal{B}(L^2_{\alpha}(\mathbb{R}; U); L^2_{\alpha}(\mathbb{R}; Y))$ is a non-increasing function of $\alpha$. 

Theorem 8.1.41. \( \omega \) on \( C \)

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\[ \text{bound} \]

Theorem 8.1.41.

Let \( \mathcal{D} \in TIC(\mathcal{U}; \mathcal{Y}) \) be exponentially bounded with growth bound \( \omega(\mathcal{D}) \). The symbol \( \tilde{\omega} \) of \( \mathcal{D} \) is the analytic \( \mathcal{B}(\mathcal{U}; \mathcal{Y}) \)-valued function defined on \( \mathbb{C}_\omega^+ \), whose restriction to \( \mathbb{C}_\alpha^+ \) for each \( \alpha > \omega(\mathcal{D}) \) is equal to the function \( \tilde{\omega} \) in Theorem 8.1.41.

8.1.44. Lemma. If \( \tilde{\mathcal{D}} \) is the symbol of an exponentially bounded operator \( \mathcal{D} \in TIC(\mathcal{U}; \mathcal{Y}) \) with growth bound \( \omega(\mathcal{D}) \), then the restriction of \( \mathcal{D} \) to every right half-plane \( C_\alpha^+ \) with \( \alpha > \omega(\mathcal{D}) \) is a \( \mathcal{B}(\mathcal{U}; \mathcal{Y}) \)-valued \( H^\infty \) function over \( \mathbb{C}_\alpha^+ \), but \( \mathcal{D} \) cannot be extended to an \( H^\infty \) function over any half-plane \( \mathbb{C}_\alpha^+ \) with \( \alpha < \omega(\mathcal{D}) \).

Proof. This follows from Theorem 8.1.41 and Definitions 8.1.37 and 8.1.43.

8.1.45. Definition. Let \( \Sigma \) be a well-posed i/o system.

(i) The symbol \( \mathcal{D} \) of the i/o map \( \mathcal{D} \) of \( \Sigma \) is called the transfer function of \( \Sigma \).

(ii) If the i/o system \( \Sigma \) is a realization of a given operator \( \mathcal{D} \in TIC(\mathcal{U}; \mathcal{Y}) \) with finite growth bound, then we also call \( \Sigma \) a realization of the symbol \( \mathcal{D} \) of \( \mathcal{D} \).

8.1.46. Theorem. A well-posed i/o system \( \Sigma \) with growth bound \( \omega(\Sigma) \) is a realization of the exponentially bounded operator \( \mathcal{D} \in TIC(\mathcal{U}; \mathcal{Y}) \) with growth bound \( \omega(\mathcal{D}) \) if and only if \( \omega(\mathcal{D}) \leq \omega(\Sigma) \) and the i/o resolvent \( \tilde{\mathcal{D}} \) of \( \Sigma \) coincides with the symbol \( \mathcal{D} \) of \( \mathcal{D} \) in the half-plane \( \mathbb{C}_{\omega(\Sigma)}^+ \).

Proof. If \( \Sigma \) is a well-posed realization of \( \mathcal{D} \), then by Lemma 8.1.23 \( \omega(\mathcal{D}) \leq \omega(\Sigma) \), and it follows from Theorem 8.1.29 that \( \mathcal{D}(\lambda) = \mathcal{D}(\lambda) \) for every \( \lambda \in \mathbb{C}_{\omega(\Sigma)}^+ \). Conversely, if \( \mathcal{D}(\lambda) = \mathcal{D}(\lambda) \) for every \( \lambda \in \mathbb{C}_{\omega(\Sigma)}^+ \), then the i/o map of \( \Sigma \) must be equal to the given operator \( \mathcal{D} \), since the restriction of the symbol of an exponentially bounded operator in \( TIC(\mathcal{U}; \mathcal{Y}) \) to any right half-plane determines the operator \( \mathcal{D} \) uniquely.

8.1.47. Theorem. Let \( \Sigma = \left( \mathcal{S}; \mathcal{X}; \mathcal{U}; \mathcal{Y} \right) \) and \( \Sigma_1 = \left( \mathcal{S}_1; \mathcal{X}_1; \mathcal{U}; \mathcal{Y} \right) \) be two well-posed i/o systems (with the same input and output spaces). Then the following conditions are equivalent:

(i) \( \Sigma \) and \( \Sigma_1 \) are externally equivalent (see Definition 2.5.24).

(ii) \( \Sigma \) and \( \Sigma_1 \) have the same future behavior (see Definition 2.5.43).

(iii) \( \Sigma \) and \( \Sigma_1 \) have the same i/o map \( \mathcal{D} \).

(iv) \( \Sigma \) and \( \Sigma_1 \) have the same transfer function \( \tilde{\mathcal{D}} \).

(v) The i/o resolvents of \( \Sigma \) and \( \Sigma_1 \) coincide in some right-half plane.

Proof. (i) \( \Leftrightarrow \) (ii): See Lemmas 2.5.44 and 8.1.11 iv).

(ii) \( \Leftrightarrow \) (iii): This follows from Definition 2.5.43 and Lemma 8.1.11 combined with the fact that the restriction of \( \mathcal{D} \) to \( L_2^{\infty}(\mathbb{R}^+; \mathcal{U}) \) defines \( \mathcal{D} \) uniquely.

(iii) \( \Leftrightarrow \) (iv): See Theorem 8.1.41 and Definition 8.1.45.

(iv) \( \Leftrightarrow \) (v): See Definition 8.1.45 and Theorem 8.1.46.

8.1. The fundamental i/o solutions of transformed i/o systems.

In Chapter 2 we defined a number of transformation that can be applied to general i/o systems. In the well-posed case the question arises of what can be said about the fundamental i/o solutions of these transformations.
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8.1.48. **Theorem.** Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system with evolution semigroup \( \mathfrak{A} \), and let \( \Sigma = (S, X, U, Y) \) be the time reflection of \( \Sigma \).

(i) \( \Sigma \) is well-posed if and only if \( \mathfrak{A} \) can be extended to a \( C_0 \) semigroup. The values of this extended semigroup on \( \mathbb{R}^- \) are given by \( \mathfrak{A}^t = (\mathfrak{A}^{-t})^{-1}, t \in \mathbb{R}^- \).

(ii) The evolution semigroup of \( \Sigma \) is the \( C_0 \) semigroup \( t \mapsto (\mathfrak{A}^t)^{-1}, t \in \mathbb{R}^+ \).

(iii) The fundamental i/s/o solution \( \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \) of \( \Sigma \) with initial time \( s \) and final time \( t, -\infty < s \leq t < \infty \), is given by

\[
\begin{align*}
(\mathfrak{A}^s & \mathfrak{A}^t)^{-1} = \begin{bmatrix} 1 & \mathfrak{C}^t \mathfrak{B}^t \mathfrak{A}^t \mathfrak{R}_t \\ \mathfrak{R}_t \mathfrak{C}^t \mathfrak{A}^t \mathfrak{D}^t \end{bmatrix} \end{align*}
\]

where \( \mathfrak{R}_t \) is the shifted time reflection operator which maps the interval \([s, t]\) onto itself, i.e.,

\[
(\mathfrak{R}_t f)(v) = f(s + t - v), \quad v \in [s, t], \quad f \in L^2([s, t]).
\]

More explicitly,

\[
\begin{align*}
\mathfrak{A}^t & = (\mathfrak{A}^{-t})^{-1}, \\
\mathfrak{B}^t & = (\mathfrak{A}^{-t})^{-1} \mathfrak{B}^t \mathfrak{R}_t, \\
\mathfrak{C}^t & = \mathfrak{R}_t \mathfrak{C}^t (\mathfrak{A}^{-t})^{-1}, \\
\mathfrak{D}^t & = \mathfrak{R}_t (\mathfrak{D}^t - \mathfrak{C}^t (\mathfrak{A}^{-t})^{-1} \mathfrak{B}^t \mathfrak{R}_t).
\end{align*}
\]

**Proof.** See [Staffans 2005] Definition 6.4.2 and Theorem 6.4.3. \( \square \)

8.1.49. **Lemma.** Let \( \Sigma = (S; X, U, Y) \) be an i/s/o system, let \( \gamma > 0 \), and let \( \Sigma_\gamma = (S; X, U, Y) \) be the time \( \gamma \)-rescaled i/s/o system.

(i) \( \Sigma_\gamma \) is well-posed if and only if \( \Sigma \) is well-posed.

(ii) Suppose that \( \Sigma \) is well-posed (and hence also \( \Sigma_\gamma \) is well-posed), and denote the fundamental i/s/o solutions of \( \Sigma \) and \( \Sigma_\gamma \) by \( \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \) and \( \begin{bmatrix} \mathfrak{A}_\gamma & \mathfrak{B}_\gamma \\ \mathfrak{C}_\gamma & \mathfrak{D}_\gamma \end{bmatrix} \).

Let \( \delta_\gamma \) be the time \( \gamma \)-rescaling operator defined by

\[
(\delta_\gamma f)(t) = f(\gamma t), \quad t \in \mathbb{R}, \quad f \in L^2_{loc}, \quad \gamma > 0.
\]

Then

\[
\begin{align*}
\mathfrak{A}_\gamma & = \mathfrak{A}^\gamma, \\
\mathfrak{B}_\gamma & = \mathfrak{B} \delta_1 / \gamma, \\
\mathfrak{C}_\gamma & = \delta_\gamma \mathfrak{C}, \\
\mathfrak{D}_\gamma & = \delta_\gamma \mathfrak{D} \delta_1 / \gamma.
\end{align*}
\]

The growth bounds \( \omega(\Sigma) \) and \( \omega(\Sigma_\gamma) \) of \( \Sigma \) respectively \( \Sigma_\gamma \) satisfy \( \omega(\Sigma_\gamma) = \gamma \omega(\Sigma) \).

**Proof.** See [Staffans 2005] Example 2.3.6 and 4.8.2. \( \square \)

8.1.50. **Lemma.** Let \( \Sigma = (S; X, U, Y) \) be an i/s/o system, let \( \alpha \in \mathbb{C} \), and let \( \Sigma_\alpha = (S_\alpha; X, U, Y) \) be the exponentially \( \alpha \)-weighted i/s/o system.

(i) \( \Sigma_\alpha \) is well-posed if and only if \( \Sigma \) is well-posed.

(ii) Suppose that \( \Sigma \) is well-posed (and hence also \( \Sigma_\alpha \) is well-posed), and denote the fundamental i/s/o solutions of \( \Sigma \) and \( \Sigma_\alpha \) by \( \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} \) and \( \begin{bmatrix} \mathfrak{A}_\alpha & \mathfrak{B}_\alpha \\ \mathfrak{C}_\alpha & \mathfrak{D}_\alpha \end{bmatrix} \). Let \( e_\alpha \) be the multiplication operator

\[
(e_\alpha u)(t) = e^{\alpha t} u(t), \quad t \in \mathbb{R}, \quad u \in L^2_{loc}.
\]
Then

\[ A^t = e^{at}, \quad B^t = Be^{-at}, \quad C^t = e^{at}C, \quad D^t = e^{at}De^{-at}. \]

The growth bounds \( \omega(\Sigma) \) and \( \omega(\Sigma_\alpha) \) of \( \Sigma \) respectively satisfy \( \omega(\Sigma_\alpha) = \omega(\Sigma) + \Re \alpha. \)

**Proof.** See [Staffans, 2005, Example 2.3.5 and 4.8.2].

8.1.51. **Lemma.** Let \( \Sigma = (S; X, U, Y) \) be an i/s/o system, and let the i/s/o system \( \Sigma_1 = (S_1; X_1, U_1, Y_1) \) be \((P, Q, R)\)-similar to \( \Sigma \).

(i) \( \Sigma_1 \) is well-posed if and only if \( \Sigma \) is well-posed.

(ii) Suppose that \( \Sigma \) is well-posed (and hence also \( \Sigma_1 \) is well-posed), and denote the fundamental i/s/o solutions of \( \Sigma \) and \( \Sigma_1 \) by \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) and \( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \).

Then

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}. \]

The growth bounds of \( \Sigma \) and \( \Sigma_\alpha \) are the same.

**Proof.** This follows from Lemmas 2.3.13 and 8.1.11 and Definition 8.1.12. Add details!

8.1.52. **Lemma.** Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system, and let \( \Sigma_1 = (S_1; X_1, U_1, Y_1) \) be the static output feedback connection of \( \Sigma \) with feedback operator \( K \). Denote the fundamental i/s/o solution of \( \Sigma \) by \( \begin{bmatrix} A \end{bmatrix} \).

(i) \( \Sigma_1 \) is well-posed if and only if \( 1_{\mathbb{R}_{\infty}}(\mathbb{R}; U) - KD \) has an inverse in TIC(\( U \)), or equivalently, if and only if \( 1_{\mathbb{R}_{\infty}}(\mathbb{R}; U) - DK \) has an inverse in TIC(\( Y \)).

(ii) The fundamental i/s/o solution of \( \Sigma_1 \) is described in [Staffans, 2005, Theorem 7.1.2].

**Proof.** See [Staffans, 2005, Theorem 7.1.2].

8.1.53. **Lemma.** Let \( \Sigma \) be an i/s/o node, and let \( \Sigma_1 = (S_1; X_1, U_1, Y_1) \) be a bounded i/o extension of \( \Sigma \) with control operator \( B_1 \), observation operator \( C_1 \), and feedthrough operator \( D_1 \).

(i) \( \Sigma_1 \) is well-posed if and only if \( \Sigma \) is well-posed.

(ii) Suppose that \( \Sigma \) is well-posed (and hence also \( \Sigma_1 \) is well-posed), and denote the fundamental i/s/o solutions of \( \Sigma \) and \( \Sigma_1 \) by \( \begin{bmatrix} A \end{bmatrix} \) respectively.
(8.1.36)  
$$
\begin{align*}
\mathcal{A}_1 &= A, \\
\mathcal{B}_1 \begin{bmatrix} u \\ u_1 \end{bmatrix} &= \mathcal{B}u + \int_{-\infty}^{0} A^{-s} B_1 u_1(s) \, ds, \\
\mathcal{C}_1 \begin{bmatrix} u \\ u_1 \end{bmatrix} &= \mathcal{C}x^0(t), \\
\mathcal{D}_1 \begin{bmatrix} u \\ u_1 \end{bmatrix} &= \mathcal{D}u(t) + D_{00} u(t) + D_{01} u_1(t) \\
&+ D_{10} u(t) + D_{11} u_1(t), \\
\begin{bmatrix} u \\ u_1 \end{bmatrix} &\in L^2_{c, \text{loc}}(\mathbb{R}; \mathcal{Y}_i).
\end{align*}
$$

The growth bounds of $\Sigma$ and $\Sigma_1$ are the same.

\textbf{Proof.} Still missing. \hfill \Box

8.1.8. The fundamental i/s/o solutions of interconnected systems.

8.1.54. \textbf{Lemma.} Let $\Sigma_i = \bigl( [A_i B_i] \, \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i \bigr)$, $i = 1, 2$ be two i/s/o systems, and let $\Sigma_{\times} := \Sigma_1 \times \Sigma_2$ be the cross product of $\Sigma_1$ and $\Sigma_2$.

(i) $\Sigma_{\times}$ is well-posed if and only if both $\Sigma_1$ and $\Sigma_2$ are well-posed.

(ii) Suppose that $\Sigma_{\times}$ is well-posed (and hence both $\Sigma_1$ and $\Sigma_2$ are well-posed).

Denote the fundamental i/s/o solution of $\Sigma_i$ by $\begin{bmatrix} \mathcal{A}_i \\ \mathcal{B}_i \\ \mathcal{C}_i \end{bmatrix}$, $i = 1, 2$. Then the fundamental i/s/o solution $\begin{bmatrix} \mathcal{A}_{\times} \\ \mathcal{B}_{\times} \\ \mathcal{C}_{\times} \end{bmatrix}$ of $\Sigma_{\times}$ is given by

\begin{equation}
\begin{bmatrix}
\mathcal{A}_{\times} & \mathcal{B}_{\times} \\
\mathcal{C}_{\times} & \mathcal{D}_{\times}
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}_1 & 0 & 0 & \mathcal{B}_1 & 0 \\
0 & \mathcal{A}_2 & 0 & \mathcal{B}_2 \\
\mathcal{C}_1 & 0 & \mathcal{D}_1 & 0 \\
0 & \mathcal{C}_2 & 0 & \mathcal{D}_2
\end{bmatrix}.
\end{equation}

The growth bound of $\Sigma_{\times}$ is equal to the maximum of the growth bounds of $\Sigma_1$ and $\Sigma_2$.

\textbf{Proof.} This follows from Lemmas 2.3.34 and 8.1.11 and Definition 8.1.12. \hfill \Box

Add details!

8.1.55. \textbf{Lemma.} Let $\Sigma_i = \bigl( [A_i B_i] \, \mathcal{X}_i, \mathcal{U}_i, \mathcal{Y}_i \bigr)$, $i = 1, 2$ be two well-posed i/s/o systems (with the same input and output spaces) with fundamental i/s/o solutions $\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}$. Denote the parallel and difference connections of $\Sigma_1$ and $\Sigma_2$ by $\Sigma_{\parallel} := \Sigma_1 \parallel \Sigma_2$ respectively $\Sigma_{\ominus} := \Sigma_1 \ominus \Sigma_2$.

(i) Both $\Sigma_{\parallel}$ and $\Sigma_{\ominus}$ are well-posed, and they have the same growth bound, namely the maximum of the growth bounds of $\Sigma_1$ and $\Sigma_2$. 

\textbf{Proof.} Still missing. \hfill \Box
The fundamental i/s/o solutions $\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix}$ and $\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix}$ of $\Sigma_{\parallel}$ respectively $\Sigma_{\parallel}$ are given by

$$\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$  

Proof. This follows from Lemmas 2.3.41 and 8.1.11 and Definition 8.1.12. Add details!

8.1.56. Lemma. Let $\Sigma_1 = (S_1; X_1, U, Z)$ and $\Sigma_2 = (S_2; X_2, Z, Y)$ be two well-posed i/s/o systems with fundamental i/s/o solutions $\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix}$. Denote the cascade connection of $\Sigma_2$ and $\Sigma_1$ by $\Sigma_\circ = \Sigma_1 \circ \Sigma_2$. Then $\Sigma_\circ$ is well-posed, and the fundamental i/s/o solution $\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix}$ of $\Sigma_\circ$ is given by (Fix the following formula!)

$$\begin{bmatrix} A \parallel B \\ C \parallel D \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad t \in \mathbb{R}^+.$$  

Proof. This follows from Lemmas 2.3.41 and 8.1.11 and Definition 8.1.12. Add details!
8.2. Intertwinements, Compressions, and Dilations (Jan 02, 2016)

In this section we discuss intertwinements, compressions, and dilations of well-posed s/s system, as well as the notions of strongly invariant and unobservably invariant subspaces. These notions were defined for arbitrary i/s/o systems in Chapter 2.

8.2.1. Strongly invariant and unobservably invariant subspaces. At this point the reader may want to recall the notions “strongly invariant” and “unobservably invariant” introduced in Definition 2.5.8.

8.2.1. Lemma. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system with evolution semigroup \( \mathcal{A} \), input map \( \mathcal{B} \), reachable subspace subspace \( \mathcal{R} \), and unobservable subspace \( \mathcal{U} \), and let \( Z \) be a subspace of \( X \).

(i) \( Z \) is strongly invariant for \( \Sigma \) if and only if

\[
\text{rng}(\mathcal{B}) \subset Z \text{ and } \mathcal{A}^t Z \subset Z \text{ for all } t \in \mathbb{R}^+.
\]

(ii) If \( Z \) is closed, then \( Z \) is strongly invariant for \( \Sigma \) if and only if

\[
\mathcal{R} \subset Z \text{ and } \mathcal{A}^t Z \subset Z \text{ for all } t \in \mathbb{R}^+.
\]

(iii) \( Z \) is unobservable invariant for \( \Sigma \) if and only if

\[
Z \subset \mathcal{U} \text{ and } \mathcal{A}^t Z \subset Z \text{ for all } t \in \mathbb{R}^+.
\]

Proof. This follows from Lemma 8.1.11(ii), Definition 2.5.8 and Lemma 8.2.4.

8.2.2. Lemma. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system

(i) If \( Z \) is a strongly invariant or unobservable invariant subspace for \( \Sigma \), then the closure of \( Z \) is also strongly invariant respectively unobservably invariant for \( \Sigma \).

(ii) If both \( Z_1 \) and \( Z_2 \) are strongly invariant for \( \Sigma \), then \( Z_1 + Z_2 \) and \( Z_1 \cap Z_2 \) are strongly invariant for \( \Sigma \).

(iii) If both \( Z_1 \) and \( Z_2 \) are unobservably invariant for \( \Sigma \), then \( Z_1 \cap Z_2 \) is unobservably invariant for \( \Sigma \).

Proof. If the invariance conditions listed in Lemmas 8.2.1 hold for some subspace \( Z \) of \( X \), then they also hold if we replace \( Z \) by \( Z \). From this (i) follows. Also the assertions (ii) and (iii) follow from Lemma 8.2.1.

8.2.3. Lemma. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system with fundamental i/s/o solution \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), exactly reachable subspace \( \mathcal{R}^{\text{exact}} \), reachable subspace \( \mathcal{R} \), and unobservable subspace \( \mathcal{U} \). Then

\[
\mathcal{R}^{\text{exact}} = \text{rng}(\mathcal{B}), \quad \mathcal{R} = \frac{\text{rng}(\mathcal{B})}{\mathcal{U}}, \quad \text{and } \mathcal{U} = \text{ker}(\mathcal{C}).
\]

In particular, \( \mathcal{U} \) is closed in \( X \).

Proof. This follows from Definition 2.5.3 and Lemmas 2.5.4, 8.1.11 and 8.1.14.

8.2.4. Lemma. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system with fundamental i/s/o solution \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), exactly reachable subspace \( \mathcal{R}^{\text{exact}} \), reachable subspace \( \mathcal{R} \), and unobservable subspace \( \mathcal{U} \).
(i) The minimal strongly invariant subspace for \( \Sigma \) is equal to \( \mathbb{R}^{\text{exact}} \), i.e., \( \mathbb{R}^{\text{exact}} \) is strongly invariant for \( \Sigma \), and \( \mathbb{R}^{\text{exact}} \) is contained in every other strongly invariant subspace for \( \Sigma \).

(ii) The minimal closed strongly invariant subspace for \( \Sigma \) is equal to \( \mathcal{R} \), i.e., \( \mathcal{R} \) is closed and strongly invariant for \( \Sigma \), and \( \mathcal{R} \) is contained in every other closed strongly invariant subspace for \( \Sigma \).

(iii) The maximal unobservable invariant subspace for \( \Sigma \) is equal to \( \mathcal{U} \), i.e., \( \mathcal{U} \) is unobservable invariant, and \( \mathcal{U} \) contains every other unobservable invariant subspace for \( \Sigma \).

Proof. (i) If \( \mathcal{Z} \) is a strongly invariant subspace for \( \Sigma \), then it follows from Lemma 8.1.11(ii) that \( \text{rng} (\mathcal{B}) \subset \mathcal{Z} \). That \( \text{rng} (\mathcal{B}) \) is strongly invariant for \( \Sigma \) follows from Lemma 8.1.11(ii), Theorem 8.1.15(ii) and Definition 2.5.8(i).

(ii) Claim (ii) follows from (i) and Lemma 8.2.3.

(iii) Claim (iii) follows from Lemma 8.1.11(ii), Theorem 8.1.15(iii), Lemma 8.2.3 and Definition 2.5.8(iii). \( \square \)
8.2.2. Intertwinements of well-posed i/s/o systems. We now return to the notion of intertwinement of two i/s/o systems which was introduced in Definition 2.5.22.

8.2.5. Lemma. Let $\Sigma_i = (S_i; X_i, U_i, Y_i), i = 1, 2$, be two well-posed i/s/o systems, and denote the exactly reachable subspaces and the unobservable subspaces of $\Sigma_i$ by $R^\text{exact}_{\Sigma_i}$ and $\Omega_{\Sigma_i}, i = 1, 2$. If $\Sigma_1$ and $\Sigma_2$ are intertwined by some $P \in \mathcal{ML}(X_1; X_2)$, then the following claims hold:

(i) $\Sigma_1$ and $\Sigma_2$ are externally $\Omega$-equivalent.
(ii) dom ($P$) is strongly invariant for $\Sigma_1$. In particular, $R^\text{exact}_{\Sigma_1} \subset \text{dom}(P)$.
(iii) rng ($P$) is strongly invariant for $\Sigma_2$. In particular, $R^\text{exact}_{\Sigma_2} \subset \text{rng}(P)$.
(iv) ker ($P$) is unobservably invariant for $\Sigma_1$. In particular, ker ($P$) $\subset \Omega_{\Sigma_1}$.
(v) mul ($P$) is unobservably invariant for $\Sigma_2$. In particular, mul ($P$) $\subset \Omega_{\Sigma_2}$.

Proof. This is a special case of Lemma 2.5.27.

8.2.6. Lemma. Two well-posed i/s/o systems $\Sigma_i = (S_i; X_i, U_i, Y_i), i = 1, 2$, (with the same input and output spaces) are intertwined by $P \in \mathcal{ML}(X_1; X_2)$ if and only if the following condition holds:

(i) If $\begin{bmatrix} z_1 \atop u \atop y_1 \end{bmatrix}$ and $\begin{bmatrix} z_2 \atop u \atop y_2 \end{bmatrix}$ are generalized future trajectories of $\Sigma_1$ respectively $\Sigma_2$ (with the same input function $u$) satisfying $x_2(0) \in Px_1(0)$, then $y_1 = y_2$ and $x_2(t) \in Px_1(t)$ for all $t \in \mathbb{R}^+$. 

Proof. Suppose first that $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$. Let $u \in L^2_{\text{loc}}(\mathbb{R}^+; U)$, and let $\begin{bmatrix} z_1 \atop u \atop y_1 \end{bmatrix}$ and $\begin{bmatrix} z_2 \atop u \atop y_2 \end{bmatrix}$ be trajectories of $\Sigma_1$ respectively $\Sigma_2$ satisfying $x_2(0) \in Px_1(0)$. By condition (i) in Definition 2.5.22, $\Sigma_2$ also has a trajectory $\begin{bmatrix} z_3 \atop u \atop y_1 \end{bmatrix}$ with $x_3(0) = x_2(0)$. Since a generalized future trajectory of $\Sigma_i, i = 1, 2$ is uniquely determined by its initial state and input function, we must have $x_3 = x_2$ and $y_3 = y_2$. Thus, condition (i) in Lemma 8.2.6 is satisfied whenever $\Sigma_1$ and $\Sigma_2$ are intertwined by $P$.

Conversely, suppose that condition (i) in Lemma 8.2.6 is satisfied. Since $\Sigma$ has the continuation property (see Lemma 8.1.4), by Lemma 2.5.44(iv) it suffices to show that conditions (i) and (ii) in Definition 2.5.22 hold with $I = \mathbb{R}^+$. Let $\begin{bmatrix} z_1 \atop u \atop y \end{bmatrix}$ be a future generalized trajectory of $\Sigma_1$ with $x_1(0) \subset \text{dom}(P)$. Let $x_2^0 \in Px_1(0)$, and let $\begin{bmatrix} z_2 \atop u \atop y \end{bmatrix}$ be the generalized future trajectory of $\Sigma_2$ with initial state $x_2^0$. By the condition in Lemma 8.2.6, $y_2 = y$. This shows that for every generalized future trajectory $\begin{bmatrix} z_1 \atop u \atop y \end{bmatrix}$ of $\Sigma_1$ with $x_1(0) \subset \text{dom}(P)$ and for every $x_2^0 \in Px_1(0)$ there exists a generalized future trajectory $\begin{bmatrix} z_2 \atop u \atop y \end{bmatrix}$ of $\Sigma_2$ satisfying $x_2(0) = x_2^0$. By interchanging the role of $\Sigma_1$ and $\Sigma_2$ and replacing $P$ by $P^{-1}$ we find that if $\begin{bmatrix} z_2 \atop u \atop y \end{bmatrix}$ is a generalized future trajectory of $\Sigma_2$, and if $x_2(0) \in Px_1^0$ for some $x_1^0 \in \text{dom}(P)$, then there exists a generalized future trajectory $\begin{bmatrix} z_1 \atop u \atop y \end{bmatrix}$ of $\Sigma_1$ satisfying $x_1(0) = x_1^0$.

8.2.7. Lemma. Let let $\Sigma_i = (S_i; X_i, U_i, Y_i), i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces). Denote the fundamental i/s/o solutions of $\Sigma_i$ by $\begin{bmatrix} A_i \atop B_i \atop C_i \atop D_i \end{bmatrix}, i = 1, 2$. Then $\Sigma_1$ and $\Sigma_2$ are intertwined by $P \in \mathcal{ML}(X_1; X_2)$ if and only if the following four conditions hold:
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(i) $A^*_1 x_2 \in P A^*_1 x_1$ for all $x_2 \in Px_1$ and all $t \in \mathbb{R}^+$.
(ii) $B_2 u \in P B_1 u$ for all $u \in L^2_0(\mathbb{R}^-;U)$.
(iii) $C x_2 = C x_1$ for all $x_2 \in Px_1$.
(iv) $D_2 = D_1$.

**Proof.** This follows from Lemma 8.2.6 and the representation formulas for the trajectories of the two systems given in Lemma 8.1.11.

8.2.8. **Corollary.** If the two well-posed i/s/o systems are intertwined by some multi-valued operator $P$, then they are also intertwined by the closure of $P$.

**Proof.** This follows from Lemma 8.2.7 and the continuity of the fundamental i/s/o solution $\begin{bmatrix} P & 0 \end{bmatrix}$ of $\Sigma$.

8.2.9. **Corollary.** Let let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces). Denote the fundamental i/s/o solutions of $\Sigma_i$ by $\begin{bmatrix} A_i & B_i \end{bmatrix}$, $i = 1, 2$. Then $\Sigma_1$ and $\Sigma_2$ are intertwined by $P \in ML(X_1; X_2)$ if and only if the following four conditions hold:

(i) $A_i \vert \text{dom}(P) \subset \text{dom}(P)$ and $A_2P = P A_1 \vert \text{dom}(P)$ for all $t \in \mathbb{R}^+$.
(ii) $\text{rang}(B_1) \subset \text{dom}(P)$ and $B_2 = PB_1$.
(iii) $C x_2P = C x_1 \vert \text{dom}(P)$.
(iv) $D_2 = D_1$.

**Proof.** This is a reformulation of Lemma 8.2.7.

8.2.10. **Lemma.** Let $\Sigma_i = (X_i; X_i, U, Y)$ be three well-posed i/s/o systems. If $\Sigma_1$ and $\Sigma_2$ are intertwined by $P_1 \in ML(X_1; X_2)$ and $\Sigma_2$ and $\Sigma_3$ are intertwined by $P_2 \in ML(X_2; X_3)$, then $\Sigma_1$ and $\Sigma_3$ are intertwined by the operator $P_3 := P_2 P_1 \in ML(X_1; X_3)$, and hence also by the closure of $P_3$.

**Proof.** Let $\begin{bmatrix} x_1 \\ u \end{bmatrix}$ be generalized future trajectories of $\Sigma_i$, $i = 1, 3$, satisfying $x_3(0) \in P_2 P_1 x_1(0)$. By the definition of the composition of two multi-valued operators, this means that there exists some $x^0 \in X_2$ such that $x^0 \in P_1 x_1(0)$ and $x_3(0) \in P_2 x^0$. Let $\begin{bmatrix} x_u \\ u \end{bmatrix}$ be the generalized future trajectory of $\Sigma_2$ whose initial state is $x_2(0) = x^0$. Since $\Sigma_1$ and $\Sigma_2$ are intertwined by $P_1$, and $\Sigma_2$ and $\Sigma_3$ are intertwined by $P_2$, this implies that $y_1 = y_2$ and $y_2 = y_3$, and that $x_3(t) \in P_1 x_1(t)$ and $x_3(t) \in P_2 x_3(t)$ for all $t \in \mathbb{R}^+$. Thus, again by the definition of the composition of two multi-valued operators, $x_3(t) \in P_2 P_1 x_1(t)$ for all $t \in \mathbb{R}^+$. This means that $\Sigma_1$ and $\Sigma_3$ are intertwined by $P_2 P_1$.

8.2.11. **Lemma.** Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces), and let $P \in ML(X_1; X_2)$. Let $\Sigma_\varphi = \Sigma_2 \varphi \Sigma_1$ be the difference connection of $\Sigma_2$ and $\Sigma_1$ (see Definition 2.3.38). Then $\Sigma_1$ and $\Sigma_2$ are intertwined by $P \in ML(X_1; X_2)$ if and only if the input/output map $D$ of $\Sigma$ satisfies $D = 0$, and in addition, $\text{gph}(P)$ is both a strongly invariant and an unobservably invariant subspace for $\Sigma$.

**Proof.** This follows from Lemmas 8.1.55, 8.2.1, and 8.2.7.

8.2.12. **Lemma.** Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces), let $P \in ML(X_1; X_2)$ be closed, and let $\Sigma = (S; \text{gph}(P), U, Y)$ be the $\text{gph}(P)$-short circuit connection of $\Sigma_2$ and $\Sigma_1$.
(see Definition 2.3.37). Then \( \Sigma_1 \) and \( \Sigma_2 \) are \( \Omega \)-intertwined by \( P \) if an only if \( \Sigma \) is well-posed.

**Proof.** Complete this proof, after discussing short circuit connections in general for well-posed i/s/o systems! \( \square \)

8.2.13. Theorem. Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \), \( i = 1, 2 \), be two well-posed i/s/o systems (with the same input and output spaces), and let \( \Sigma = \Sigma_2 \parallel \Sigma_1 \) be the difference connection of \( \Sigma_2 \) and \( \Sigma_1 \) (see Definition 2.3.38). Denote the fundamental i/s/o solutions of \( \Sigma_i \) by \([\begin{array}{ccc} \mathfrak{A}_i & \mathfrak{B}_i \\ \mathfrak{C}_i & \mathfrak{D}_i \end{array}]\), \( i = 1, 2 \), and the fundamental i/s/o solutions of \( \Sigma_\parallel \) by \([\begin{array}{ccc} \mathfrak{A}_\parallel & \mathfrak{B}_\parallel \\ \mathfrak{C}_\parallel & \mathfrak{D}_\parallel \end{array}]\). Then the following claims are true.

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

(ii) If \( \Sigma_1 \) and \( \Sigma_2 \) are externally \( \Omega \)-equivalent, then the following claims are true.

(a) There exists a unique minimal intertwining multi-valued operator \( P_{\min} \), i.e., there exists a unique \( P_{\min} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) such that \( P_{\min} \subset P \) for any other \( P \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( P_{\min} \) is given by

\[
\text{gph}(P_{\min}) = \left[ \begin{array}{ccc} \mathfrak{B}_2 \\ \mathfrak{B}_1 \end{array} \right] L^2_{\mathfrak{D}}(\mathbb{R}^+; \mathcal{U}) = \mathfrak{B}_\parallel L^2_{\mathfrak{D}}(\mathbb{R}^+; \mathcal{U}).
\]

(b) The closure \( \overline{P_{\min}} \) of \( P_{\min} \) is the minimal closed multi-valued operator which intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( \overline{P_{\min}} \) of \( P_{\min}^\Omega \) equal to the reachable subspace of \( \Sigma_\parallel \).

(c) There also exists a unique maximal intertwining multi-valued operator \( P_{\max} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \), i.e., there exists a unique \( P_{\max} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2) \) such that \( P \subset P_{\max} \) for any other \( P \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( P_{\max} \) is equal to the unobservable subspace of \( \Sigma_\parallel \), and it is given by

\[
\text{gph}(P_{\max}) = \ker \left( \begin{bmatrix} \mathfrak{C}_2 & -\mathfrak{C}_1 \end{bmatrix} \right) = \ker \left( \mathfrak{C}_\parallel \right).
\]

**Proof.** (i) By Lemma 2.5.27 if \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some multi-valued operator \( P \), then they are externally equivalent. The converse part of (i) follows from (ii). (The proof of (ii) does not use (i).)

Proof of (ii)(a)–(b): Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. Let \( P_{\min} \) be the operator whose graph is equal to the right-hand side of (8.2.5). By Lemma 8.2.4 \( \text{gph}(P_{\min}) \) is the minimal strongly invariant subspace for \( \Sigma_\parallel \), and the closure of \( \text{gph}(P_{\min}) \), which is equal to the reachable subspace of \( \Sigma_\parallel \), is the minimal closed strongly invariant subspace for \( \Sigma_\parallel \). By Theorem 8.1.47 the i/o maps of \( \Sigma_1 \) and \( \Sigma_2 \) coincide, and therefore \( \mathfrak{D}_\parallel = \mathfrak{D}_2 - \mathfrak{D}_1 = 0 \).

We claim that both \( \text{gph}(P_{\min}) \) and \( \overline{\text{gph}(P_{\min})} \) are unobservably invariant for \( \Sigma \). We know that they are both invariant for the evolution semigroup \( \mathfrak{A}_\parallel \) of \( \Sigma_\parallel \); since they are strongly invariant for \( \Sigma_\parallel \), so by Lemma 8.2.1 we still have to show that they are both contained in \( \text{gph}(P_{\max}) \), where \( P_{\max} \) be the operator whose graph is equal to the right-hand side of (8.2.6). Since \( \text{gph}(P_{\max}) \subset \text{gph}(P_{\min}) \), it suffices to show that

\[
\mathfrak{C}_\parallel \mathfrak{B}_\parallel = \begin{bmatrix} \mathfrak{C}_2 & -\mathfrak{C}_1 \end{bmatrix} \begin{bmatrix} \mathfrak{B}_2 \\ \mathfrak{B}_1 \end{bmatrix} = 0.
\]
But this follows from (8.1.16e) and the fact that \( D_2 - D_1 = 0 \). Thus by Lemma 8.2.1, \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by both \( P_{\min} \) and \( P_{\max} \). This proves the converse part of claim (i), as well as claim (ii)(a) and (ii)(b).

Proof of (ii)(c): Suppose again that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent, and define \( P_{\max} \) by (8.2.6). By Lemma 8.2.4, \( \text{gph}(P_{\max}) \) is the maximal unobservably invariant subspace for \( \Sigma_\beta \). We again have \( D_1 = D_2 \), and we know that \( \text{gph}(P_{\max}) \) is invariant for the evolution semigroup \( \mathfrak{A}_\beta \) of \( \Sigma_\beta \), so it only remains to show that \( C_\beta B_\beta = 0 \). But this was already done in (8.2.7) above. \( \square \)

The definition of pseudo-similarity of two i/s/o systems is given in Definition 2.5.23.

8.2.14. Corollary. Let \( \Sigma_i = (S_i; X_i, U, Y), i = 1, 2, \) be two well-posed i/s/o systems (with the same input and output spaces). Moreover, suppose that both \( \Sigma_1 \) and \( \Sigma_2 \) are controllable and observable. Then \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. Among all the pseudo-similarities between \( \Sigma_1 \) and \( \Sigma_2 \) there is a (unique) minimal one \( P_{\min} \) and a (unique) maximal one \( P_{\max} \), namely those defined in Theorem 8.2.13 (both of which in this case are single-valued densely defined injective operators with dense range).

Proof. If \( \Sigma_1 \) and \( \Sigma_2 \) are pseudo-similar, then it follows from Theorem 8.2.13 that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

Conversely, suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent. By Theorem 8.2.13, \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some multi-valued operator \( P \). By Lemma 8.2.5, \( \text{rng}(B_1) \subset \text{dom}(P) \) and \( \text{rng}(B_2) \subset \text{rng}(P) \), and hence the controllability of \( \Sigma_1 \) and \( \Sigma_2 \) implies that \( \text{dom}(P) \) is dense in \( X_1 \) and \( \text{rng}(P) \) is dense in \( X_2 \). By the same lemma, \( \text{ker}(P) \subset U_1 \) and \( \text{mul}(P) \subset U_2 \), where \( U_i \) is the unobservable subspace of \( \Sigma_i, i = 1, 2, \) and hence the observability of \( \Sigma_1 \) and \( \Sigma_2 \) implies that \( P \) is injective and single-valued. \( \square \)
8.2.3. Compressions, restrictions, and projections. At this point the reader may want to review the definitions of compressions, dilations, restrictions, and projections given in Definitions 2.5.28, 2.5.33, and 2.5.37.

8.2.15. Lemma. Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces), where $X_1$ is a closed subspace of $X_2$, and let $Z_1$ be a direct complement to $X_1$ in $X_2$. Then $\Sigma_1$ is a compression of $\Sigma_2$ onto $X_1$ along $Z_1$ if and only if the following condition holds:

(i) For each $x^0 \in X_1$ and each $u \in L^2_{\text{loc}}(\mathbb{R}^+; U)$, if we denote the generalized future trajectories of $\Sigma_1$ and $\Sigma_2$ with initial state $x^0$ and input function $u$ by $\begin{bmatrix} x_1 \cr y_1 \end{bmatrix}$ respectively $\begin{bmatrix} x_2 \cr y_2 \end{bmatrix}$, then $y_1 = y_2$ and $x_1(t) = P_{X_1}^Z x_2(t)$ for all $t \in \mathbb{R}^+$.

Proof. Suppose first that $\Sigma_1$ is the compression of $\Sigma_2$ onto $X_1$ along $Z_1$. Let $\begin{bmatrix} x_1 \cr y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \cr y_2 \end{bmatrix}$ be the two trajectories in (i). By condition (i) in Definition 2.5.28, $\Sigma_2$ has also a trajectory $\begin{bmatrix} x_2 \cr y_2 \end{bmatrix}$ with initial state $x_3(0) = x_0$ satisfying $x_1 = P_{X_1}^Z x_3$. By the uniqueness part of Lemma 8.1.5, $x_2 = x_3$ and $y_2 = y$, and hence condition (i) above holds.

Conversely, suppose that condition (i) above holds. This implies that both condition (i) and condition (ii) in Definition 2.5.28 hold with $I = \mathbb{R}^+$. Since well-posed i/s/o systems have the continuation property (see Lemma 8.1.4), it follows from Lemma 2.5.44 that $\Sigma_1$ is a compression of $\Sigma_2$ onto $X_1$ along $Z_1$. \qed

By Theorem 8.2.13 and Lemma 2.5.29, if $\Sigma_1$ is a compression of $\Sigma_2$, then there exists a closed multi-valued operator $P$ which intertwines $\Sigma_1$ and $\Sigma_2$. We shall give an explicit formula for how to construct such a multi-valued operator $P$ in Lemma 8.2.23 below.

8.2.16. Lemma. Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces), where $X_1$ is a closed subspace of $X_2$, and let $Z_1$ be a direct complement to $X_1$ in $X_2$.

(i) $\Sigma_1$ is the restriction of $\Sigma_2$ to $X_1$ if and only if every generalized future trajectory of $\Sigma_2$ is also a generalized future trajectory of $\Sigma_2$.

(ii) $\Sigma_1$ is the projection of $\Sigma_2$ onto $X_1$ along $Z_1$ if and only if $\begin{bmatrix} x \cr y \end{bmatrix}$ is a generalized future trajectory of $\Sigma_1$ whenever $\begin{bmatrix} x \cr y \end{bmatrix}$ is a generalized future trajectory of $\Sigma_2$.

Proof. The proof is analogous to the proof of Lemma 8.2.15. \qed

8.2.17. Lemma. Let $\Sigma_i = (S_i; X_i, U, Y)$, $i = 1, 2$, be two well-posed i/s/o systems where $X_2 = X_1 + Z_1$, with fundamental i/s/o solutions $\begin{bmatrix} \mathcal{A}_1 \cr \mathcal{B}_1 \end{bmatrix}, i = 1, 2$.

(i) $\Sigma_1$ is the compression of $\Sigma_2$ onto $X_1$ along $Z_1$ if and only if

(a) $\mathcal{A}_1^t = P_{X_1}^Z \mathcal{A}_2^t |_{X_1}$ for all $t \in \mathbb{R}^+$.

(b) $\mathcal{B}_1 = P_{X_1}^Z \mathcal{B}_2$.

(c) $\mathcal{C}_1 = \mathcal{C}_2 |_{X_1}$.

(d) $\mathcal{D}_1 = \mathcal{D}_2$.

(ii) $\Sigma_1$ is the restriction of $\Sigma_2$ to $X_1$ if and only if
(a) $A_1^t = A_2^t |_{X_1}$ for all $t \in \mathbb{R}^+$.  
(b) $B_1 = B_2$.  
(c) $C_1 = C_2 |_{X_1}$.  
(d) $D_1 = D_2$.

(iii) $\Sigma_1$ is the projection of $\Sigma_2$ onto $X_1$ along $Z_1$ if and only if
(a) $A_1^t P_{X_1}^Z = P_{X_1}^Z A_2^t$ for all $t \in \mathbb{R}^+$.  
(b) $B_1 = P_{X_1}^Z B_2$.  
(c) $C_1 P_{X_1}^Z = C_2$.  
(d) $D_1 = D_2$.

Proof. This follows from Lemma 8.2.15 and the representation formulas for the trajectories of the two systems given in Lemma 8.1.11.

8.2.18. Theorem. Let $\Sigma = (S; X, U, Y)$ be a well-posed i/s/o system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$.

(i) If $\Sigma$ has a restriction $\Sigma_1$ to $X_1$, then $X_1$ is strongly invariant for $\Sigma$. Conversely, if $X_1$ is strongly invariant for $\Sigma$, then $\Sigma$ has a unique well-posed restriction to $X_1$. The fundamental i/s/o solution $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ of $\Sigma_1$ can be obtained from Lemma 8.2.17(ii).

(ii) If $\Sigma$ has an $\Omega$-projection $\Sigma_1$ onto $X_1$ along $Z_1$, then $Z_1$ is unobservably $\Omega$-invariant for $\Sigma$. Conversely, if $Z_1$ is unobservably $\Omega$-invariant for $\Sigma$, then $\Sigma$ has a unique well-posed projection onto $X_1$ along $Z_1$. The fundamental i/s/o solution $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ of $\Sigma_1$ can be obtained from Lemma 8.2.17(iii).

Proof. By Lemmas 2.5.36 and 2.5.39 if $\Sigma$ has a restriction to $X_1$, then $X_1$ is strongly invariant for $\Sigma$, and if $\Sigma$ has a projection onto $X_1$ along $Z_1$, then $Z_1$ is unobservably invariant for $\Sigma$.

Suppose next that $X_1$ is strongly invariant for $\Sigma$. Define $A_1$, $B_1$, $C_1$, and $D_1$ by formulas (a)-(d) in Lemma 8.2.17(ii). Then it follows from Theorem 8.1.15 and Lemma 8.2.1 that $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ is a formal fundamental i/s/o solution in $(X, U, Y)$, and by Theorem 8.1.33 there exists a unique well-posed i/s/o system $\Sigma_1 = (S_1; X_1, U, Y)$ such that $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$ is the fundamental i/s/o solution of $\Sigma_1$. By Lemma 8.2.17(ii), this system is a compression of $\Sigma$ onto $X_1$ along $Z_1$.

The proof of the remaining part of (ii) is analogous to the proof given above with Lemma 8.2.17(ii) replaced by Lemma 8.2.17(iii).
8.2.4. The general structure of a well-posed i/s/o compression.

8.2.19. Lemma. Let \( \Sigma = (S; X, U, Y) \) be a well-posed i/s/o system with fundamental i/s/o solution \( \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \), and let \( X = X_1 + Z_1 \) be a direct sum decomposition of \( X \). Let \( \Sigma_{\text{ext}} = (S_{\text{ext}}; X, \begin{bmatrix} X_1 \\ X \end{bmatrix}, \begin{bmatrix} Y \\ X \end{bmatrix}) \) be the i/o extension of \( \Sigma \) with control operator equal to the embedding operator \( I_{X_1} : X_1 \rightarrow X \), observation operator \( P_{Z_1} \), and feedthrough operator zero, i.e., \( S_{\text{ext}} \) is given by (6.1.15). Denote the reachable and unobservable subspaces of \( \Sigma_{\text{ext}} \) respectively by \( R_{\Sigma_{\text{ext}}} \) and \( U_{\Sigma_{\text{ext}}} \).

(i) There exists a (unique) minimal closed strongly invariant subspace \( X_{\text{min}} \) for \( \Sigma \) which contains \( X_1 \) (i.e., \( X_{\text{min}} \) is closed and strongly invariant for \( \Sigma \), and \( X_{\text{min}} \) is contained in every other closed strongly invariant subspace of \( \Sigma \) which contains \( X_1 \)). This subspace has the following alternative descriptions:

\[
(8.2.8a) \quad X_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \left\{ x_2(t) \begin{bmatrix} x \\ y \end{bmatrix} \right. \text{ is a generalized future trajectory of } \Sigma \left. \text{ with initial state } x(0) \in X_1 \right\},
\]

\[
(8.2.8b) \quad X_{\text{min}} = R_{\Sigma_{\text{ext}}},
\]

\[
(8.2.8c) \quad X_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng} \left( \begin{bmatrix} A^t | X_1 \\ B \end{bmatrix} \right).
\]

(ii) The space \( X_{\text{min}} \) has the direct sum decomposition \( X_{\text{min}} = X_1 + Z_{\text{min}} \), where

\[
(8.2.9a) \quad Z_{\text{min}} = X_{\text{min}} \cap Z_1 = P_{Z_1} X_{\text{min}} = P_{Z_1} R_{\Sigma_{\text{ext}}}.
\]

This subspace also given by

\[
(8.2.9b) \quad Z_{\text{min}} = \bigvee_{t \in \mathbb{R}^+} \text{rng} \left( P_{Z_1} \begin{bmatrix} X_1 \\ Y \end{bmatrix} \right).
\]

(iii) There exists a (unique) maximal unobservably invariant subspace \( Z_{\text{max}} \) for \( \Sigma \) which is contained in \( Z_1 \) (i.e., \( Z_{\text{max}} \) is unobservably invariant for \( \Sigma \), and \( Z_{\text{max}} \) contains every other unobservably invariant subspace for \( \Sigma \) which is contained in \( Z_1 \)). This subspace has the following alternative descriptions:

\[
(8.2.10a) \quad Z_{\text{max}} = \left\{ x_2(0) \in Z_1 \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} \text{ is an generalized unobservable future trajectory of } \Sigma_2 \text{ satisfying } x_2(t) \in Z_1 \text{ for all } t \in \mathbb{R}^+ \right\},
\]

\[
(8.2.10b) \quad Z_{\text{max}} = Z_1 \cap U_{\Sigma_{\text{ext}}},
\]

\[
(8.2.10c) \quad Z_{\text{max}} = \bigcap_{t \in \mathbb{R}^+} \ker \left( P_{Z_1} \begin{bmatrix} X_1 \\ Y \end{bmatrix} \right)_{Z_1}.
\]

In particular, \( Z_{\text{max}} \) is closed.

Proof. Complete this proof after finishing the discussion i/o extensions of well-posed i/s/o systems! Clearly both \( X_1 + Z_{\text{min}} \) and \( Z_{\text{max}} \) are closed subspaces of \( X_2 \). It is also easy to see that \( Z_{\text{max}} \) is unobservably invariant for \( \Sigma_2 \) (a left-shifted generalized future trajectory of \( \Sigma_2 \) is still a generalized future trajectory of \( \Sigma_2 \)). Moreover, it is easy to see from Definition 2.5.8 that if \( Z_u \subset Z_1 \) is closed
and \( X_1 + Z_u \) is strongly invariant for \( \Sigma_2 \) then \( Z_{\min} \subset Z_u \), and that if \( Z_u \) is an arbitrary unobservably invariant subspace for \( \Sigma_2 \) contained in \( Z_1 \) then \( Z_u \subset Z_{\max} \).

Thus, it only remains to show that \( X_1 + Z_{\min} \) is strongly invariant for \( \Sigma_2 \).

By Lemma 8.1.11(ii), the trajectory \( \begin{bmatrix} x_2 \\ u \end{bmatrix} \) in (8.2.9) satisfies \( x_2(t) = A^t_1 x^0 + B_2 \tau^t \pi_+ u \), where \( x^0 \in X_1 \) and \( u \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}^+; U) \) can be chosen arbitrarily, and \( X_1 + Z_{\min} \) is the closed linear span of all such vectors \( x_2(t) \). In particular, this implies that \( \text{rng} (B)_2 \subset X_1 + Z_{\min} \). Applying \( A_3 \) with \( s \in \mathbb{R}^+ \) to \( x_2(t) \) and using the intertwinement conditions in Theorem 8.1.15 we get

\[
A_3^t x_2(t) = A_3^t A^t_1 x^0 + A_3^t B_2 \tau^t \pi_+ u = A_3^t A^t_2 x^0 + A_3^t B_2 \tau^t \pi_+ \pi_{[0,t]} u.
\]

This is equal to \( x_2(t+s) \) if we replace \( u \) by \( \pi_{[0,t]} u \). Thus \( A_3^t x_2(t) \subset X_1 + Z_{\min} \). After taking the closed linear span of all such vectors \( x_2(t) \) we find that \( A_3^t(x_1 + Z_{\min}) \subset X_1 + Z_{\min} \) for all \( s \in \mathbb{R}^+ \). Thus by Lemma 8.2.1, \( X_1 + Z_{\min} \) is strongly invariant for \( \Sigma_2 \).

8.2.20. Theorem. Let \( \Sigma = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system, and let \( \mathcal{X} = X_1 + Z_1 \) be a direct sum decomposition of \( \mathcal{X} \). Let \( X_{\min} \) be the minimal strongly invariant subspace of \( \Sigma \) which contains \( X_1 \), let \( Z_{\max} \) be the maximal unobservably invariant subspace of \( \Sigma \) which is contained in \( Z_1 \), and let \( Z_{\min} = X_1 \cap Z_1 \) (cf. Lemma 8.2.19). Then the following conditions are equivalent:

(i) \( \Sigma \) has a (unique) well-posed compression \( \Sigma_1 = (\mathcal{S}_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}) \) onto \( X_1 \) along \( Z_1 \).

(ii) \( Z_1 \) contains some closed subspace \( Z \) such that \( Z \) is unobservably invariant for \( \Sigma \) and \( X_1 + Z \) is strongly invariant for \( \Sigma \).

(iii) \( Z_{\min} \) defined is unobservably invariant for \( \Sigma \).

(iv) \( X + Z_{\max} \) is strongly invariant for \( \Sigma \).

(v) \( Z_{\min} \subset Z_{\max} \).

Two possible choices of the subspace \( Z \) in (ii) are to take either \( Z = Z_{\min} \) or \( Z = Z_{\max} \), and every possible subspace \( Z \) in (ii) satisfies \( Z_{\min} \subset Z \subset Z_{\max} \).

Proof. Throughout this proof we denote the fundamental i/s/o solution of \( \Sigma \) by \( \begin{bmatrix} A \mid B \end{bmatrix} \).

(i) \( \Rightarrow \) (iii): Using the intertwinement conditions listed Theorem 8.1.15 for both systems \( \Sigma_1 \) and \( \Sigma \) we get for all \( s, t \in \mathbb{R}^+ \), all \( x^0 \in X_1 \), and all \( u \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; U) \),

\[
P_{X_1} A^t (A^t x^0 + Bu) = P_{X_1} A^t (A^{t+s} x^0 + B \tau^t \pi_+ u)
= A^{t+s} x^0 + B_1 \pi_+ u
= A^t (A^t x^0 + B_1 u)
= P_{X_1} A^t (A^t x^0 + Bu),
\]

\[
C (A^t x^0 + Bu) = \pi_+ \tau^t C x^0 + \pi_+ D \pi_- u
= \pi_+ \tau^t C_1 x^0 + \pi_+ D_1 \pi_- u
= C_1 (A^t x^0 + B_1 u)
= CP_{X_1} (A^t x^0 + Bu),
\]
and hence
\[
(8.2.11) \quad P_{X_t}^a \mathcal{A}^t P_{Z_t}^a \left( \mathcal{A}^t x^0 + \mathcal{B} u \right) = 0 \quad \text{and} \quad \mathcal{C} P_{Z_t}^a \left( \mathcal{A}^t x^0 + \mathcal{B} u \right) = 0
\]
for all \( s, t \in \mathbb{R}^+ \), all \( x^0 \in X_t \), and all \( u \in L^2_{t, \text{loc}}(\mathbb{R}; \mathcal{U}) \). Taking the closed linear span over all \( t \in \mathbb{R}^+ \), all \( x^0 \in X_t \), and all \( u \in L^2_{t, \text{loc}}(\mathbb{R}; \mathcal{U}) \) we get from \( 8.2.9b \)
\[
P_{X_t}^a \mathcal{A}^a z = 0, \quad \mathcal{C} z = 0, \quad s \in \mathbb{R}^+, \quad z \in Z_{\text{min}}.
\]
Thus by Lemma \( 8.2.3 \), \( Z_{\text{min}} \) is unobservably invariant.

(iii) \( \Rightarrow \) (ii): This follows from Lemma \( 8.2.19 \).

(ii) \( \Rightarrow \) (i): By Lemma \( 8.2.1 \), condition (ii) is equivalent to the following four conditions, for all \( t \in \mathbb{R}^+ \):
\[
\mathcal{A}^t (X_t + Z) \subset X_t + Z, \quad \text{rng} \left( \mathcal{B} \right) \subset X_t + Z,
\]
\[
\mathcal{A}^t Z \subset Z, \quad Z \subset \ker (\mathcal{C}).
\]
This implies that, for all \( t \in \mathbb{R}^+ \),
\[
P_{Z_t}^a \text{rng} \left( \left[ \mathcal{A}^t \quad \mathcal{B} \right] \left[ \begin{array}{c} \mathcal{X}^t \\ \mathcal{U} \end{array} \right] \right) \subset Z,
\]
and hence for all \( s, t \in \mathbb{R}^+ \),
\[
\left[ \begin{array}{cc} P_{Z_t}^a & 0 \\ 0 & 1_y \end{array} \right] \left[ \begin{array}{c} \mathcal{A}^s \\ \mathcal{C} \end{array} \right] P_{Z_t}^a \left[ \mathcal{A}^t \quad \mathcal{B} \right] \left[ \begin{array}{c} \mathcal{X}^t \\ \mathcal{U} \end{array} \right] = 0,
\]
or equivalently,
\[
(8.2.12) \quad \left[ \begin{array}{cc} P_{X_t}^a & 0 \\ 0 & 1_y \end{array} \right] \left[ \begin{array}{c} \mathcal{A}^s \\ \mathcal{C} \end{array} \right] P_{Z_t}^a \left[ \mathcal{A}^t \quad \mathcal{B} \right] \left[ \begin{array}{c} \mathcal{X}^t \\ \mathcal{U} \end{array} \right] = 0.
\]

Define \( \mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \) and \( \mathcal{D}_1 \) by conditions (a)–(d) in Lemma \( 8.2.17(i) \). By using \( 8.2.12 \) it is a straightforward exercise to show that \( \mathcal{A}_1 \) is a \( C_0 \) semigroup, and that the operators \( \mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \) and \( \mathcal{D}_1 \) satisfy the same intertwinement conditions (i)–(iv) listed in Theorem \( 8.1.15 \) as the operators \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \) and \( \mathcal{D} \). By Theorem \( 8.1.33 \) there exists a well-posed system \( \Sigma_1 = (S_1; X_1, U, Y) \) such that the fundamental i/s/o solution of \( \Sigma \) is \( \left[ \begin{array}{c} \mathcal{A}^t(0) \\ 0 \end{array} \right] \left[ \begin{array}{c} \mathcal{B} \\ \mathcal{D} \end{array} \right] \). By Lemma \( 8.2.17 \), \( \Sigma_1 \) is the compression of \( \Sigma \) onto \( X_1 \) along \( Z_1 \).

(i) \( \Rightarrow \) (iv): Assume that (i) holds. By Lemma \( 8.2.1(ii) \), in order to show that \( X_1 + Z_{\text{max}} \) is strongly invariant it suffices to show that for all \( x \) of the form \( x = x^0 + z^0 \) with \( x^0 \in X_1 \) and \( z^0 \in Z_{\text{max}} \), for all \( u \in L^2_{t, \text{loc}}(\mathbb{R}; \mathcal{U}) \), and for all \( t \in \mathbb{R}^+ \) we have
\[
\mathcal{A}^t(x^0 + z^0) + \mathcal{B} u \in X_1 + Z_{\text{max}},
\]
or equivalently,
\[
P_{Z_t}^a \left( \mathcal{A}^t(x^0 + z^0) + \mathcal{B} u \right) \in Z_{\text{max}}.
\]
By \( 8.2.10c \), this is equivalent to the requirement that for all \( x^0, z^0, \) and \( u \) of the type above and for all \( s, t \in \mathbb{R}^+ \) we have
\[
P_{X_t}^a \mathcal{A}^s P_{Z_t}^a \left( \mathcal{A}^t(x^0 + z^0) + \mathcal{B} u \right) = 0,
\]
\[
\mathcal{C} P_{Z_t}^a \left( \mathcal{A}^t(x^0 + z^0) + \mathcal{B} u \right) = 0.
\]
If \( z^0 = 0 \), then these two identities follow from \( 8.2.11 \), so it only remains to prove that these two identities also hold when \( x^0 = 0 \) and \( u = 0 \). However, that this
is true follows from the unobservable invariance of $Z_{\text{max}}$, established in Lemma 8.2.19. Thus (i) ⇒ (iv).

(iv) ⇒ (ii): This follows from Lemma 8.2.19.

(iii) ⇒ (v): This follows from Lemma 8.2.19.

(v) ⇒ (iii): We know from Lemma 8.2.19 that $X_1 + Z_{\text{min}}$ is strongly invariant and that $Z_{\text{max}}$ is unobservably invariant for $\Sigma$. By Lemma 8.2.1 the strong invariance of $X_1 + Z_{\text{min}}$ gives $\mathcal{U}^t Z_{\text{min}} \subset X_1 + Z_{\text{min}}, t \in \mathbb{R}^+$, whereas the unobservable invariance of $Z_{\text{max}}$ together with the condition $Z_{\text{min}} \subset Z_{\text{max}}$ gives for all $z_0 \in Z_{\text{min}}$ and all $t \in \mathbb{R}^+$

$$\mathcal{U}^t z_0 \subset \mathcal{U}^t Z_{\text{max}} \subset Z_{\text{max}} \subset Z_1, \quad t \in \mathbb{R}^+, \quad \mathcal{C}z_0 = 0.$$ 

Thus $Z_{\text{min}} \subset \ker(\mathcal{C})$ and $\mathcal{U}^t Z_{\text{min}} \subset (X_1 + Z_{\text{min}}) \cap Z_1 = Z_{\text{min}}, t \in \mathbb{R}^+$. By Lemma 8.2.1, $Z_{\text{min}}$ is unobservably invariant for $\Sigma$.

We have now completed the proof of the equivalence of the conditions (i)–(v).

From this proof also follows that (ii) holds with $Z$ replaced by $Z_{\text{min}}$ and by $Z_{\text{max}}$, and the inclusions $Z_{\text{min}} \subset Z \subset Z_{\text{max}}$ follow from Lemma 8.2.19.

8.2.21. Corollary. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a well-posed i/s/o system, and let $X = X_1 + Z_1$ be a direct sum decomposition of $X$. Then $\Sigma$ has a well-posed compression $\Sigma_1 = (S_1; X_1, \mathcal{U}, \mathcal{Y})$ onto $X_1$ along $Z_1$ with fundamental i/s/o solution

$$\left[ \begin{array}{c|c} 0 & \mathcal{A} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right]$$

if and only if $Z_1$ has a direct sum decomposition $Z_1 = Z + Z_c$ such that the fundamental i/s/o solution

$$\left[ \begin{array}{c|c|c} 0 & \mathcal{A} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right]$$

of $\Sigma$ has the following structure with respect to the decomposition $X = X + X_1 + Z_c$ of $X$ (where irrelevant entries have been denoted by $*$):

$$(8.2.13) \left[ \begin{array}{c|c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right] = \left[ \begin{array}{c|c|c} \mathcal{A}_Z & * & * \\ \hline 0 & \mathcal{A}_1 & * \\ \hline 0 & \mathcal{C}_1 & * \\ \hline 0 & 0 & \mathcal{B}_1 \end{array} \right].$$

Here $Z + X_1$ is strongly invariant for $\Sigma$, $Z$ is unobservably invariant for $\Sigma$, $\mathcal{A}_Z$ is the restriction of $\mathcal{A}$ to $Z$, and $\mathcal{A}_Zc$ is the projection of $\mathcal{A}$ onto $Z_c$ along $X_1 + Z$.

The subspace $Z$ in this decomposition can be chosen to be the same as the subspace $\mathcal{Z}$ in condition (ii) in Theorem 8.2.20, and the subspace $Z_c$ can be chosen to be an arbitrary direct complement to $Z$ in $Z_1$. In particular, two possible choices of $Z$ are $Z = Z_{\text{min}}$, and $Z = Z_{\text{max}}$, where $Z_{\text{min}}$ and $Z_{\text{max}}$ are the subspaces defined in Lemma 8.2.12.

Proof. This follows from the equivalence of (i) and (ii) in Theorem 8.2.20 (take $Z_c$ to be an arbitrary direct complement to $Z$ in $Z_1$).

Clearly, the subspace $Z$ is not unique without any further assumptions. For example, as the following theorem shows, every compression can be decomposed into a restriction followed by a projection, and also into a projection followed by a restriction.

8.2.22. Theorem. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a well-posed i/s/o system, let $X = X_1 + Z_1$ be a direct sum decomposition of $X$, and suppose that $\Sigma_1 = (S_1; X_1, \mathcal{U}, \mathcal{Y})$ is a well-posed compression of $\Sigma$ onto $X_1$ along $Z_1$. Let $Z$ satisfy the conditions listed in (ii) in Theorem 8.2.20, and let $Z_c$ be an arbitrary direct complement to $Z$ in $Z_1$. 

8.2. COMPRESSIONS, DILATIONS, AND INTERTWINEMENTS

(i) Let $\Sigma_2$ be the well-posed restriction of $\Sigma$ to the strongly invariant subspace $X_1 + Z$ for $\Sigma$ given by Theorem 8.2.18(i). Then $Z$ is unobservably invariant for $\Sigma_2$, and $\Sigma_1$ is the projection onto $X_1$ along $Z^\Omega$ of $\Sigma_2$.

(ii) Let $\Sigma_3$ be the $\Omega$-resolvable $\Omega$-projection of $\Sigma$ onto $X_1 + Z_c$ along $Z^\Omega$ given by Theorem 6.1.44(ii). Then $X_1$ is strongly $\Omega$-invariant for $\Sigma_3$, and $\Sigma_1$ is the $\Omega$-restriction to $X_1$ of $\Sigma_3$.

Proof. The proof is analogous to the proof of Theorem 6.1.50. □

8.2.23. Lemma. Let $\Sigma_i = (S_i, X_i, U, Y)$ be two well-posed i/s/o systems, $i = 1, 2$, with $X_2 = X_1 + Z_1$. Then the following two conditions are equivalent.

(i) $\Sigma_1$ is the compression of $\Sigma_2$ onto $X_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z$ such that $\Sigma_2$ and $\Sigma_1$ are intertwined by the operator $P = P_{X_1}^Z$ with $\text{dom}(P) = X_1 + Z$.

Condition (ii) above holds for some particular subspace $Z$ if and only condition (ii) in Theorem 8.2.20 holds for the same subspace $Z$. Thus, in particular, two possible choices of the subspace $Z$ in (ii) are to take either $Z = Z_{\text{min}}$ or $Z = Z_{\text{max}}$ defined in (8.2.9) and (8.2.9), and every possible subspace $Z$ satisfies $Z_{\text{min}} \subset Z \subset Z_{\text{max}}$.

Proof. The proof is analogous to the proof of Lemma 6.1.51. □

8.2.24. Theorem. A well-posed i/s/o system $\Sigma = (S, X, U, Y)$ is minimal if and only if $\Sigma$ is both controllable and observable.

Proof. The proof is analogous to the proof of Theorem 6.2.7. □
8.3. Well-Posed I/S/O Systems in the Frequency Domain (Jan 02, 2016)

In Chapter 6 we developed a frequency domain theory for I/S/O systems with nonempty resolvent sets. Since every well-posed I/S/O system has a nonempty resolvent set, there ought to be a close connection between these two theories, and indeed, this is the case.

8.3.1. Time and frequency domain invariance.

8.3.1. Lemma. Let $\Sigma = (S; X, U, Y)$ be a well-posed I/S/O system with fundamental i/s/o solution $[\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$ and i/s/o resolvent matrix $[\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$. Then there is a one-to-one correspondence between each one of $A$, $B$, $C$, and $D$ and the restriction of $A$, $B$, $C$, and $D$, respectively, to $\rho(\Sigma)$.

Proof. Let $\omega(\Sigma)$ be the growth bound of $\Sigma$. It follows from Theorem 8.1.31 that the restriction of each one of $A$, $B$, $C$, and $D$ to $C^\omega(\Sigma)$ is determined uniquely by $A$, $B$, $C$, and $D$, respectively. Since all of these functions are analytic in $\rho(\Sigma)$ and $\rho(\Sigma)$ is connected, it follows by analytic continuation that the restrictions of these functions to $\rho(\Sigma)$ are determined uniquely by their restrictions of $C^\omega(\Sigma)$.

Conversely, suppose that we know the restriction of $A$, $B$, $C$, or $D$ to $C^\omega(\Sigma)$. Since a Laplace transformable function is determined uniquely by its Laplace transform, these functions to $\rho(\Sigma)$ and $\rho(\Sigma)$ by $A$, $B$, $C$, and $D$, respectively. Since all of these functions are analytic in $\rho(\Sigma)$ and $\rho(\Sigma)$ is connected, it follows by analytic continuation that the restrictions of these functions to $\rho(\Sigma)$ are determined uniquely by their restrictions of $C^\omega(\Sigma)$.

8.3.2. Lemma. Let $\Sigma$ be a well-posed I/S/O system. Then the reachable subspace $R$ and the unobservable subspace of $\Sigma$ coincide with the $\rho(\Sigma)$-reachable subspace respectively the $\rho(\Sigma)$-unobservable subspace of $\Sigma$ (cf. Lemma 6.1.54).

Proof. Let us denote the fundamental i/s/o solution of $\Sigma$ by $[\begin{bmatrix} A & B \\ C & D \end{bmatrix}]$, the reachable subspace by $R$, and the unobservable subspace by $\mathfrak{U}$. By Lemma 8.2.3 $R = \text{rng}(B)$ and $\mathfrak{U} = \ker(C)$. Fix some $\omega > \omega(\Sigma)$. Then $L^2_\omega(\mathbb{R}^-; U)$ is dense in $L^2_\omega(\mathbb{R}^-; U)$, and by Lemma 8.1.23 $B$ can be extended to a bounded linear operator $L^2_\omega(\mathbb{R}^-; U) \to X$. This combined with Lemma 8.2.3 implies that $R$ is equal to the closure of the range of $B$, regarded as an operator in $B(L^2_\omega(\mathbb{R}^-; U); X)$. By Theorem 8.1.31 with the notation of that theorem, $B(\lambda)u_0 = B(e_{\lambda}u_0 \in R$ for all $\lambda \in \mathbb{C}_+^{\omega(\Sigma)}$ and all $u_0 \in U$. Moreover, since the linear span in $L^2_\omega(\mathbb{R}^-; U)$ of the set $\{e_{\lambda}u_0 \mid \lambda \in \mathbb{C}_+^{\omega(\Sigma)}, u_0 \in U\}$ is dense in $L^2_\omega(\mathbb{R}^-; U)$, we actually have

$$R = \bigvee_{\lambda \in \mathbb{C}_+^{\omega(\Sigma)}, u_0 \in U} B(e_{\lambda}u_0 = \bigvee_{\lambda \in \mathbb{C}_+^{\omega(\Sigma)}, u_0 \in U} B(\lambda)u_0 = \bigvee_{\lambda \in \mathbb{C}_+^{\omega(\Sigma)}} \text{rng}(B(\lambda)).$$

This together with Lemma 6.1.54 shows that $R$ coincides with the $\rho(\Sigma)$-reachable subspace of $\Sigma$.

As we noticed above, $x_0 \in \mathfrak{U}$ if and only if $C x_0 = 0$. By Theorem 8.1.31, this implies that $x_0 \in \ker(C(\lambda))$ for all $\lambda \in \mathbb{C}_+^{\omega(\Sigma)}$. The converse is also true since a Laplace transformable function is determined uniquely by its Laplace transform.
Hence by Lemma 6.1.54, $\mathcal{U}$ coincides with the $\rho_+(\Sigma)$-unobservable subspace of $\Sigma$. □

8.3.3. Lemma. Let $\Sigma = (S; X, U, Y)$ be a well-posed i/s/o system with fundamental i/s/o solution $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$, and let $Z$ be a closed subspace of $X$. Then the following claims are true.

(i) $Z$ is invariant for $\mathfrak{A}$ if and only if $Z$ is $\rho_+(\Sigma)$-invariant for $\Sigma$ (see Definitions 6.1.15(i) and 4.1.24).

(ii) $Z$ is strongly invariant for $\Sigma$ if and only if $Z$ is strongly $\rho_+(\Sigma)$-invariant for $\Sigma$ (see Definitions 2.5.8(i) and 6.1.15(ii)).

(iii) $Z$ is unobservably invariant for $\Sigma$ if and only if $Z$ is unobservably $\rho_+(\Sigma)$-invariant for $\Sigma$ (see Definitions 2.5.8(ii) and 6.1.15(iii)).

Proof. In this proof we denote the i/s/o resolvent matrix of $\Sigma$ by $[\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{smallmatrix}]$.

(i) Suppose that $Z$ is an invariant subspace for $A$. Then by Lemma 4.1.14, for all $\lambda \in C^{+\omega(A)}$ and all $x_0 \in Z$,

$$A^t x_0 = (\lambda - A)^{-1} x_0 = \int_0^\infty e^{-\lambda s} \hat{A}^s x_0 \, ds \in Z.$$ 

Thus $\mathfrak{A}^t Z \subset Z$ for all $\lambda \in \mathbb{C}^{+\omega(\mathfrak{A})}$. By Lemma 6.4.17, $Z$ is $\rho_+(\Sigma)$-invariant for $\Sigma$.

Conversely, if $Z$ is $\rho_+(\Sigma)$-invariant for $\Sigma$, then by Lemma 6.1.16, $\mathfrak{A}^t Z \subset Z$ for all $\lambda \in \rho_+(\Sigma)$. It then follows from Lemma 4.1.14(ii) that $\mathfrak{A}^t Z \subset Z$ for all $t \in \mathbb{R}^+$, i.e., $Z$ is invariant for $\mathfrak{A}$.

(ii)–(iii) Claims (ii) and (iii) follows from (i) combined with Lemmas 6.1.16, 6.1.54, 8.2.1, 8.2.3, and Lemma 8.3.2. □

8.3.4. Corollary. Let $\Sigma = (S; X, U, Y)$ be a well-posed i/s/o system with i/s/o resolvent matrix $[\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{smallmatrix}]$, and let $\lambda_0 \in \rho_+(\Sigma)$. Then

(i) the reachable subspace of $\Sigma$ is the minimal closed invariant subspace for $\mathfrak{A}$ which contains $\text{rng} (\hat{B}(\lambda_0))$,

(ii) the unobservable subspace of $\Sigma$ is the maximal invariant subspace for $\mathfrak{A}$ which is contained in $\ker (\hat{C}(\lambda_0))$.

Proof. This follows from Lemmas 6.1.54 and 8.3.3. □

8.3.2. Time and frequency domain intertwinements and compressions.

8.3.5. Lemma. Two well-posed i/s/o systems $\Sigma_i = (S_i; X_i, U, Y), i = 1, 2$, are intertwined by a closed $P \in \mathcal{ML}(X_1; X_2)$ if and only if $\Sigma_1$ and $\Sigma_2$ are $\Omega_+\rho_+(\Sigma)$-intertwined by $P$, where $\Omega_+\rho_+(\Sigma)$ is the (connected) component of $\rho(\Sigma_1) \cap \rho(\Sigma_2)$ which contains some right half-plane (see Definition 6.1.19 and Lemma 8.2.6).

Proof. This follows from Lemmas 6.1.26, 8.2.11, 8.3.3, and 8.3.1. □

8.3.6. Lemma. Let $\Sigma_j, j = 1, 2$, be two externally equivalent well-posed i/s/o system. Then the minimal and maximal closed intertwining relations $P^\Omega_{\min}$ and $P^\Omega_{\max}$ defined in Theorem 8.2.13 coincide with the minimal and maximal closed $\rho_+(\Sigma)$-intertwining relations $P^\Omega_{\min}$ and $P^\Omega_{\max}$ defined in Theorem 6.1.28 with $\Omega = \rho_+(\Sigma)$. □
8.3.7Lemma. Let $\Sigma_i = (\Sigma_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y})$, $i = 1, 2$, be two well-posed i/s/o systems (with the same input and output spaces), where $\mathcal{X}_1$ is a closed subspace of $\mathcal{X}_2$, and let $\mathcal{Z}_1$ be a direct complement to $\mathcal{X}_1$ in $\mathcal{X}_2$. Then the following claims are true:

(i) $\Sigma_1$ is the restriction of $\Sigma_2$ to $\mathcal{X}_1$ if and only if $\Sigma_1$ is the $\rho_+\infty(\Sigma)$-restriction of $\Sigma_2$ to $\mathcal{X}_1$ (see Definition 6.4.8 and Lemma 8.2.16).

(ii) $\Sigma_1$ is the projection of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ if and only if $\Sigma_1$ is the $\rho_+\infty(\Sigma)$-projection of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ (see Definition 6.4.8 and Lemma 8.2.16).

(iii) $\Sigma_1$ is the compression of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ if and only if $\Sigma_1$ is the $\rho_+\infty(\Sigma)$-compression of $\Sigma_2$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ (see Definition 6.4.8 and Lemma 8.2.16).

Proof. This follows from Lemma 6.1.41, Theorem 8.1.31, and Lemmas 8.2.17 and 8.3.1.

8.3.8Lemma. Let $\Sigma = (\Sigma; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a well-posed i/s/o system, and let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{Z}_1$. Then the subspaces $\mathcal{Z}_\text{min}$ and $\mathcal{Z}_\text{max}$ defined in Lemma 8.2.19 coincide with the subspaces $\mathcal{Z}_{\text{min}}^\Omega$ and $\mathcal{Z}_{\text{max}}^\Omega$ defined in Lemma 6.1.46, with $\Omega = \rho_+\infty(\Sigma)$.

Proof. This follows from Lemma 6.1.46, Theorem 8.1.31, and Lemmas 8.2.17 and 8.3.1.

8.3.9Lemma. Every well-posed of compression a bounded i/s/o system is bounded.

Proof. Let $\Sigma = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be a bounded i/s/o system, and let $\Sigma_1 = (\Sigma_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ be a well-posed compression of $\Sigma$ onto $\mathcal{X}_1$ along $\mathcal{Z}_1$ with main operator $A_1$. By Lemma 8.3.7, $\Sigma_1$ is also the $\rho_+\infty(\Sigma)$-compression of $\Sigma$. In particular, the resolvents of $A_1$ and $A$ satisfy $(\lambda - A_1)^{-1} = P_{\mathcal{Z}_1}(\lambda - A)^{-1}|_{\mathcal{X}_1}$, $\lambda \in \rho_+\infty(A)$. By Lemma A.3.1, $(\lambda - A)^{-1}$ is analytic at infinity, and therefore also $(\lambda - A_1)^{-1} = P_{\mathcal{Z}_1}(\lambda - A)^{-1}|_{\mathcal{X}_1}$ is analytic at infinity. Applying Lemma A.3.1 once more we see that $A_1$ is bounded, and hence $\Sigma_1$ is bounded.

8.3.3. The system operators of restrictions, projections, and compressions. As we saw in Lemma 8.3.7, a well-posed i/s/o system $\Sigma_1$ is the restriction, or projection, or compression of a well-posed i/s/o system $\Sigma_2$ if and only if $\Sigma_1$ is the $\rho_+\infty(\Sigma_2)$-restriction, or $\rho_+\infty(\Sigma_2)$-projection, or $\rho_+\infty(\Sigma_2)$-compression of $\Sigma_2$. In particular, this means that it is possible to use Theorems 6.2.1, 6.2.2, and 6.2.4 to compute the system operator $S_1$ of $\Sigma_1$ from the system operator $S_2$ of $\Sigma_2$. However, in the well-posed case it is possible to say something more about these system operators, as we shall see below.

8.3.10Definition. Let $S : [\mathcal{X}] \supset \text{dom}(S) \rightarrow [\mathcal{Y}]$ be a closed operator, where $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are $H$-spaces, and let $\mathcal{Z}$ be a subspace of $\mathcal{X}$. We denote the main operator and the observation operator of $S$ by $A$ respectively $C$. 

(i) By the notation $S|_{[\mathcal{Z}]}$ we mean the operator which is the restriction of $S$ to $[\mathcal{Z}]$ in the sense that $\text{dom}(S|_{[\mathcal{Z}]}) = \text{dom}(S) \cap [\mathcal{Z}]$ and $S|_{[\mathcal{Z}]}[n] = S[n]$ for all $n \in \text{dom}(S|_{[\mathcal{Z}]})$.

(ii) $\mathcal{Z}$ is called an strongly i/s/o-invariant subspace for $S$ if $[1_{\mathcal{X}} \ 0] \ S[n] \in \mathcal{Z}$ for every $[n] \in \text{dom}(S) \cap [\mathcal{Z}]$.

(iii) $\mathcal{Z}$ is called an unobservably i/s/o-invariant subspace for $S$ if $Ax \in \mathcal{Z}$ and $Cx = 0$ for every $x \in \text{dom}(A) \cap \mathcal{Z}$.

8.3.11. Lemma. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o system, and let $\mathcal{Z}$ be a closed subspace of $\mathcal{X}$. Let $\Sigma_{\text{part}} = (\Sigma_{\text{part}}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be the part of $\Sigma$ in $[\mathcal{Z}]$ (see Definition 8.3.20), and let $S_{[\mathcal{Z}]}$ be the restriction of $S$ to $[\mathcal{Z}]$ (cf. Definition 8.3.10). Then $\Sigma_{\text{part}} = S_{[\mathcal{Z}]}$ if and only if $\mathcal{Z}$ is a strongly i/s/o-invariant subspace for $S$.

Proof. This can be seen by comparing Definitions (6.4.21(i)) and (4.1.24) to each other. □

8.3.12. Lemma. Let $\mathfrak{A}$ be a $C_0$ semigroup on a $H$-space $\mathcal{X}$ with generator $A$ and growth bound $\omega(\mathfrak{A})$, let $\mathcal{Z}$ be a closed subspace of $\mathcal{X}$, and let $\mathcal{X}_0$ and $\mathcal{X}_*$ be the interpolation respectively extrapolation spaces induced by $A$. Then the two following conditions are equivalent, and they are also equivalent to conditions (i)–(v) in Theorem 4.1.26.

(vi) $\mathcal{Z} \cap \mathcal{X}_*$ is dense in $\mathcal{Z}$ and closed in $\mathcal{X}_*$, and it is an invariant subspace of the semigroup $\mathfrak{A}_*$.

(vii) the closure of $\mathcal{Z}$ in $\mathcal{X}_0$ is an invariant subspace of the semigroup $\mathfrak{A}_0$.

If these equivalent conditions hold, then the interpolation space $\mathcal{Z}_*$ induced by $A|_{\mathcal{Z}}$ is equal to $\mathcal{Z} \cap \mathcal{X}_*$, and the extrapolation space $\mathcal{Z}_0$ induced by $A|_{\mathcal{Z}}$ can be identified with the closure of $\mathcal{Z}$ in $\mathcal{X}_0$.

Proof. In this proof, by (i) we mean condition (i) in Theorem 4.1.26.

(i) $\Rightarrow$ (vi): That $\text{dom}(A|_{\mathcal{Z}}) = \mathcal{Z}_* := \mathcal{Z} \cap \mathcal{X}_*$ is dense in $\mathcal{Z}$ follows from condition (iv) in Theorem 4.1.26. To see $\mathcal{Z}_*$ is closed in $\mathcal{X}_*$ we argue as follows: For all $x \in \mathcal{Z}_*$ we have $A|_{\mathcal{Z}}x = Ax$, and hence the restriction of the graph norm induced by $A$ on $\mathcal{X}_*$ to $\text{dom}(A|_{\mathcal{Z}}) = \mathcal{Z}_*$ is equal to the graph norm of $A_{\mathcal{Z}}$. Since $A_t^*x \subset \mathcal{Z}$ for all $x \in \mathcal{Z}$ and $t \in \mathbb{R}^+$ and $A_t^*x \subset \mathcal{X}_*$ for all $x \in \mathcal{X}_*$ and $t \in \mathbb{R}^+$, we have $A_t^*x \subset \mathcal{Z} \cap \mathcal{X}_* = \mathcal{Z}_*$ for all $x \in \mathcal{Z}_*$. Thus, $\mathcal{Z}_*$ is an invariant subspace for $\mathfrak{A}_*$.

(vi) $\Rightarrow$ (i): Let $z \in \mathcal{Z}_*$, and choose some sequence $z_n \in \mathcal{Z}_*$ such that $z_n \to z$ in $\mathcal{X}$. Then, for all $t \in \mathbb{R}^+$, $A_t^*z_n = A_t^*z_n \in \mathcal{Z} \subset \mathcal{Z}_*$, and $A_t^*z_n \to A_t^*z$ as $n \to \infty$. Thus also $A_t^*z \in \mathcal{Z}_*$.

(vi) $\Leftrightarrow$ (vii): This follows from the equivalence of (i) and (vi), with $\mathfrak{A}$ and $A_*$ replaced by $\mathfrak{A}_0$ and $A_0$ respectively $\mathfrak{A}$.

That $\mathcal{Z}_* = \mathcal{X}_0 \cap \mathcal{Z}$ is obvious. The claim about $\mathcal{Z}_0$ follows from the fact that the extrapolated semigroup of $A|_{\mathcal{Z}}$ can be identified with the restriction of $\mathfrak{A}_0$ to the closure of $\mathcal{Z}$ in $\mathcal{X}_0$, and the domain of the generator of this extrapolated semigroup can be identified with $A_0|_{\mathcal{Z}}$.

8.3.13. Theorem. Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a well-posed i/s/o system and let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}$. Then the following conditions are equivalent:

(i) $\mathcal{X}_1$ is a strongly invariant subspace for $\Sigma$ (see Definition 2.5.8).
(ii) $\Sigma$ has a (unique) well-defined well-posed restriction $\Sigma_1$ to $X_1$ (see Lemma 8.2.16).

(iii) $X_1$ is a strongly $\rho_{+\infty}(\Sigma)$-invariant subspace for $\Sigma$ (see Definition 6.1.15).

(iv) $\Sigma$ has a (unique) well-defined $\rho_{+\infty}(\Sigma)$-restriction to $X_1$ (see Definition 6.1.30).

When these equivalent conditions hold, then

(a) The well-posed restriction $\Sigma_1$ of $\Sigma$ mentioned in (ii) coincides with the $\rho_{+\infty}(\Sigma)$-restriction of $\Sigma$ mentioned in (iv).

(b) $X_1$ is a strongly i/s/o-invariant subspace for $S$ (see Definition 8.3.10).

(c) $S[\begin{bmatrix} z_1 \\ u \end{bmatrix}]$ coincides with the generator $S_{\text{part}}$ of the part $\Sigma_{\text{part}} = (S_{\text{part}}; X_1, U, Y)$ of $\Sigma$ in $[\begin{bmatrix} X_1 \\ Y \end{bmatrix}]$ (see Definitions 2.3.20 and 8.3.10).

(d) the system operator of the system $\Sigma_1$ in (ii) is the operator $S_1 = S[\begin{bmatrix} \dot{x}_1 \\ \dot{u} \end{bmatrix}]$.

(e) $\text{dom}(X_1)$ is dense in $[\begin{bmatrix} Y_1 \\ u \end{bmatrix}]$.

(f) $\rho_{+\infty}(\Sigma) \subset \rho(X_1)$.

(g) The main operators $A_i$, the control operators $B_i$, and the observation $C_i$ of $\Sigma_1$, $i = 1, 2$ satisfy

$$A_1 = A|_{X_1}, \quad C_1 = C|_{X_1}, \quad B_1 = B,$$

where we in the identity $B_1 = B$ have identified the extrapolation space $X_{1e}$ with the closure of $X_1$ in the extrapolation space $X_e$ as described in Lemma 8.3.13.

**Proof.** (i) $\Leftrightarrow$ (ii): See Lemma 8.2.16 and Theorem 8.2.18.

(i) $\Leftrightarrow$ (iii): See Lemma 8.3.3.

(iii) $\Leftrightarrow$ (iv): See Theorem 6.1.44.

Proof of (a): See Lemma 8.3.7.

(ii) $\Rightarrow$ (b): Let $[\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}] \in \text{dom}(S) \cap [\begin{bmatrix} X_1 \\ Y \end{bmatrix}]$, and let $[\begin{bmatrix} x(0) \\ u(0) \\ y(0) \end{bmatrix}]$ be a classical future trajectory of $\Sigma$ with $[\begin{bmatrix} x(0) \\ u(0) \\ y(0) \end{bmatrix}] = [\begin{bmatrix} x_0 \\ u_0 \\ 0 \end{bmatrix}]$. Then there also exists a generalized future trajectory $[\begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix}]$ of $\Sigma_1$ with $x_1(0) = x_0$ (and with the same input function $u$). Since $\Sigma_1$ is a restriction of $\Sigma$, and since $x$ and $y$ are uniquely determined by $x(0) = x_0$ and $u$, we must have $[\begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix}] = [\begin{bmatrix} x \\ u \\ y \end{bmatrix}]$. By Theorem 2.4.32, $[\begin{bmatrix} x \\ u \\ y \end{bmatrix}]$ is also a classical trajectory of $\Sigma_1$. By taking $t = 0$ we get $S[\begin{bmatrix} x_0 \\ u_0 \\ 0 \end{bmatrix}] = [\begin{bmatrix} x(0) \\ u(0) \\ y(0) \end{bmatrix}] = [\begin{bmatrix} x_1 \\ u_1 \\ y_1 \end{bmatrix}] \in [\begin{bmatrix} X_1 \\ Y \end{bmatrix}]$. This shows that $X_1$ is strongly i/s/o-invariant for $S$.

(b) $\Rightarrow$ (c): See Lemma 8.3.11.

(a) & (c) $\Rightarrow$ (d): See Theorem 6.2.1.

(d) $\Rightarrow$ (e): See Definition 2.1.1.

Proof of (g): The formulas for $A_1$ and $C_1$ in (g) follow from (d), and the formula for $B_1$ holds because $A_1e$ is the restriction of $A_e$ to $X_1$, and for $\lambda \in \rho(A)$ we have $B_1 = (\lambda - A_1)\mathcal{B}_1(\lambda) = (\lambda - A)\mathcal{B}(\lambda) = B$. \[\Box\]

8.3.14. **Lemma.** Let $\Sigma = (S; X, U, Y)$ be an i/s/o system, where $X$ be the direct sum $X = Z + Y$. Then the projection $S_{\text{proj}}$ of $S$ onto $V$ along $Z$ is single-valued if and only if $Z$ is an unobservably invariant subspace for $S$ (cf. Definitions 2.3.20 and 8.3.10).
8.3.15. **Theorem.** Let \( \Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}) \) be a well-posed i/s/o system, and let \( \mathcal{X}_2 = \mathcal{X}_1 + \mathcal{Z}_1 \). Then the following conditions are equivalent:

(i) \( \mathcal{Z}_1 \) is an unobservably invariant subspace for \( \Sigma_2 \) (see Definition 2.5.8).

(ii) \( \Sigma_2 \) has a (unique) well-defined well-defined projection \( \Sigma_1 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) (see Lemma 8.2.16).

(iii) \( \mathcal{Z}_1 \) is an unobservably \( \rho_+(\Sigma) \)-invariant subspace for \( \Sigma_2 \) (see Definition 6.1.15).

(iv) \( \Sigma_2 \) has a (unique) well-defined \( \rho_+(\Sigma_2) \)-projection onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) (see Definition 6.1.30).

When these equivalent conditions hold, then

(a) The well-defined projection \( \Sigma_1 \) of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) mentioned in (ii) coincides with the \( \rho_+(\Sigma_2) \)-projection of \( \Sigma_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) mentioned in (iv).

(b) \( \mathcal{Z}_1 \) is an unobservably i/s/o-invariant subspace for \( \Sigma_2 \) (see Definition 8.3.10).

(c) The projection \( S_{\text{proj}} \) of \( S_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) is single-valued (see Definition 2.3.20).

(d) The system operator of the system \( \Sigma_1 \) in (ii) is the projection \( S_{\text{proj}} \) of \( S_2 \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \).

(e) \( \rho_+(\Sigma_2) \subset \rho(\Sigma_1) \).

(f) The main operator of \( \Sigma_1 \) is the operator \( A_{\text{proj}} \) defined in (2.3.20) (where \( A_2 \) is the main operator of \( \Sigma_2 \)), and the observation operators \( C_i \) of \( S_i \), \( i = 1, 2 \) satisfy \( C_1 P_{\mathcal{X}_2} x = C_2 x \), \( x \in \text{dom}(A_2) \).

**Proof.** (i) \( \Leftrightarrow \) (ii): See Lemma 8.2.16 and Theorem 8.2.18

(i) \( \Leftrightarrow \) (iii): See Lemma 8.3.3

(iii) \( \Leftrightarrow \) (iv): See Theorem 6.1.44

**Proof of (a):** See Lemma 8.3.7

**Proof of (c):** Let us denote the system operator of \( \Sigma_1 \) by \( S_1 \). Since \( \Sigma_2 \) is solvable, for every \( [x_0 \ u_0] \in \text{dom}(S_2) \) there exists a classical future trajectory \( [x_2 \ u_2] \)

of \( \Sigma_2 \) such that \( [x_2(0) \ u_2(0)] = [x_0 \ u_0] \). By Lemma 8.2.16 \( [p_{x_1} x_2 \ u_2 \ y_2] \) is a generalized future trajectory of \( \Sigma_1 \), and it follows from Theorem 2.4.32 that this trajectory is even classical. Thus, for all \( t \in \mathbb{R}^+ \) we have

\[
S_1 \begin{bmatrix} \frac{dx_2(t)}{dt} \\ \frac{dy_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dx_2(t)}{dt} \\ \frac{dy_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} P_{x_1} \frac{dx_2(t)}{dt} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} S_2 [x_2(t) \ u_2(t)].
\]

Taking \( t = 0 \) we see that \( \begin{bmatrix} p_{x_1} x_0 \ u_0 \end{bmatrix} \) dom \( (S_2) \subset \text{dom}(S_1) \), and that \( S_1 \begin{bmatrix} p_{x_1} x_0 \ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} S_2 [x_0 \ u_0] \) for all \( [x_0 \ u_0] \in \text{dom}(S_2) \). This implies that the restriction of \( S_1 \) to \( \text{dom}(S_{\text{proj}}) \) coincides with \( S_{\text{proj}} \). Since \( S_1 \) is single-valued, also \( S_{\text{proj}} \) must be single-valued.

(c) \( \Rightarrow \) (b): See Lemma 8.3.14

(iv)\&(a) \( \Rightarrow \) (d)\&(e)\&(f): These implications follow from Theorem 6.2.2 and Lemma 8.3.7

(c) \( \Rightarrow \) (d): See Definition 2.1.1
8.3.16. **Theorem.** Let \( \Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}) \) be a i/s/o system with \( \rho(\Sigma_2) \neq \emptyset \), and let \( \mathcal{X}_2 = \mathcal{X}_1 + \mathcal{Z}_1 + \mathcal{Z}_c \) be a direct sum decomposition of \( \mathcal{X}_2 \). Then the following conditions are equivalent:

(i) \( \mathcal{Z}_1 \) is unobservably invariant for \( \Sigma_2 \) and \( \mathcal{X}_1 + \mathcal{Z}_1 \) is strongly invariant for \( \Sigma_2 \).

(ii) \( \mathcal{Z}_1 \) is unobservably \( \rho_+ \infty(\Sigma_2) \)-invariant for \( \Sigma_2 \) and \( \mathcal{X}_1 + \mathcal{Z}_1 \) is strongly \( \rho_+ \infty(\Sigma_2) \)-invariant for \( \Sigma_2 \).

When these equivalent conditions hold, then \( \Sigma_2 \) has both a (unique) well-posed compression \( \Sigma_1 = (S_1; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) and \( \mathcal{Z}_c \), and a (unique) \( \rho_+ \infty(\Sigma_2) \)-compression onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) and \( \mathcal{Z}_c \), and these two compressions coincide. Conversely, if \( \Sigma_2 \) has either a well-posed compression \( \Sigma_1 = (S_1; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) onto \( \mathcal{X}_1 \) along some direct complement \( \mathcal{Z}_2 \) to \( \mathcal{X}_1 \), then \( \mathcal{Z}_2 \) has a decomposition \( \mathcal{Z}_2 = \mathcal{Z}_1 + \mathcal{Z}_c \) for which conditions (i) and (ii) hold. The system operator \( S_1 \) of this \( \Sigma_1 \) is the projection onto \( \mathcal{X}_1 \) along \( \mathcal{Z}_1 \) of the restriction of \( S_2 \) to \( \mathcal{X}_1 + \mathcal{Z}_1 \), and it is also the restriction to \( \mathcal{X}_1 \) of the projection onto \( \mathcal{X} + \mathcal{Z}_1 \) along \( \mathcal{Z}_c \) of \( S_2 \).

**Proof.** That (i) and (ii) are equivalent follows from Lemma 8.3.3 and the other claims from Theorem 6.2.3.

8.3.4. **Time and frequency domain intertwinements.**

8.3.17. **Theorem.** Let \( \Sigma_i = (S_i; \mathcal{X}_i, \mathcal{U}, \mathcal{Y}) \), \( i = 1, 2 \) be two well-posed i/s/o systems, and let \( P \in \mathcal{ML} \) \( (\mathcal{X}_1; \mathcal{X}_2) \) be closed. Let \( \Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \), where \( \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_1 \), be the difference connection of \( \Sigma_1 \) and \( \Sigma_2 \), and denote the part of \( \Sigma \) in \( \text{gph} (P) \) by \( \Sigma_{\text{part}} = (S_{\text{part}}; \text{gph} (P), \mathcal{U}, \mathcal{Y}) \). Then the following conditions are equivalent.

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by \( P \).

(ii) \( \Sigma_1 \) and \( \Sigma_2 \) are \( \rho_+ \infty(\Sigma) \)-intertwined by \( P \).

When these conditions hold, then \( \text{gph} (P) \) is both a strongly i/s/o-invariant and an unobservably i/s/o-invariant subspace for the system operator \( S \) of \( \Sigma \).

**Proof.** That (i) and (ii) are equivalent follows from Lemma 8.3.5. The second claim follows from Lemma 8.2.11 and Theorems 8.3.13 and 8.3.15.

8.3.5. **Compressions into minimal well-posed i/s/o systems.** Still to be written.
All the results in this chapter remain valid if we throughout replace $L^2$ by $L^p$ for some $p$, $1 \leq p < \infty$.

Almost all the results in this chapter also remain true if we allow the state space, the input space, and the output space to be $B$-spaces instead of $H$-spaces. The only exception is the result on the existence of a minimal well-posed compression of a well-posed i/s/o system (using the definition of compression that we use in this book). See also the comments in Section 3.6.

The spaces $L^p_{\alpha,\text{loc}}(\mathbb{R}; \mathcal{Z})$ in Definition 8.1.22 and the spaces $BUC_{\alpha,\text{loc}}(\mathbb{R}; \mathcal{Z})$ in Definition 8.1.27 are Fréchet spaces (like the spaces $C(I; \mathcal{Z})$ and $L^p_{\text{loc}}(I; \mathcal{Z})$; cf. Section 1.7). The spaces $L^p(I; \mathcal{Z})$ in Definition 8.1.10 are so called strict (LB)-spaces, i.e., these spaces are strict inductive limits of the Banach spaces $L^p([T, t_0]; \mathcal{Z})$, $T < t_0$ if we identify $L^p([T, t_0]; \mathcal{Z})$ with the set of all functions in $L^p((-\infty, t_0]; \mathcal{Z})$ whose support is contained in $[T, t_0]$ (see Köthe 1969, Section 19.5). In the same way the spaces $L^p_{\alpha}(\mathbb{R}; \mathcal{Z})$ in Definition 8.1.22 and the spaces $BUC_{\alpha}(\mathbb{R}; \mathcal{Z})$ in Definition 8.1.27 are strict (LB)-spaces. The space $L^p_{\alpha,\text{loc}}(\mathbb{R}; \mathcal{Z})$ in Definition 8.1.22 is a so called strict (LF)-spaces, i.e., it is a strict inductive limits of the Fréchet spaces $L^p_{\text{loc}}([T, \infty); \mathcal{Z})$ if we identify $L^p_{\text{loc}}([T, \infty); \mathcal{Z})$ with the set of all functions in $L^p_{\text{loc}}(\mathbb{R}; \mathcal{Z})$ whose support is contained in $[T, \infty)$. In the same way the space $BC_{\alpha,\text{loc}}(\mathbb{R}; \mathcal{Z})$ can be interpreted as a (LF)-space.
CHAPTER 9

Well-Posed State/Signal Systems and their Behaviors
9.1. Well-Posed State/Signal Systems (Jan 02, 2016)

9.1.1. Basic definitions.

9.1.1. Definition.
(i) A s/s system $\Sigma$ is well-posed if it has at least one well-posed i/o representation.
(ii) An i/o representation $((U, Y))$ of the signal space $W$ of a well-posed s/s system $\Sigma$ is called i/o-well-posed for $\Sigma$ if it is i/o-admissible for $\Sigma$ and the corresponding i/o representation is well-posed.
(iii) The growth bound of $\Sigma$ is the infimum over the growth bounds of all well-posed i/o representations of $\Sigma$.
(iv) An i/o pseudo-system $\Sigma_{i/s/o}$ is well-posable if the s/s system induced by $\Sigma_{i/s/o}$ is a well-posed s/s system.

9.1.2. Lemma. Every well-posed s/s system and every well-posable i/o pseudo-system is forward uniquely solvable.

Proof. This follows from Proposition 2.5.49 and Definition 9.1.1.

9.1.3. Lemma. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a well-posed s/s system.
(i) If $[\hat{\mathbf{x}}_1]$ is a generalized trajectory of $\Sigma$ on the interval $I_1$ with left end-point $t_1$ and $x_1(t_1) = 0$, then the pair $[\hat{\mathbf{x}}]$ defined by

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ w_1(t) \end{bmatrix} \text{ for } t \in I_1,$$

and

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = 0 \text{ for } t < t_1$$

is a generalized trajectory of $\Sigma_{i/s/o}$ on $I := (-\infty, t_1] \cup I_1$.
(ii) Let $[\hat{\mathbf{x}}_1]$ be either a generalized trajectory of $\Sigma$ on the interval $I_1 = (-\infty, t_1]$ whose support is bounded to the left, or a generalized trajectory of $\Sigma_{i/s/o}$ on the finite closed interval $I_1 = [t_0, t_1]$, let $[\hat{\mathbf{x}}_2]$ be a generalized trajectory of $\Sigma_{i/s/o}$ on some interval $I_2$ with left end-point $t_1$, and suppose that $x_1(t_1) = x_2(t_1)$. Then the pair $[\hat{\mathbf{x}}]$ defined by

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ w_1(t) \end{bmatrix} \text{ for } t \in I_1, t < t_1,$$

and

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ w_2(t) \end{bmatrix} \text{ for } t \in I_2,$$

is a generalized trajectory of $\Sigma_{i/s/o}$ on $I = I_1 \cup I_2$.

Proof. This follows from Proposition 2.5.49, Lemma 8.1.4, and Definition 9.1.1.

9.1.4. Lemma. The conclusion of Lemma 9.1.3 remains true also for well-posable i/o systems.

Proof. This follows from Proposition 2.5.49, Lemma 9.1.3, and Definition 9.1.1.
(ii) There exist $T > 0$ and $K_T > 0$ such that all classical trajectories $[\frac{x}{w}]$ of $\Sigma$ on the interval $[0,T]$ satisfy

\[
\|x(t)\|^2 + \|P_T^X w\|^2_{L^2([0,T];X)} \leq K_T \left( \|x(0)\|^2 + \|P_T^Y w\|^2_{L^2([0,T];U)} \right),
\]

where $\|\cdot\|_X$, $\|\cdot\|_U$, and $\|\cdot\|_Y$, are some admissible norms in $X$, $U$, and $Y$.

If these conditions hold, then the $i/o$ decomposition $(U, Y)$ of $W$ is $i/s/o$-well-posed for $\Sigma$.

**Proof.** See Kurula and Staffans [2010].

9.1.6. **Theorem.** Let $\Sigma = (V; X, W)$ be a well-posed $s/s$ system, and let $\Sigma_{i/s/o} = (S; X, U, Y)$ be a well-posed $i/s/o$ representation of $\Sigma$ with $i/s/o$ maps $[A B C D]_i$. Let $(U_1, Y_1)$ be an $i/o$ representation of $W$, and define $\Theta$ and $\bar{\Theta}$ by \((2.2.8)\) and \((2.2.10)\).

(i) The following conditions are equivalent:

(a) The representation $(U_1, Y_1)$ is $i/s/o$-well-posed for $\Sigma$.

(b) The operator $\Theta_{11} + \Theta_{12} D$ maps $L^2_{\text{loc}}(\mathbb{R}^+; U)$ one-to-one onto $L^2_{\text{loc}}(\mathbb{R}^+; U)$.

(c) The operator $\bar{\Theta}_{22} - D \Theta_{12}$ maps $L^2_{\text{loc}}(\mathbb{R}^+; Y)$ one-to-one onto $L^2_{\text{loc}}(\mathbb{R}^+; Y)$.

(ii) If the representation $(U_1, Y_1)$ is $i/s/o$-well-posed for $\Sigma$ then the map $\Theta_{11} + \bar{\Theta}_{12} D$ is a bicontinuous map of $L^2_{\text{loc}}(\mathbb{R}^+; U)$ onto $L^2_{\text{loc}}(\mathbb{R}^+; U)$, the block matrix operator $\begin{bmatrix} 1_X & 0 \\ \Theta_{11} + \Theta_{12} D \end{bmatrix}$ is a bicontinuous map of $L^2_{\text{loc}}(\mathbb{R}^+; U)$ onto $L^2_{\text{loc}}(\mathbb{R}^+; U)$, and the $i/s/o$ maps of the $i/s/o$ representation $\Sigma_1 = (S_1; X, U_1, Y_1)$ of $\Sigma$ are given by

\[
\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1_X & 0 \\ \Theta_{22} C + \Theta_{22} D \end{bmatrix}^{-1},
\]

or equivalently,

\[
\begin{align*}
A_1 &= A - B(\Theta_{11} + \Theta_{12} D)^{-1} C, \\
B_1 &= B(\Theta_{11} + \Theta_{12} D)^{-1}, \\
C_1 &= \Theta_{22} C - (\Theta_{22} + \Theta_{22} D)(\Theta_{11} + \Theta_{12} D)^{-1} C, \\
D_1 &= (\Theta_{22} + \Theta_{22} D)(\Theta_{11} + \Theta_{12} D)^{-1}.
\end{align*}
\]

(iii) If the representation $(U_1, Y_1)$ is $i/s/o$-well-posed for $\Sigma$ then the map $\bar{\Theta}_{22} - D \bar{\Theta}_{12}$ is a bicontinuous map of $L^2_{\text{loc}}(\mathbb{R}^+; Y)$ onto $L^2_{\text{loc}}(\mathbb{R}^+; Y)$, the block matrix operator $\begin{bmatrix} 1_X & 0 \\ \Theta_{22} - \bar{\Theta}_{12} D \end{bmatrix}$ is a bicontinuous map of $L^2_{\text{loc}}(\mathbb{R}^+; Y)$ onto $L^2_{\text{loc}}(\mathbb{R}^+; Y)$, and the $i/s/o$ maps of the $i/s/o$ representation $\Sigma_1 = (S_1; X, U_1, Y_1)$ of $\Sigma$ are given by

\[
\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1_X & -B \bar{\Theta}_{12} \\
0 & \Theta_{22} - \bar{\Theta}_{12} D \end{bmatrix}^{-1} \begin{bmatrix} A & B \bar{\Theta}_{11} \\ C & -\bar{\Theta}_{21} + D \bar{\Theta}_{11} \end{bmatrix},
\]

or equivalently,

\[
\begin{align*}
A_1 &= A + B \bar{\Theta}_{21}(\Theta_{22} - \bar{\Theta}_{12} D)^{-1} C, \\
B_1 &= B \bar{\Theta}_{11} + \bar{\Theta}_{12}(\Theta_{22} - \bar{\Theta}_{12} D)^{-1}(-\bar{\Theta}_{21} + D \bar{\Theta}_{11}), \\
C_1 &= (\Theta_{22} - \bar{\Theta}_{12} D)^{-1} C, \\
D_1 &= (\Theta_{22} - \bar{\Theta}_{12} D)^{-1}(-\bar{\Theta}_{21} + D \bar{\Theta}_{11}).
\end{align*}
\]
9.1.1. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a well-posed \( i/s \) system.

(i) By the future behavior of \( \Sigma \) we mean the set of all \( w \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \) such that \([ \frac{\pi}{w} ]\) is a generalized externally generated future trajectory of \( \Sigma \) for some \( x \in C_0(\mathbb{R}^+; \mathcal{X}) \). The future behavior of \( \Sigma \) is denoted by \( \mathfrak{B}_\Sigma^+ \).

(ii) By the past behavior of \( \Sigma \) we mean the set of all \( w \in L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{W}) \) such that \([ \frac{\pi}{w} ]\) is a generalized past trajectory of \( \Sigma \) for some \( x \in C(\mathbb{R}^-; \mathcal{X}) \) with compact support. The past behavior of \( \Sigma \) is denoted by \( \mathfrak{B}_\Sigma^- \).

(iii) By the two-sided behavior of \( \Sigma \) we mean the set of all \( w \in L^2_{\text{loc}}^c(\mathbb{R}; \mathcal{W}) \) such that \([ \frac{\pi}{w} ]\) is a generalized two-sided trajectory of \( \Sigma \) for some \( x \in C(\mathbb{R}; \mathcal{X}) \) whose support is bounded to the left. The two-sided behavior of \( \Sigma \) is denoted by \( \mathfrak{B}_\Sigma \).

9.1.8. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a well-posed \( s/s \) system, and let \( \Sigma_{i/o} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be a well-posed \( i/s/o \) representation of \( \Sigma \) with \( i/o \) map \( \mathfrak{D}_{\Sigma_{i/o}} \). Then the future, past, and two-sided behaviors of \( \Sigma \) have the following representations:

\[
\mathfrak{B}_\Sigma^+ = \left\{ \left[ \begin{array}{c} u \\ \mathfrak{D}_{\Sigma_{i/o}} u \end{array} \right] \mid u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \right\},
\]

\[
\mathfrak{B}_\Sigma^- = \left\{ \left[ \begin{array}{c} u \\ \mathfrak{D}_{\Sigma_{i/o}} u \end{array} \right] \mid u \in L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{U}) \right\},
\]

\[
\mathfrak{B}_\Sigma = \left\{ \left[ \begin{array}{c} \pi^{-} u \\ \pi^{-} \mathfrak{D}_{\Sigma_{i/o}} u \end{array} \right] \mid u \in L^2_{\text{loc}}^c(\mathbb{R}; \mathcal{U}) \right\}.
\]

Proof. This follows from Proposition 2.5.49, Lemma 8.1.11, and Definitions 9.1.1 and 9.1.7.

9.1.9. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a well-posed \( s/s \) system with future, past, and two-sided behaviors \( \mathfrak{B}_\Sigma^+, \mathfrak{B}_\Sigma^-, \) and \( \mathfrak{B}_\Sigma \), respectively. Then any one of the three behaviors \( \mathfrak{B}_\Sigma^+, \mathfrak{B}_\Sigma^-, \) and \( \mathfrak{B}_\Sigma \) determine the other two behaviors uniquely. More precisely,

(i) \( \mathfrak{B}_\Sigma^+ = \mathfrak{B}_\Sigma \cap L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \),

(ii) \( \mathfrak{B}_\Sigma^- = \pi^{-} \mathfrak{B}_\Sigma = \{ \pi^{-} w \mid w \in \mathfrak{B}_\Sigma \} \),

(iii) \( w \in \mathfrak{B}_\Sigma \) if and only if \( \pi^{+} w \in \mathfrak{B}_\Sigma^+ \) for some \( t \in \mathbb{R} \),

(iv) \( w \in \mathfrak{B}_\Sigma \) if and only if \( \pi^{-} t w \in \mathfrak{B}_\Sigma^- \) for all \( t \in \mathbb{R} \).

(In (i) and (iii) above we identify \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \) with the subspace of functions in \( L^2_{\text{loc}}^c(\mathbb{R}; \mathcal{W}) \) which vanish on \( \mathbb{R}^- \).)

Proof. This follows from Lemma 9.1.8 and the fact that \( \mathfrak{D}_{\Sigma_{i/s/o}} \) satisfies \( \pi^{-} \mathfrak{D} \pi^{+} = 0 \) and \( \pi^{+} \mathfrak{D}_{\Sigma_{i/s/o}} \pi^{-} = \mathfrak{D}_{\Sigma_{i/s/o}} \), \( t \in \mathbb{R} \).
9.1.11. Definition. The two \( \Sigma_1 = (V_1; X_1, \mathcal{W}) \) and \( \Sigma_2 = (V_2; X_2, \mathcal{W}) \) (with the same signal space \( \mathcal{W} \)) are \textit{externally equivalent} if the equivalent conditions (i)–(iii) in Lemma 9.1.10 hold.

9.1.12. Lemma. Let \( \Sigma = (V; X, \mathcal{W}) \) be a well-posed \( s/s \) system with unobservable subspace \( U_\Sigma \) and reachable subspace \( R_\Sigma \), let \( X_0 \) be a direct complement to \( R_\Sigma \) in \( X \).

(i) There exists a unique \( C_0 \) semigroup \( A_{U_\Sigma} \) in \( U_\Sigma \) with the property that if \( \left[ \begin{array}{l} 0 \\ \end{array} \right] \) is a generalized future unobservable trajectory of \( \Sigma \), then \( x(t) = A_{U_\Sigma}^t x(0), \, t \in \mathbb{R}^+ \).

(ii) There exists a unique \( C_0 \) semigroup \( A_{X_0} \) in \( X_0 \) with the property that if \( \left[ \begin{array}{l} x \\ y \end{array} \right] \) is a generalized future trajectory of \( \Sigma \), then \( P^{R_\Sigma} x(t) = A_{X_0}^t A_{X_0}^{-1} P^{R_\Sigma} x(0), \, t \in \mathbb{R}^+ \).

(iii) If \( \Sigma_{i/s/o} = (S; X, U, Y) \) is an arbitrary \( i/s/o \) representation of \( \Sigma \) with evolution semigroup \( \mathfrak{A} \), then the two semigroups in (i) and (ii) are given by \( A_{U_\Sigma} = \mathfrak{A}|_{U_\Sigma} \) respectively \( A_{X_0} = P_{X_0}^{R_\Sigma} \mathfrak{A} |_{X_0} \).

Proof. Let \( \Sigma_{i/s/o} = (S; X, U, Y) \) be a well-posed \( i/s/o \) representation of \( \Sigma \) with evolution semigroup \( \mathfrak{A} \).

We first prove the claim about \( A_{U_\Sigma} \). Let \( \left[ \begin{array}{l} 0 \\ x \end{array} \right] \) be an unobservable trajectory of \( \Sigma \). Then \( \left[ \begin{array}{l} 0 \\ x \end{array} \right] \) is an unobservable trajectory of \( \Sigma_{i/s/o} \), and hence \( x(t) = \mathfrak{A}^t x(0), \, t \geq 0 \). We have \( x(t) \in U_{\Sigma_{i/s/o}} = U_\Sigma \) for all \( t \geq 0 \), so if we define \( A_{U_\Sigma} := A_{U_\Sigma} \), then \( x(t) = A_{U_\Sigma}^t x(0) \) for all \( t \in \mathbb{R}^+ \). By Theorem 4.1.16, \( A_{U_\Sigma} \) is a \( C_0 \) semigroup in \( U_\Sigma \). Finally, \( A_{U_\Sigma} \) is unique since an unobservable future generalize trajectory of \( \Sigma \) is determined uniquely by its initial state.

We next prove the claim about \( A_{X_0} \). Let \( \left[ \begin{array}{l} x \\ y \end{array} \right] \) be a generalized future trajectory of \( \Sigma \), and define \( u = P_{X_0}^{R_\Sigma} w \) and \( y = P_{X_0}^{R_\Sigma} y \). Then \( \left[ \begin{array}{l} x \\ y \end{array} \right] \) is a generalized future trajectory of \( \Sigma_{i/s/o} \), and hence \( x(t) = \mathfrak{A}^t x(0) + \mathfrak{B}^t u, \, t \geq 0 \). Since \( \text{rng}(\mathfrak{B}) \subset R_{\Sigma_{i/s/o}} = R_\Sigma \) we get \( P_{X_0}^{R_\Sigma} x(t) = \mathfrak{A}_{X_0}^t x(0) \). Define \( \mathfrak{A}_{X_0} = P_{X_0}^{R_\Sigma} \mathfrak{A} |_{X_0} \). Then \( P_{X_0}^{R_\Sigma} x(t) = \mathfrak{A}_{X_0}^t P_{X_0}^{R_\Sigma} x(0) \) for all \( t \in \mathbb{R}^+ \). By Theorem 4.1.16, \( \mathfrak{A}_{X_0} \) is a \( C_0 \) semigroup in \( X_0 \). That \( \mathfrak{A}_{X_0} \) is unique follows from the fact that the requirement that \( x(t) = P_{X_0}^{R_\Sigma} x(t) = \mathfrak{A}_{X_0}^t + \mathfrak{B}^t u, \, t \geq 0 \). Since \( \text{rng}(\mathfrak{B}) \subset R_{\Sigma_{i/s/o}} = R_\Sigma \) we get \( P_{X_0}^{R_\Sigma} x(t) = \mathfrak{A}_{X_0}^t P_{X_0}^{R_\Sigma} x(0) \). Define \( \mathfrak{A}_{X_0} = P_{X_0}^{R_\Sigma} \mathfrak{A} |_{X_0} \). Then \( P_{X_0}^{R_\Sigma} x(t) = \mathfrak{A}_{X_0}^t P_{X_0}^{R_\Sigma} x(0) \) for all \( t \in \mathbb{R}^+ \). By Theorem 4.1.16, \( \mathfrak{A}_{X_0} \) is a \( C_0 \) semigroup in \( X_0 \). That \( \mathfrak{A}_{X_0} \) is unique follows from the fact that the requirement that \( x(t) = P_{X_0}^{R_\Sigma} x(t) = \mathfrak{A}_{X_0}^t x(0) \) whenever \( \left[ \begin{array}{l} x \\ y \end{array} \right] \) is a generalized future trajectory of \( \Sigma \) with \( x(0) \in X_0 \) (and hence \( P_{X_0}^{R_\Sigma} x(t) = x(t) \) implies that necessarily \( P_{X_0}^{R_\Sigma} x(t) = P_{X_0}^{R_\Sigma} \mathfrak{A}_{X_0} x(t) \), and hence \( \mathfrak{A}_{X_0} \) must satisfy \( \mathfrak{A}_{X_0} = P_{X_0}^{R_\Sigma} \mathfrak{A} |_{X_0} \). □

9.1.13. Definition. The two semigroups \( A_{U_\Sigma} \) and \( A_{X_0} \) in Lemma 9.1.12 are called the \textit{unobservable evolution semigroup} respectively the \textit{unreachable evolution semigroup} in \( Z \) of \( \Sigma \).

9.1.14. Lemma. Let \( \Sigma = (V; X, \mathcal{W}) \) be a well-posed \( s/s \) system.

(i) For every \( w \in U_\Sigma \) there exists a unique generalized two-sided trajectory \( \left[ \begin{array}{l} x \\ y \end{array} \right] \) of \( \Sigma \) (with signal part equal to \( w \)) whose support is bounded to the left. Moreover, there exist a unique continuous linear operator \( \mathfrak{B}_\Sigma \) from
\( \mathcal{B}_\Sigma \), equipped with the topology inherited from \( L^2_{c,\text{loc}}(\mathbb{R}; \mathcal{W}) \), to \( \mathcal{X} \) such that \( x \) is given by \( x(t) = \mathcal{B}_\Sigma \tau^t w, \ t \in \mathbb{R} \).

(ii) For every \( w \in \mathcal{B}_\Sigma^+ \) there exists a unique generalized past trajectory \([\frac{x}{w}]\) of \( \Sigma \) (with signal part equal to \( w \)) whose support is bounded to the left. The state component \( x \) of this trajectory is given by \( x(t) = \mathcal{B}_\Sigma \tau^t w, \ t \in \mathbb{R}^- \) (where we interpret \( w \) as the restriction to \( \mathbb{R}^- \) of a function in \( L^2_{c,\text{loc}}(\mathbb{R}; \mathcal{W}) \)).

(iii) For every \( w \in \mathcal{B}_\Sigma^+ \) there exists a unique externally generated generalized future trajectory \([\frac{x}{w}]\) of \( \Sigma \) (with signal part equal to \( w \)). The state component \( x \) of this trajectory is given by \( x(t) = \mathcal{B}_\Sigma \tau^t w, \ t \in \mathbb{R}^+ \) (where we interpret \( w \) as a function in \( L^2_{c,\text{loc}}(\mathbb{R}; \mathcal{W}) \) which vanishes on \( \mathbb{R}^- \)).

(iv) If \( \Sigma_{i/s/o} = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) is a well-posed \( i/s/o \) representation of \( \Sigma \), then the operator \( \mathcal{B}_\Sigma \) in (i) is given in terms of the coordinate representation \([\frac{y}{w}]\) of \( \mathcal{W} \) by

\[
(9.1.9) \quad \mathcal{B}_{\Sigma_{i/s/o}} = \mathcal{B}_\Sigma \left[ \begin{array}{c} 1 \\ \mathcal{P} \end{array} \right] \quad \text{and} \quad \mathcal{B}_\Sigma = \mathcal{B}_{\Sigma_{i/s/o}} \mathcal{P}_\mathcal{U}^\mathcal{Y}.
\]

**Proof.** This follows from Proposition 2.5.49, Theorem 8.1.11 and Definition 9.1.1. \( \square \)

**9.1.15. Definition.** The map \( \mathcal{B}_\Sigma \) in Lemma 9.1.14 is called the **input map** of the well-posed \( s/s \) system \( \Sigma \).

**9.1.16. Lemma.** Let \( \mathcal{B}_\Sigma \) be the input map of a well-posed \( s/s \) system \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) with reachable subspace \( \mathcal{R}_\Sigma \). Then the following claims hold.

(i) \( \mathcal{B}_\Sigma \pi_+ = 0 \),

(ii) \( \mathcal{R}_\Sigma = \text{rng}(\mathcal{B}_\Sigma) \).

**Proof.** This follows from Lemmas 8.2.3 and 9.1.14. \( \square \)

**9.1.17. Lemma.** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a well-posed \( s/s \) system.

(i) For every \( x_0 \in \mathcal{X} \) there exists a generalized future trajectory \([\frac{x}{w}]\) of \( \Sigma \) satisfying \( x(0) = x_0 \).

(ii) If \([\frac{x_1}{w_1}]\) and \([\frac{x_2}{w_2}]\) are two generalized future trajectories of \( \Sigma \) with the same initial state \( x(0) = x_1(0) = x_2(0) \), then \( w_1 - w_2 \in \mathcal{B}_\Sigma^+ \).

(iii) Let \([\frac{x_1}{w}]\) be a generalized future trajectory of \( \Sigma \), let \( w_2 \in \mathcal{B}_\Sigma^+ \), and define \( w = w_1 + w_2 \). Then there exists a (unique) function \( x \in C(\mathbb{R}^+; \mathcal{X}) \) with \( x(0) = x_1(0) \) such that \([\frac{x}{w}]\) is a generalized future trajectory of \( \Sigma \).

**Proof.** That (i) holds follows from Proposition 2.5.49, Lemma 8.1.11 and Definition 9.1.1. That (ii) is true follows from the fact that \([\frac{x_1 - x_2}{w_1 - w_2}]\) is a generalized externally generated future trajectory of \( \Sigma \). That (iii) holds follows from Lemma 9.1.14. \( \square \)

**9.1.18. Definition.** Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a well-posed \( s/s \) system. Then the **output map** of \( \Sigma \) is the multi-valued operator \( \mathcal{C}_\Sigma \in \mathcal{ML}(\mathcal{X}; L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W})) \) whose graph is given by

\[
(9.1.10) \quad \text{gph}(\mathcal{C}_\Sigma) = \left\{ \left[ \begin{array}{c} w \\ x(0) \end{array} \right] \in \left[ L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \right]_{\mathcal{X}} \mid \left[ \begin{array}{c} x \\ w \end{array} \right] \text{ is a generalized future trajectory of } \Sigma \right\}.
\]
9.1.19. Lemma. Let \( \mathcal{C}_\Sigma \) be the output map of a well-posed s/s system \( \Sigma = (\mathcal{V}; \mathcal{X}, \mathcal{W}) \) with unobservable subspace \( \mathcal{U}_\Sigma \) and future behavior \( \mathcal{V}_\Sigma^+ \), and let \( \Sigma_{i/s/o} = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \) be an i/s/o representation of \( \Sigma \) with output map \( \mathcal{C}_{\Sigma_{i/s/o}} \).

(i) \( \mathcal{C}_\Sigma \) is closed, \( \text{dom}(\mathcal{C}_\Sigma) = \mathcal{X} \), \( \ker(\mathcal{C}_\Sigma) = \mathcal{U}_\Sigma \), and \( \text{mul}(\mathcal{C}_\Sigma) = \mathcal{V}_\Sigma^+ \).

(ii) \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \) can be written as the direct sum \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) = \mathcal{V}_\Sigma^+ + L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \), and the single-valued part of \( \mathcal{C}_\Sigma \) with respect to the above decomposition of \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \) is equal to \( \mathcal{C}_{\Sigma_{i/s/o}} \).

Proof. We first show that \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) = \mathcal{V}_\Sigma^+ + L^2_{\text{loc}}(\mathbb{R}; \mathcal{Y}) \). Let \( w \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \), and let \( u = P^W_u w \) and \( y = P^Y_y w \), so that \([y\ y]\) is the representation of \( w \) in \( \begin{bmatrix} L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \\ L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \end{bmatrix} \) with respect to the i/o decomposition \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \). Denote the output map of \( \Sigma_{i/s/o} \) by \( \mathcal{D}_{\Sigma_{i/s/o}} \). Then \([y\ y]\) = \( \begin{bmatrix} x_0 \\ \pi_+ \mathcal{D}_{\Sigma_{i/s/o}} u \end{bmatrix} + \begin{bmatrix} 0 \\ y - \pi_0 \mathcal{D}_{\Sigma_{i/s/o}} u \end{bmatrix} \). Here \( \left[ y - \pi_0 \mathcal{D}_{\Sigma_{i/s/o}} u \right] \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \), and by Lemma 9.1.8 \( \left[ \pi_+ \mathcal{D}_{\Sigma_{i/s/o}} u \right] \in \mathcal{V}_\Sigma^+ \). Thus \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) = \mathcal{V}_\Sigma^+ + L^2_{\text{loc}}(\mathbb{R}; \mathcal{Y}) \).

That \( \mathcal{C}_\Sigma \) is closed and that the single-valued part of \( \mathcal{C}_\Sigma \) is \( \mathcal{C}_{\Sigma_{i/s/o}} \) follows from the fact that the graph of \( \mathcal{C}_\Sigma \) has the representation

\[
\text{gph}(\mathcal{C}_\Sigma) = \left\{ \begin{bmatrix} x_0 \\ \pi_+ \mathcal{D}_{\Sigma_{i/s/o}} u + \pi_0 \mathcal{D}_{\Sigma_{i/s/o}} u \end{bmatrix} \right| x_0 \in \mathcal{X} \text{ and } u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \},
\]

where \( \mathcal{C}_{\Sigma_{i/s/o}} \) is continuous \( \mathcal{X} \to L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \) and \( \mathcal{D}_{\Sigma_{i/s/o}} \) is continuous \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \to L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y}) \). This representation also implies that \( \text{dom}(\mathcal{C}_\Sigma) = \mathcal{X} \). That \( \ker(\mathcal{C}_\Sigma) = \mathcal{U}_\Sigma \) follows from Proposition 2.5.49, Lemma 8.2.3 and Definition 9.1.18, and that \( \text{mul}(\mathcal{C}_\Sigma) = \mathcal{V}_\Sigma^+ \) follows from Definitions 9.1.7 and 9.1.18. \( \square \)

9.1.20. Definition. Let \( \Sigma = (\mathcal{V}; \mathcal{X}, \mathcal{W}) \) be a well-posed s/s system with two-sided behavior \( \mathcal{V}_\Sigma \). Then the past/future map of \( \Sigma \) is the multi-valued operator \( \Gamma_\Sigma \in \mathcal{ML}(L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{W}); L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W})) \) whose graph is given by

\[
\text{gph}(\Gamma_\Sigma) = \left\{ \begin{bmatrix} \pi^+ u \\ \pi^- u \end{bmatrix} \in \begin{bmatrix} L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \\ L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{W}) \end{bmatrix} \right| w \in \mathcal{V}_\Sigma \right\},
\]

9.1.21. Lemma. Let \( \Sigma = (\mathcal{V}; \mathcal{X}, \mathcal{W}) \) be a well-posed s/s system with input map \( \mathcal{B}_\Sigma \), output map \( \mathcal{C}_\Sigma \), and past/future map \( \Gamma_\Sigma \). Then the following claims hold:

(i) \( \Gamma_\Sigma \) is closed, \( \text{dom}(\Gamma_\Sigma) = \mathcal{V}_\Sigma \), and \( \text{mul}(\Gamma_\Sigma) = \mathcal{V}_\Sigma^+ \).

(ii) \( \Gamma_\Sigma = \mathcal{C}_\Sigma \mathcal{B}_\Sigma \).

(iii) \( \ker(\mathcal{B}_\Sigma) \subset \ker(\Gamma_\Sigma) \). If \( \Sigma \) is observable, then \( \ker(\mathcal{B}_\Sigma) = \ker(\Gamma_\Sigma) \).

(iv) \( \text{rng}(\Gamma_\Sigma) \subset \text{rng}(\mathcal{C}_\Sigma) \). If \( \Sigma \) is controllable, then \( \text{rng}(\Gamma_\Sigma) \) is dense in \( \text{rng}(\mathcal{C}_\Sigma) \).

Proof. To be added (easy). \( \square \)
9.2. Well-Posed Behaviors (Jan 02, 2016)

9.2.1. Definition. Let \( \mathcal{W} \) be a vector space with a Hilbert space topology. By a well-posed future behavior in \( \mathcal{W} \) we mean a subset \( \mathcal{U}^+ \) of \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \) with the following properties:

(i) \( \mathcal{U}^+ \) is closed in \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{W}) \).

(ii) \( \tau^*_t \mathcal{U}^+ \subset \mathcal{U}^+ \) for all \( t \in \mathbb{R}^+ \) (i.e., \( \mathcal{U}^+ \) is right shift-invariant).

(iii) There exists an input/output representation \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) with the following two properties:

(a) \( P^\mathcal{U}_t \) maps \( \mathcal{U}^+ \) onto \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \), i.e., for every \( u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \) there exists some \( w \in \mathcal{U}^+ \) such that \( u = P^\mathcal{U}_t w \).

(b) There exist constants \( \omega \in \mathbb{R} \) and \( M > 0 \) and inner products in \( \mathcal{U} \) and \( \mathcal{Y} \) which are compatible with the topology inherited from \( \mathcal{W} \) such that every \( w \in \mathcal{U}^+ \) satisfies

\[
\| P^\mathcal{U}_t w \|_{L^2_{\text{loc}}([0,T]; \mathcal{Y})} \leq M \| P^\mathcal{Y}_T w \|_{L^2_{\text{loc}}([0,T]; \mathcal{U})}, \quad T > 0.
\]

9.2.2. Definition. Let \( \mathcal{W} \) be a vector space with a Hilbert space topology. By a two-sided well-posed behavior in \( \mathcal{W} \) we mean a subset \( \mathcal{U} \) of \( L^2_{\text{loc}}(\mathbb{R}; \mathcal{W}) \) with the following properties:

(i) \( \mathcal{U} \) is closed in \( L^2_{\text{loc}}(\mathbb{R}; \mathcal{W}) \).

(ii) \( \tau^*_t \mathcal{U} = \mathcal{U} \) for all \( t \in \mathbb{R} \) (i.e., \( \mathcal{U} \) is bilaterally shift-invariant).

(iii) There exists an input/output representation \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) with the following two properties:

(a) \( P^\mathcal{U}_t \) maps \( \mathcal{U} \) onto \( L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}) \), i.e., for every \( u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U}) \) there exists some \( w \in \mathcal{U} \) such that \( u = P^\mathcal{U}_t w \).

(b) There exist constants \( \omega \in \mathbb{R} \) and \( M > 0 \) and inner products in \( \mathcal{U} \) and \( \mathcal{Y} \) which are compatible with the topology inherited from \( \mathcal{W} \) such that every \( w \in \mathcal{U} \) whose support is contained in the interval \([a, \infty)\) satisfies

\[
\| P^\mathcal{U}_t w \|_{L^2_{\text{loc}}([a,b]; \mathcal{Y})} \leq M \| P^\mathcal{Y}_b w \|_{L^2_{\text{loc}}([a,b]; \mathcal{U})}, \quad b > a.
\]

9.2.3. Definition. Let \( \mathcal{W} \) be a vector space with a Hilbert space topology. By a well-posed past behavior in \( \mathcal{W} \) we mean a subset \( \mathcal{U}^- \) of \( L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{W}) \) with the following properties:

(i) \( \mathcal{U}^- \) is closed in \( L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{W}) \).

(ii) \( \tau^*_t \mathcal{U}^- = \mathcal{U}^- \) for all \( t \in \mathbb{R} \) (i.e., \( \mathcal{U}^- \) is right shift-invariant).

(iii) There exists an input/output representation \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) with the following two properties:

(a) \( P^\mathcal{U}_t \) maps \( \mathcal{U}^- \) onto \( L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{U}) \), i.e., for every \( u \in L^2_{\text{loc}}(\mathbb{R}^-; \mathcal{U}) \) there exists some \( w \in \mathcal{U}^- \) such that \( u = P^\mathcal{U}_t w \).

(b) There exist constants \( \omega \in \mathbb{R} \) and \( M > 0 \) and inner products in \( \mathcal{U} \) and \( \mathcal{Y} \) which are compatible with the topology inherited from \( \mathcal{W} \) such that every \( w \in \mathcal{U}^- \) whose support is contained in the interval \([a, 0)\) satisfies

\[
\| P^\mathcal{U}_t w \|_{L^2_{\text{loc}}([a,b]; \mathcal{Y})} \leq M \| P^\mathcal{Y}_b w \|_{L^2_{\text{loc}}([a,b]; \mathcal{U})}, \quad a < b \leq 0.
\]

9.2.4. Definition. Let \( \mathcal{W} \) be a vector space with a Hilbert space topology. An i/o decomposition \((\mathcal{U}, \mathcal{Y})\) of \( \mathcal{W} \) is called i/o well-posed for a future well-posed
behavior $\mathfrak{Y}^+$ in $\mathcal{W}$, or for a two-sided well-posed behavior $\mathfrak{Y}$ in $\mathcal{W}$, or for a past well-posed behavior $\mathfrak{Y}^-$ in $\mathcal{W}$, if it satisfies condition (iii) in Definitions 9.2.1 or 9.2.3 respectively.

9.2.5. Lemma. Let $\Sigma = (V; X, \mathcal{W})$ be a well-posed s/s system with future, two-sided, and past behaviors $\mathfrak{X}_\Sigma^+, \mathfrak{X}_\Sigma$, and $\mathfrak{X}_\Sigma^-$ respectively.

(i) $\mathfrak{X}_\Sigma^+, \mathfrak{X}_\Sigma$, and $\mathfrak{X}_\Sigma^-$ are well-posed future, two-sided, and past behaviors in $\mathcal{W}$ respectively, in the sense of Definitions 9.2.1, 9.2.2, and 9.2.3.

(ii) For each i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ the following conditions are equivalent:

(a) $(\mathcal{U}, \mathcal{Y})$ is i/s/o-well-posed for $\Sigma$;

(b) $(\mathcal{U}, \mathcal{Y})$ is i/o-well-posed for $\mathfrak{X}_\Sigma^+$;

(c) $(\mathcal{U}, \mathcal{Y})$ is i/o-well-posed for $\mathfrak{X}_\Sigma$;

(d) $(\mathcal{U}, \mathcal{Y})$ is i/o-well-posed for $\mathfrak{X}_\Sigma^-$.

Proof. The nontrivial part of this proof is based on Theorem 9.1.5.

9.2.6. Corollary. Let $\Sigma_1(V_1; X_1; \mathcal{W})$ and $\Sigma_2(V_2; X_2; \mathcal{W})$ be two externally equivalent well-posed s/s systems. Then an i/o representation $(\mathcal{U}, \mathcal{Y})$ of $\mathcal{W}$ is i/s/o-well-posed for $\Sigma_1$ if and only if it is i/s/o-well-posed for $\Sigma_1$.

Proof. This follows from Lemma 9.2.5.

9.2.7. Lemma. Let $\mathcal{W}$ be a vector space with a Hilbert space topology.

(i) Let $\mathfrak{Y}$ be a well-posed two-sided behavior in $\mathcal{W}$, and define $\mathfrak{Y}^+ := \mathcal{W} \cap L^2_{loc}(\mathbb{R}^+; \mathcal{W})$. Then $\mathfrak{Y}^+$ is a well-posed future behavior in $\mathcal{W}$, and $\mathfrak{Y} = \bigcup_{t \in \mathbb{R}} t^+ \mathfrak{Y}^+$.

(ii) Let $\mathfrak{Y}$ be a well-posed two-sided behavior in $\mathcal{W}$, and define $\mathfrak{Y}^- := \pi_- \mathcal{W}$. Then $\mathfrak{Y}^-$ is a well-posed past behavior in $\mathcal{W}$, and

\[
\mathfrak{Y}^- = \{ w \in L^2_c(\mathbb{R}; \mathbb{W}) \mid \pi_- w \in \mathcal{W}^- \text{ for all } t \in \mathbb{R} \}.
\]

(iii) Let $\mathfrak{Y}^+$ be a well-posed future behavior in $\mathcal{W}$, and define $\mathfrak{Y} := \bigcup_{t \in \mathbb{R}} t^+ \mathfrak{Y}^+$. Then $\mathfrak{Y}$ is a well-posed two-sided behavior in $\mathcal{W}$, and $\mathfrak{Y}^+ = \mathcal{W} \cap L^2_{loc}(\mathbb{R}^+; \mathcal{W})$.

(iv) Let $\mathfrak{Y}^-$ be a well-posed past behavior in $\mathcal{W}$, and define $\mathfrak{Y}$ by 9.2.1.

\[
\mathfrak{Y} := \{ w \in L^2_c(\mathbb{R}; \mathbb{W}) \mid \pi_- w \in \mathcal{W}^- \text{ for all } t \in \mathbb{R} \}.
\]

Then $\mathfrak{Y}$ is a well-posed two-sided behavior in $\mathcal{W}$, and $\mathfrak{Y}^- = \pi_- \mathcal{W}$.

Proof. This follows immediately from Definitions 9.2.1, 9.2.2, and 9.2.3.

9.2.8. Definition. Let $\mathcal{W}$ be a vector space with a Hilbert space topology.

(i) If $\mathfrak{Y}^+$ is a well-posed future behavior in $\mathcal{W}$, then by the corresponding well-posed two-sided and past behaviors in $\mathcal{W}$ we mean the well-posed two-sided behavior $\mathfrak{Y} := \bigcup_{t \in \mathbb{R}} \mathfrak{Y}^+$ and the well-posed past behavior $\mathfrak{Y}^- := \pi_- \mathfrak{Y}$.

(ii) If $\mathfrak{Y}$ is a well-posed two-sided behavior in $\mathcal{W}$, then by the corresponding well-posed future and past behaviors in $\mathcal{W}$ we mean the well-posed future behavior $\mathfrak{Y}^+ := \mathfrak{Y} \cap L^2_{loc}(\mathbb{R}^+; \mathcal{W})$ and the well-posed past behavior $\mathfrak{Y}^+ := \pi_- \mathfrak{Y}$.
9.2. WELL-POSED BEHAVIORS (Jan 02, 2016) 505

(iii) $\mathfrak{V}^-$ is a well-posed past behavior in $\mathcal{W}$, then by the corresponding well-posed two-sided and future behaviors in $\mathcal{W}$ we mean the well-posed two-sided behavior $\mathfrak{V}$ defined by (9.2.1) and the well-posed future behavior $\mathfrak{V}^+ := \mathfrak{V} \cap L^2_{\text{loc}}(\mathbb{R}^+, \mathcal{W})$.

9.2.9. LEMMA. Let $\mathfrak{V}$ be a well-posed two-sided behavior in $\mathcal{W}$, and let $(\mathcal{U}, \mathcal{Y})$ be an i/o well-posed i/o representation of $\mathcal{W}$ for $\mathfrak{V}$. Then there exists an exponentially bounded causal shift-invariant operator $\mathfrak{D} : L^2_{\text{c}, \text{loc}}(\mathbb{R}; \mathcal{W})$ such that $\mathfrak{V}$ has the graph representation

\[ \mathfrak{V} = \left\{ w \in L^2_{\text{c}, \text{loc}}(\mathbb{R}; \mathcal{W}) \mid P^U \pi \mathfrak{D} \pi^Y w = \mathfrak{D} P^Y w \right\}. \]  

Proof. It follows from condition (c) in (9.2.2) that $\mathfrak{V}$ is the graph of a continuous linear operator $\mathfrak{D} : L^2_{\text{c}, \text{loc}}(\mathbb{R}; \mathcal{U}) \rightarrow L^2_{\text{c}, \text{loc}}(\mathbb{R}; \mathcal{Y})$, and that the restriction of $\mathfrak{D}$ to $L^2_{\text{c}, \text{loc}}(\mathbb{R}; \mathcal{U})$ maps this space continuously into $L^2_{\text{c}, \text{loc}}(\mathbb{R}; \mathcal{Y})$. The shift invariance of $\mathfrak{V}$ implies that $\mathfrak{D}$ commutes with the shift $\tau^t$ for all $t \in \mathbb{R}$, and the second condition in (c) implies that $\pi_{(-\infty, T]} \mathfrak{D} u = 0$ whenever $\pi_{(-\infty, T]} u = 0$. □

9.2.10. DEFINITION. Let $\mathcal{W}$ be a vector space with a Hilbert space topology. By a well-posed realization of a well-posed future behavior $\mathfrak{V}^+$ in $\mathcal{W}$, or a well-posed two-sided behavior $\mathfrak{V}$ in $\mathcal{W}$, or a well-posed past behavior $\mathfrak{V}^-$ in $\mathcal{W}$, we mean a well-posed s/s system $\Sigma = (\mathfrak{V}; \mathcal{X}, \mathcal{W})$ whose future, two-sided, or past behavior is equal to $\mathfrak{V}^+$, $\mathfrak{V}$, and $\mathfrak{V}^-$, respectively.

9.2.11. LEMMA. Let $\mathfrak{V}^+$, $\mathfrak{V}$, and $\mathfrak{V}^-$ be well-posed future, two-sided, and past behaviors in $\mathcal{W}$ that correspond to each other in the sense of Definition 9.2.8, and let $\Sigma = (\mathcal{V}; \mathcal{X}, \mathcal{W})$ be a well-posed s/s system. Then the following conditions are equivalent:

(i) $\Sigma$ is a well-posed realization of $\mathfrak{V}^+$;
(ii) $\Sigma$ is a well-posed realization of $\mathfrak{V}$;
(iii) $\Sigma$ is a well-posed realization of $\mathfrak{V}^-$.

Proof. This follows from Lemmas 9.1.9 and 9.2.7 and Definition 9.2.8. □

9.2.12. THEOREM. Every well-posed future, two-sided, or past behavior has a well-posed s/s realization.

Proof. By Lemma 9.2.11 it suffices to prove that every well-posed two-sided behavior has a well-posed s/s realization. Let $\mathfrak{O}$ be the exponentially bounded causal shift-invariant operator given by Lemma 9.2.9. By Theorem 8.1.40 $\mathfrak{O}$ has a well-posed realization $\Sigma_{i/s/o} = (\mathcal{S}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$. The s/s system $\Sigma$ induced by $\Sigma_{i/s/o}$ is then a well-posed s/s realization of $\mathfrak{V}$. □
9.3. Intertwinements, Compressions and Dilations (Jan 02, 2016)

In this section we discuss intertwinements, compressions, and dilations of well-posed s/s system, as well as the notions of strongly invariant and unobservably invariant subspaces. These notions were defined for arbitrary s/s systems in Chapter 1. See, in particular, Definitions 1.5.21, 1.5.22, 1.5.23, 1.5.8, 1.5.28, 1.5.33, 1.5.37, and 1.5.41.

9.3.1. Strongly invariant and unobservably invariant subspaces. At this point the reader may want to recall the notions “strongly invariant” and “unobservably invariant” introduced in Definition 1.5.8.

9.3.1. Lemma. Let $\Sigma = (V; X, W)$ be a well-posed s/s system with input map $\mathcal{B}$.

(i) The minimal strongly invariant subspace for $\Sigma$ is equal to $\text{rng (B)}$, i.e., $\text{rng (B)}$ is strongly invariant for $\Sigma$, and $\text{rng (B)}$ is contained in every other strongly invariant subspace for $\Sigma$.

(ii) The minimal closed strongly invariant subspace for $\Sigma$ is equal to the reachable subspace $\mathcal{R}$ of $\Sigma$, i.e., $\mathcal{R}$ is closed and strongly invariant for $\Sigma$, and $\mathcal{R}$ is contained in every other closed strongly invariant subspace for $\Sigma$.

(iii) The maximal unobservable invariant subspace for $\Sigma$ is equal to the unobservable subspace $\mathcal{U}$, i.e., $\mathcal{U}$ is unobservable invariant, and $\mathcal{U}$ contains every other unobservable invariant subspace for $\Sigma$.

Proof. This follows from Theorem 8.2.4, Lemma 9.1.14, and Proposition 2.5.49. □

9.3.2. Lemma. Let $\Sigma = (V; X, W)$ be a well-posed s/s system with unobservable evolution semigroup $\mathcal{R}_{\mathcal{X}}$ and unreachable evolution semigroup $\mathcal{A}_{X_0}$ in $X_0$, where $X_0$ is a direct complement to the reachable subspace $\mathcal{R}$.

(i) A closed subspace $Z$ of $X$ is strongly invariant for $\Sigma$ if and only if

\[ \mathcal{R} \subset Z \text{ and } \mathcal{A}_{X_0}^t P_0 Z \subset Z \text{ for all } t \in \mathbb{R}^+. \]

(ii) A subspace $Z$ of $X$ is unobservable invariant for $\Sigma$ if and only if

\[ Z \subset \mathcal{U} \text{ and } \mathcal{A}_{X_0}^t Z \subset Z \text{ for all } t \in \mathbb{R}^+. \]

Proof. (i) Suppose first that $Z$ is closed and strongly invariant. Then by Lemma 9.3.1 $\mathcal{R} \subset Z$, and $\mathcal{R}$ is strongly invariant. Let $[ x_0 \parallel w ]$ be an arbitrary generalized future trajectory of $\Sigma$ with $x(0) \in Z$, and define $x_0 = P_0 x$ and $x_1 = P_0 x$. Then $x = x_0 + x_1$. We have $x_1(t) \in \mathcal{R} \subset Z$ for all $t \in \mathbb{R}^+$, and since $Z$ is strongly invariant also $x(t) \in Z$ for all $t \in \mathbb{R}^+$. This implies that $x_0(t) = x(t) - x_1(t) \in Z$ for all $t \in \mathbb{R}^+$. As this is true for all generalized future trajectories of $[ x_0 \parallel w ]$, it is contained in every other strongly invariant subspace for $\Sigma$.

Conversely, suppose that condition (9.3.1) holds. Let $[ x_0 \parallel w ]$ be an arbitrary generalized future trajectory of $\Sigma$ with $x(0) \in Z$, and define $x_0$ and $x_1$ as above. We again have $x_1(t) \in \mathcal{R} \subset Z$, and also $x_2(t) \in Z$ since $x_2(t) = \mathcal{A}_{X_0}^t P_0 x(0)$. This implies that $Z$ is strongly invariant.

(ii) That (ii) holds follows immediately from Definitions 1.5.8 and 9.1.12. □

9.3.3. Lemma. If $Z$ is a strongly invariant or unobservable invariant subspace for a well-posed s/s system $\Sigma$, then the closure $\overline{Z}$ of $Z$ in $X$ is also strongly invariant respectively unobservable invariant for $\Sigma$. 
9.3.2. Compressions, dilations, and minimality. At this point the reader may want to review the definitions of compressions, dilations, restrictions, and projections given in Definitions 1.5.28, 1.5.33, and 1.5.37.

9.3.4. Lemma. Let \( \Sigma_i = (V_i; X_i, W) \), \( i = 1, 2 \), be two well-posed s/s systems (with the same input and output spaces), where \( X_1 \) is a closed subspace of \( X_2 \), and let \( Z_1 \) be a direct complement to \( X_1 \) in \( X_2 \).

(i) \( \Sigma_1 \) is the restriction of \( \Sigma_2 \) to \( X_1 \) if and only if every generalized future trajectory of \( \Sigma_2 \) is also a generalized future trajectory of \( \Sigma_1 \).

(ii) \( \Sigma_1 \) is the projection of \( \Sigma_2 \) onto \( X_1 \) along \( Z_1 \) if and only if the following condition holds: If \( \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \) is a generalized future trajectory of \( \Sigma_2 \), then \( \begin{bmatrix} p_{x_1}^1 & x_2 \\ u_2 \\ y_2 \end{bmatrix} \) is a generalized future trajectory of \( \Sigma_1 \).

Proof. This follows from Lemma 8.2.16 and Proposition 2.5.50.

The following theorem complements Lemmas 1.5.48 and 1.5.49.

9.3.5. Theorem. Let \( \Sigma_2 = (S_2; X_2, U, Y) \) be a well-posed s/s system, and let \( X_2 = X_1 + Z_1 \) be a direct sum decomposition of \( X_2 \).

(i) If \( X_1 \) is strongly invariant for \( \Sigma_2 \), then the restriction of \( \Sigma_2 \) to \( X_1 \) constructed in Lemma 1.5.36 is well-posed.

(ii) If \( Z_1 \) is unobservably invariant for \( \Sigma_2 \), then \( \Sigma_2 \) has a (unique) well-defined well-posed projection \( \Sigma_1 = (S_1; X_1, U, Y) \) onto \( X_1 \) along \( Z_1 \).

Proof. This follows from Lemma 8.2.18 and Proposition 2.5.49.

9.3.3. Time domain intertwinements of well-posed s/s systems.

9.3.6. Lemma. If the two well-posed s/s systems are intertwined by some multi-valued operator \( P \), then they are also intertwined by the closure of \( P \).

Proof. This follows from Proposition 2.5.50 and Corollary 8.2.8.

9.3.7. Lemma. Let \( \Sigma_i = (V_i; X_i, W) \) be three well-posed s/s systems. If \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by the \( P_1 \in M\mathcal{L}(X_1; X_2) \) and \( \Sigma_2 \) and \( \Sigma_3 \) are intertwined by \( P_2 \in M\mathcal{L}(X_2; X_3) \), then \( \Sigma_1 \) and \( \Sigma_3 \) are intertwined by \( P_3 := P_2 P_1 \in M\mathcal{L}(X_1; X_2) \), and hence also by the closure of \( P_3 \).

Proof. This follows from Lemma 8.2.10 and Proposition 2.5.50.

9.3.8. Theorem. Let \( \Sigma_i = (V_i; X_i, W) \), \( i = 1, 2 \), be two well-posed s/s systems (with the same signal space).

(i) \( \Sigma_1 \) and \( \Sigma_2 \) are intertwined by some \( P \in M\mathcal{L}(X_1; X_2) \) if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent.

(ii) If \( \Sigma_1 \) and \( \Sigma_2 \) are externally equivalent, then there exists a unique minimal intertwining \( P_{\min} \in M\mathcal{L}(X_1; X_2) \), i.e., there exists a unique \( P_{\min} \in \mathcal{M}(X_1; X_2) \) such that \( P_{\min} \subseteq P \) for any other \( P \) which intertwines \( \Sigma_1 \) and \( \Sigma_2 \). The graph of \( P_{\min} \) is given by

\[
\text{gph}(P_{\min}) = \left\{ \begin{bmatrix} \mathcal{B}_{X_{1}z} \\\ \mathcal{B}_{X_{2}z} \end{bmatrix} \middle| w \in \mathcal{Y}^+ \right\},
\]

(9.3.3)
where $\mathcal{B}_{\Sigma_1}$ and $\mathcal{B}_{\Sigma_2}$ are the input maps of $\Sigma_1$ respectively $\Sigma_2$ and $\mathcal{W}$ is the common past behavior of $\Sigma_1$ and $\Sigma_2$. The closure $P_{\text{min}}$ of $P_{\text{min}}$ is the minimal closed multi-valued operator which intertwines $\Sigma_1$ and $\Sigma_2$.

(iii) If $\Sigma_1$ and $\Sigma_2$ are externally equivalent, then there exists a unique maximal intertwining $P_{\max} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$, i.e., there exists a unique $P_{\max} \in \mathcal{ML}(\mathcal{X}_1; \mathcal{X}_2)$ such that $P \subset P_{\max}$ for any other $P$ which intertwines $\Sigma_1$ and $\Sigma_2$. The graph of $P_{\max}$ is given by

\[ \text{gph}(P_{\max}) = \ker \left( [\mathcal{C}_{\Sigma_2} - \mathcal{C}_{\Sigma_1}] \right), \]

where $\mathcal{C}_{\Sigma_i}$ are the output maps of $\Sigma_1$ respectively $\Sigma_2$.

**Proof.** This follows from Theorem 8.2.13 and Lemmas 9.1.14 and 9.1.19, and Proposition 2.5.50.

The definition of similarity of two s/s systems is given in Definition 1.5.23.

**9.3.9 Theorem.** Two controllable and observable well-posed s/s systems $\Sigma_i = (V_i, \mathcal{X}_i, \mathcal{W})$, $i = 1, 2$, are similar if and only if $\Sigma_1$ and $\Sigma_2$ are externally equivalent. Among all the similarities between $\Sigma_1$ and $\Sigma_2$ there is a (unique) minimal one $P_{\text{min}}$, and a (unique) maximal one $P_{\max}$, namely the ones defined in Theorem 9.3.8 (both of which in this case are densely defined injective operators with dense range).

**Proof.** This follows from Theorem 8.2.13 and Proposition 2.5.50.

**9.3.4. The general structure of a compression.**

**9.3.10 Lemma.** Let $\Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{W})$ be a well-posed s/s system, and let $\mathcal{X}_2 = \mathcal{X}_1 + Z_1$. Define

\[ Z_{\text{min}} = \bigcup_{t \in \mathbb{R}^+} \left\{ x_{Z_1}^{\mathcal{X}_1} x_2(t) \left| \begin{array}{c} x_{Z_1} \in X_{Z_1} \text{ is a generalized future trajectory} \\ x_2(t) \in Z_1 \text{ with initial state } x_2(0) \in \mathcal{X}_1 \end{array} \right. \right\}, \]

\[ Z_{\text{max}} = \left\{ x_1(0) \in Z_1 \left| \begin{array}{c} x_2(t) \in Z_1 \text{ for all } t \in \mathbb{R}^+ \\ x_2(t) \text{ is an generalized unobservable} \\ \text{future trajectory of } \Sigma_2 \text{ satisfying} \end{array} \right. \right\}, \]

Then $\mathcal{X}_1 + Z_{\text{min}}$ is the minimal closed strongly invariant subspace for $\Sigma_2$ which contains $\mathcal{X}_1$, and $Z_{\text{max}}$ is the maximal unobservably invariant subspace for $\Sigma_2$ which is contained in $Z_1$.

**Proof.** This follows from Lemma 8.2.19 and Proposition 2.5.49.

**9.3.11 Theorem.** Let $\Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{W})$ be a well-posed s/s system, and let $\mathcal{X}_2 = \mathcal{X}_1 + Z_1$ be a direct sum decomposition of $\mathcal{X}_2$. Then the following conditions are equivalent:

(i) $\Sigma_2$ has a well-posed s/s compression $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{W})$ onto $\mathcal{X}_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z_u$ such that $Z_u$ is unobservably invariant for $\Sigma_2$ and $\mathcal{X}_1 + Z_u$ is strongly invariant for $\Sigma_2$.

(iii) The subspace $Z_{\text{min}}$ defined in (9.3.5) is unobservably invariant for $\Sigma_2$.

(iv) $\mathcal{X} + Z_{\text{max}}$ is strongly invariant for $\Sigma_2$, where $Z_{\text{max}}$ is defined as in (9.3.6).

(v) $Z_{\text{min}} \subset Z_{\text{max}}$.

Two possible choices of the subspace $Z_u$ in (ii) are to take either $Z_u = Z_{\text{min}}$ or $Z_u = Z_{\text{max}}$, and every possible subspace $Z_u$ satisfies $Z_{\text{min}} \subset Z_u \subset Z_{\text{max}}$. 


Clearly, the subspace $Z_u$ is not unique without any further assumptions. For example, as the following theorem shows, every compression can be decomposed into a restriction followed by a projection, and also into a projection followed by a restriction.

**9.3.12. Theorem.** Let $\Sigma_i = (V_i; X_i; W_i)$, $i = 1, 2$ be two well-posed s/s systems with $X_2 = X_1 + Z_1$, and suppose that $\Sigma_1$ is the compression of $\Sigma_2$ onto $X_1$ along $Z_1$. Let $Z_u$ satisfy the conditions listed in (ii) in Theorem [9.3.11] and let $Z_c$ be an arbitrary direct complement to $Z_u$ in $Z_1$.

(i) Let $\Sigma_3$ be the restriction of $\Sigma_2$ to the strongly invariant subspace $X_1 + Z_u$ of $X_2$ given by Theorem [9.3.5](i). Then $Z_u$ is unobservably invariant for $\Sigma_3$, and $\Sigma_1$ is the projection onto $X_1$ along $Z_u$ of $\Sigma_3$.

(ii) Let $\Sigma_4$ be the projection of $\Sigma_2$ onto $X_1 + Z_c$ along $Z_u$ of $X_2$ given by Theorem [9.3.5](ii). Then $X_1$ is strongly invariant for $\Sigma_4$, and $\Sigma_1$ is the restriction to $X_1$ of $\Sigma_4$.

**Proof.** This follows from Theorem [8.2.22] and Proposition [2.5.50].

**9.3.13. Lemma.** Let $\Sigma_i = (V_i; X_i; W_i)$ be two well-posed s/s systems, $i = 1, 2$, with $X_2 = X_1 + Z_1$. Then the following two conditions are equivalent.

(i) $\Sigma_1$ is the compression of $\Sigma_2$ onto $X_1$ along $Z_1$.

(ii) $Z_1$ contains some closed subspace $Z_u$ such that $\Sigma_2$ and $\Sigma_1$ are intertwined by the operator $P = P_{X_1}^{Z_u}$ with $\text{dom}(P) = X_1 + Z_u$.

Condition (ii) above holds for some particular subspace $Z_u$ if and only condition (ii) in Theorem [9.3.11] holds for the same subspace $Z_u$. Thus, in particular, two possible choices of the subspace $Z_u$ in (ii) are to take either $Z_u = Z_{\min}$ or $Z_u = Z_{\max}$ defined in [9.3.5] and [9.3.5], and every possible subspace $Z_u$ satisfies $Z_{\min} \subset Z_u \subset Z_{\max}$.

**Proof.** This follows from Lemma [8.2.23] and Proposition [2.5.50].

**9.3.14. Theorem.** A well-posed i/s/o system $\Sigma = (S; X; U; Y)$ is minimal if and only if $\Sigma$ is both controllable and observable.

**Proof.** This follows from Lemma [8.2.24] and Proposition [2.5.50].
9.4. Frequency Domain Intertwinements, Compressions, and Dilations

9.4.1. The frequency domain input and output maps.

9.4.1. Lemma. Let $\Sigma = (V; X, W)$ be a s/s system with $\rho(\Sigma) \neq \emptyset$.

(i) For each $\lambda \in \rho(\Sigma)$ the (multi-valued) operator $\hat{B}_\Sigma(\lambda) : W \to X$ whose graph is given by

\[
gph(\hat{B}_\Sigma(\lambda)) = \begin{cases} 
\begin{bmatrix} x \\ w \end{bmatrix} & | \begin{bmatrix} x \\ w \end{bmatrix} \in \hat{E}(\lambda) \\
\end{cases}
\]

(9.4.1)

is a single-valued bounded linear operator with closed domain $\text{dom}(\hat{B}_\Sigma(\lambda)) = \hat{F}(\lambda)$.

(ii) If $(U, Y)$ is an i/o representation of $W$ which is frequency domain i/s/o-admissible for $\Sigma$ at the point $\lambda \in \rho(\Sigma)$, and if we denote the corresponding i/s/o resolvent matrix by $[\hat{A} \hat{B} \hat{C} \hat{D}]$, then

(9.4.2) $\hat{B}(\lambda) = \hat{B}_\Sigma(\lambda) \left[ \begin{array}{c} \hat{U} \\
\hat{D}(\lambda) \end{array} \right]$ and $\hat{B}_\Sigma(\lambda) = \hat{B}(\lambda)P^Y_U$.

Proof. Claim (ii) follows from the representation (5.3.8) of $\hat{F}(\lambda)$, and (i) follows from (ii) and Lemma 1.6.2. □

9.4.2. Definition. We call the family $\{\hat{B}_\Sigma(\lambda)\}_{\lambda \in \rho(\Sigma)}$ defined in Lemma 9.4.1 the frequency domain input map of $\Sigma$.

9.4.3. Lemma. Let $\Sigma = (V; X, W)$ be a s/s system with $\rho(\Sigma) \neq \emptyset$.

(i) For each $\lambda \in \rho(\Sigma)$ the (multi-valued) operator $\hat{C}_\Sigma(\lambda) : X \to W$ whose graph is given by

\[
gph(\hat{C}_\Sigma(\lambda)) = \begin{cases} 
\begin{bmatrix} w \\ x_0 \\ x \\ w \end{bmatrix} & | \begin{bmatrix} x_0 \\ x \\ w \end{bmatrix} \in \hat{E}(\lambda) \text{ for some } x \in X \\
\end{cases}
\]

(9.4.3)

is closed, with $\text{dom}(\hat{C}_\Sigma(\lambda)) = X$ and $\text{mul}(\hat{C}_\Sigma(\lambda)) = \hat{F}(\lambda)$.

(ii) If $(U, Y)$ is an i/o representation of $W$ which is frequency domain i/s/o-admissible for $\Sigma$ at the point $\lambda \in \rho(\Sigma)$, then $W$ has the direct sum decomposition $W = \hat{F}(\lambda) + Y$, and the single-valued part of $\hat{C}_\Sigma(\lambda)$ with respect to this decomposition of $W$ is the s/o resolvent $\hat{C}(\lambda)$ of $\Sigma$ at the point $\lambda$ with respect to the coordinate representation $[U \ Y]$ of $W$.

Proof. Claim (ii) follows from the representation (5.3.8) of $\hat{F}(\lambda)$, and (i) follows from (ii) and Lemma 1.6.2. □

9.4.4. Definition. We call the family $\{\hat{C}_\Sigma(\lambda)\}_{\lambda \in \rho(\Sigma)}$ defined in Lemma 9.4.3 the frequency domain output map of $\Sigma$. 
9.4.2. Frequency domain controllability and observability.

9.4.5. Lemma. Let \( \Sigma = (V; X, W) \) be a well-posed s/s system with \( \rho(\Sigma) \neq \emptyset \) with reachable subspace \( R_\Sigma \), unobservable subspace \( U_\Sigma \), and characteristic node bundle \( \hat{\mathcal{E}} \). Denote the (connected) component of \( \rho(\Sigma) \) which contains some right half-plane by \( \rho_\infty(\Sigma) \), and let \( \Omega' \) be an arbitrary subset of \( \rho_\infty(\Sigma) \) which has a cluster point in \( \rho_\infty(\Sigma) \). Then

\[
R_\Sigma = \bigvee_{\lambda \in \rho_\infty(\Sigma)} \left\{ x \in X \mid \begin{bmatrix} 0 \\ x \\ w \end{bmatrix} \in \hat{\mathcal{E}}(\lambda) \text{ for some } w \in W \right\} \\
= \bigvee_{\lambda \in \Omega'} \left\{ x \in X \mid \begin{bmatrix} 0 \\ \lambda x \\ w \end{bmatrix} \in \hat{\mathcal{E}}(\lambda) \text{ for some } w \in W \right\} \\
= \bigvee_{\lambda \in \rho_\infty(\Sigma)} \left\{ x \in X \mid \begin{bmatrix} \lambda x \\ w \end{bmatrix} \in V \text{ for some } w \in W \right\}
\]

(9.4.4)

\[
U_\Sigma = \bigwedge_{\lambda \in \rho_\infty(\Sigma)} \left\{ z \in X \mid \begin{bmatrix} z \\ -z + \lambda x \end{bmatrix} \in V \text{ for some } x \in \mathcal{X} \right\} \\
= \bigwedge_{\lambda \in \Omega'} \left\{ z \in X \mid \begin{bmatrix} z \\ -z + \lambda x \end{bmatrix} \in V \text{ for some } x \in \mathcal{X} \right\} \\
= \bigwedge_{\lambda \in \rho_\infty(\Sigma)} \left\{ z \in X \mid \begin{bmatrix} z \\ -z + \lambda x \end{bmatrix} \in V \text{ for some } x \in \mathcal{X} \right\} \\
= \bigwedge_{\lambda \in \Omega'} \left\{ z \in X \mid \begin{bmatrix} z \\ -z + \lambda x \end{bmatrix} \in V \text{ for some } x \in \mathcal{X} \right\}.
\]

(9.4.5)

Proof. Still missing. \( \square \)

9.4.3. Compressions into minimal well-posed s/s systems (Jan 02, 2016). Still to be written.
9.5. Stable state/signal systems (Jan 02, 2016)

9.5.1. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a well-posed s/s system.

(i) \( \Sigma \) is \( \alpha \)-bounded if \( \Sigma \) has some \( \alpha \)-bounded realization.

(ii) \( \Sigma \) is stable if \( \Sigma \) has a stable i/s/o representation.

(iii) The growth bound of \( \Sigma \) is the infimum of the growth bounds of all i/s/o representations of \( \Sigma \).

(iv) \( \Sigma \) is exponentially stable if the growth bound of \( \Sigma \) is (strictly) negative.

Note that a s/s system with growth bound zero need not be stable.

9.5.2. Theorem. A s/s system \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) is stable if and only if \( \Sigma \) is solvable and there exists an i/o representation \( (\mathcal{W}) \) of \( \mathcal{W} \) such that the following two conditions hold:

(i) The set of all \( u \in C_0(\mathbb{R}^+; \mathcal{U}) \) such that \( u = P^w_\Delta \) for some classical externally generated trajectory \( [\frac{x}{w}] \) of \( \Sigma_{i/o} \) is dense in \( L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U}) \).

(ii) There exist a constant \( K > 0 \) such that all classical future trajectories \( [\frac{x}{w}] \) of \( \Sigma_{i/o} \) satisfy

\[
\|x(t)\|^2_X + \|P^w_r(u)\|^2_{L^2([0,T]; \mathcal{Y})} \leq K \left( \|x(0)\|^2_X + \|P^w_r(u)\|^2_{L^2([0,T]; \mathcal{Y})} \right), \quad T > 0.
\]

Proof. Later.

9.5.3. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a stable s/s system. \(^2\)

(i) A generalized trajectory \( [\frac{x}{w}] \) of \( \Sigma \) on some interval \( I \) is called stable if \( x \) is bounded on \( I \) and \( w \in L^2(I; \mathcal{W}) \).

(ii) A generalized past or two-sided stable trajectory \( [\frac{x}{w}] \) is called externally generated if \( x(t) \to 0 \) as \( t \to -\infty \).

9.5.4. Definition. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a stable s/s system. \(^3\)

(i) By the stable future behavior of \( \Sigma \) we mean the set of all \( w \in L^2(\mathbb{R}^+; \mathcal{W}) \) such that \( [\frac{x}{w}] \) is a generalized externally generated future trajectory of \( \Sigma \) for some \( x \in C_0(\mathbb{R}^+; \mathcal{X}) \). The stable future behavior of \( \Sigma \) is denoted by \( \mathfrak{M}_\Sigma^+ \).

(ii) By the stable past behavior of \( \Sigma \) we mean the set of all \( w \in L^2(\mathbb{R}^-; \mathcal{W}) \) such that \( [\frac{x}{w}] \) is a generalized past trajectory of \( \Sigma \) for some \( x \in C(\mathbb{R}^-; \mathcal{X}) \) satisfying \( x(t) \to 0 \) as \( t \to -\infty \). The stable past behavior of \( \Sigma \) is denoted by \( \mathfrak{M}_\Sigma^- \).

(iii) By the stable two-sided behavior of \( \Sigma \) we mean the set of all \( w \in L^2(\mathbb{R}; \mathcal{W}) \) such that \( [\frac{x}{w}] \) is a generalized two-sided trajectory of \( \Sigma \) for some \( x \in C(\mathbb{R}; \mathcal{X}) \) satisfying \( x(t) \to 0 \) as \( t \to -\infty \). The two-sided behavior of \( \Sigma \) is denoted by \( \mathfrak{M}_\Sigma \).

9.5.5. Lemma. Let \( \Sigma = (V; \mathcal{X}, \mathcal{W}) \) be a stable s/s system.

(i) The stable future behavior of \( \Sigma \) is dense in the future behavior of \( \Sigma \).

(ii) The past behavior of \( \Sigma \) is dense in the stable past behavior of \( \Sigma \).

Proof. Later. \(^4\)

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\(^2\)This definition can formally be applied also to s/s systems which are not stable, but we are not able to say anything significant about these notions unless \( \Sigma \) is stable.

\(^3\)This definition can formally be applied also to s/s systems which are not stable, but we are only able to say anything significant about these notions unless \( \Sigma \) is stable.

\(^4\)This definition can formally be applied also to s/s systems which are not stable, but we are only able to say anything significant about these notions unless \( \Sigma \) is stable.
9.5.6. Lemma. Let $\Sigma = (V; X, W)$ be a stable s/s system.

(i) The input map of $\Sigma$ can be extended to a bounded linear map from the stable past behavior of $\Sigma$ to $X$.

(ii) The output map of $\Sigma$ can be reinterpreted as a closed multi-valued operator in $|\text{MUL}(X; L^2(\mathbb{R}^+; W))|$ whose multi-valued part is equal to the stable future behavior of $\Sigma$.

(iii) The past/future map can also be interpreted as a multi-valued operator from the stable past behavior of $\Sigma$ to $L^2(\mathbb{R}^+; W)$.

Proof. Reformulate this result and prove it properly. \qed
APPENDIX A

Appendix

In this appendix we present a number of results in $H$-spaces and Kreĭn spaces.
A.1. Using more than one norm in a vector space.

A.1.1. Definition. Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms in a vector space $\mathcal{X}$.

(i) We say that $\|\cdot\|_1$ is weaker than $\|\cdot\|_0$, or equivalently, the $\|\cdot\|_0$ is stronger than $\|\cdot\|_1$, if there exists a positive constant $C$ such that
\[
\|x\|_1 \leq C\|x\|_0, \quad x \in \mathcal{X}.
\]

(ii) The two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent if $\|\cdot\|_1$ is weaker than $\|\cdot\|_0$ and $\|\cdot\|_0$ is weaker than $\|\cdot\|_1$, i.e., if there exists positive constants $c$ and $C$ such that
\[
c\|x\|_0 \leq \|x\|_1 \leq C\|x\|_0, \quad x \in \mathcal{X}.
\]

A.1.2. Lemma. Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms in a vector space $\mathcal{X}$.

(i) The following conditions are equivalent:
   (a) The norm $\|\cdot\|_1$ is weaker than the norm $\|\cdot\|_0$,
   (b) Every sequence $x_n$ in $\mathcal{X}$ which converges with respect to the norm $\|\cdot\|_0$ also converges with respect to the norm $\|\cdot\|_1$.
   (c) Every subset of $\mathcal{X}$ which is open with respect to the norm $\|\cdot\|_1$ is also open with respect to the norm $\|\cdot\|_0$.

(ii) The following conditions are equivalent:
   (a) The norms $\|\cdot\|_1$ and $\|\cdot\|_0$ are equivalent.
   (b) A sequence $x_n$ in $\mathcal{X}$ which converges with respect to the norm $\|\cdot\|_0$ if and only if it converges with respect to the norm $\|\cdot\|_1$.
   (c) A subset of $\mathcal{X}$ is open with respect to the norm $\|\cdot\|_1$ if and only if it is open with respect to the norm $\|\cdot\|_0$.

Proof. Since (ii) follows from (i), it suffices to prove (i). That (a) and (c) are equivalent follows from Definition A.1.1 and the definitions of what one means by an open set in a normed space, and that (a) $\Rightarrow$ (b) follows from the definition of a converging sequence in a normed space. Conversely, suppose that (a) does not hold. Then there exists a sequence $x_n$ in $\mathcal{X}$ such that $\|x_n\|_1 = 1$ and $\|x_n\|_0 = 1/n, \quad n = 1, 2, \cdots$. Thus $x_n \to 0$ in the norm $\|\cdot\|_0$ but not in the norm $\|\cdot\|_1$, and hence (b) does not hold.

A.1.3. Lemma. Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two Banach space norms in a vector space $\mathcal{X}$. If $\|\cdot\|_0$ is weaker or stronger than $\|\cdot\|_1$, then $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent.

Proof. Suppose, for example, that $\|\cdot\|_0$ is stronger than $\|\cdot\|_1$. Then the identity map from $\mathcal{X}$ equipped with the Banach space norm $\|\cdot\|_0$ to $\mathcal{X}$ equipped with the norm $\|\cdot\|_1$ is continuous. By the closed graph theorem, the inverse of this map is also continuous, and thus the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent.

A.1.4. Lemma. Let $(\cdot, \cdot)_0$ be a Hilbert space inner product in the vector space $\mathcal{X}$.

(i) If $(\cdot, \cdot)_1$ is another Hilbert space inner product in $\mathcal{X}$ with the property that the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ induced by $(\cdot, \cdot)_0$ respectively $(\cdot, \cdot)_1$ are equivalent, then there exists a unique bicontinuous bijection $G$ in $\mathcal{X}$ (with respect to
either of the two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ such that
\begin{align}
(A.1.3) \quad (x_1, x_2)_1 &= (x_1, Gx_2)_0, \quad x_1, x_2 \in \mathcal{X}, \\
(A.1.4) \quad (x_1, x_2)_0 &= (x_1, G^{-1}x_2)_1, \quad x_1, x_2 \in \mathcal{X}.
\end{align}

The operator $G$ is self-adjoint and positive with respect to both $[\cdot, \cdot]_0$ and $[\cdot, \cdot]_1$, i.e.,
\begin{align}
(A.1.5) \quad (x_1, Gx_2)_i &= (Gx_1, x_2)_i \geq 0, \quad x_1, x_2 \in \mathcal{X}, \quad i = 1, 2.
\end{align}

(ii) Conversely, suppose that $G$ is a bicontinuous bijection in $\mathcal{X}$ with respect to the norm $\|\cdot\|_0$ induced by $[\cdot, \cdot]_0$, and that $G$ which is self-adjoint and positive operator with respect to the inner product $[\cdot, \cdot]_0$. Define $[\cdot, \cdot]_1$ by $A.1.3$, then $(\cdot, \cdot)_1$ is a Hilbert space inner product in $\mathcal{X}$ with the property that the norm $\|\cdot\|_1$ induced by $(\cdot, \cdot)_1$ is equivalent to $\|\cdot\|_0$.

Proof. (i) Let us denote the space $\mathcal{X}$ equipped with the inner product $(\cdot, \cdot)_0$ by $\mathcal{X}_0$. The inner product $(\cdot, \cdot)_1$ defines a bounded sesqui-linear functional on $\mathcal{X}_0$, and hence there exists a bounded linear operator $G$ in $\mathcal{X}_0$ such that (A.1.3) holds (see, e.g., [Rudin, 1973, Theorem 12.8]). Since both $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_1$ is a Hilbert space inner product, the operator $G$ must be self-adjoint and positive in $\mathcal{X}_0$, and since the norms induced by these two inner products are assumed to be equivalent, $G$ is surjective and has a bounded inverse. Formula (A.1.4) follows from (A.1.3) by replacing $x_1$ and $x_2$ in (A.1.3) by $G^{-1}x_2$ respectively $G^{-1}x_1$. The $G^{-1}$, and hence also $G$, is self-adjoint also with respect to the inner product $(\cdot, \cdot)_1$ follows from (A.1.4).

(ii) The easy proof of (ii) is left to the reader. \qed

A.1.5. Lemma. Let $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_1$ be two Hilbert space inner product in the vector space $\mathcal{X}$. Denote the norms induced by $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_1$ by $\|\cdot\|_0$ respectively $\|\cdot\|_1$. Then the following conditions are equivalent:
\begin{itemize}
  \item[(i)] The norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are equivalent.
  \item[(ii)] There exists a continuous bijection $R$ from $\mathcal{X}$ equipped with the norm $\|\cdot\|_0$ to $\mathcal{X}$ itself such that
\begin{align}
(A.1.6) \quad \|x\|_1 &= \|Rx\|_0, \quad x \in \mathcal{X}.
\end{align}
\end{itemize}

Proof. If (i) holds, then we can take $R$ in (A.1.6) to be equal to $R = G^{1/2}$, where $G$ is the operator in (A.1.3), and if (ii) holds, then we can take $G$ in (A.1.3) to be equal to $G = R^*R$ (where the adjoint has been computed with respect to the inner product $(\cdot, \cdot)_0$). \qed
A.1.2. Introduction to H-spaces.

A.1.6. Definition. (i) By a H-space \( X \) we mean a topological vector space which is isomorphic to a Hilbert space, i.e., the topology of \( X \) is induced by a norm that arises from some Hilbert space inner product in \( X \).

(ii) By an admissible norm for the H-space \( X \) we mean any (Banach space) norm in \( X \) which is compatible with the topology of \( X \).

(iii) By an admissible Hilbert space inner product for the H-space \( X \) we mean any Hilbert space inner product in \( X \) with the property that the norm induced by this inner product is an admissible norm in \( X \).

A.1.7. Remark. We shall frequently apply results from the literature which have been formulated and proved in a Hilbert space setting, also in the case where the spaces that we are working with are H-spaces instead of a Hilbert spaces. In this case one is simply supposed to first fix some arbitrary admissible Hilbert space inner product in each of our H-space, and then apply the appropriate Hilbert space result cited in the literature. We do this only in the case when the results that we cite are of a topological nature, i.e., they do not depend on the particular admissible inner products that we fix, so that they are valid for all possible choices of admissible Hilbert space inner products.

The following three lemmas follow directly from Definition A.1.6 and the fact that the corresponding results are true in Hilbert spaces.

A.1.8. Lemma. A closed subspace \( Z \) of a H-space \( X \) with the topology inherited from \( X \) is an H-space. Moreover, the restriction to \( Z \) of any admissible norm or Hilbert space inner product in \( X \) is an admissible norm or Hilbert space inner product in \( Z \).

A.1.9. Lemma. Every closed subspace \( Z \) of a H-space \( X \) is complemented, i.e., there exists another closed subspace \( Y \) of \( X \) such that \( X = Z \oplus Y \).

A.1.10. Lemma. The product \( [X, Y] \) of two H-space \( X \) and \( Y \) is an H-space. Moreover, if \((\cdot, \cdot)_X\) and \((\cdot, \cdot)_Y\) are arbitrary admissible Hilbert space inner products in \( X \) respectively \( Y \), then the inner product

\[
(A.1.7) \quad \left( \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} \right) = (x_1, x_2)_X + (y_1, y_2)_Y,
\]

\[ x_1, \ y_1, \ x_2, \ y_2 \in [X, Y]. \]

is an admissible Hilbert space inner product in \( [X, Y] \).

A.1.11. Lemma. Let \( X \) be an H-space, let \((\cdot, \cdot)_0\) be an admissible Hilbert space inner product in \( X \), and let \((\cdot, \cdot)_1\) be another (not necessarily admissible) Hilbert space inner product in \( X \). Then \((\cdot, \cdot)_1\) is an admissible Hilbert space inner product in \( X \) if and only if the norms \( \|\cdot\|_0 \) and \( \|\cdot\|_1 \) induced by \((\cdot, \cdot)_0\) respectively \((\cdot, \cdot)_1\) are equivalent.

Proof. This is true because \( \|\cdot\|_1 \) induces the same topology as \( \|\cdot\|_0 \) if and only if the two norms are equivalent. \( \square \)

A.1.12. Lemma. Let \( X \) be an H-space, and let \( X = U + Y \) be a direct sum decomposition of \( X \). Then there exists an admissible Hilbert space inner product in \( X \) with the property that \( U \) and \( Y \) are orthogonal with respect to this inner product.
Proof. Let $(\cdot, \cdot)_0$ be an admissible Hilbert space inner product in $\mathcal{X}$. Define

$$ (x_1, x_2)_1 = (P^Y_\mathcal{U} x_1, P^Y_\mathcal{U} x_2)_0 + (P^U_\mathcal{Y} x_1, P^U_\mathcal{Y} x_2)_0, \quad x_1, x_2 \in \mathcal{X}. $$

This is an inner product in $\mathcal{X}$, and it can be rewritten in the equivalent form (where the adjoints have been computed with respect to the inner product $(\cdot, \cdot)_0$)

$$ (x_1, x_2)_1 = (x_1, (P^Y_\mathcal{U})^* P^Y_\mathcal{U} x_2)_0 + (x_1, (P^U_\mathcal{Y})^* P^U_\mathcal{Y} x_2)_0 = (x_1, ((P^Y_\mathcal{U})^* P^Y_\mathcal{U} + (P^U_\mathcal{Y})^* P^U_\mathcal{Y}) x_2)_0, \quad x_1, x_2 \in \mathcal{X}. $$

Let $G = (P^Y_\mathcal{U})^* P^Y_\mathcal{U} + (P^U_\mathcal{Y})^* P^U_\mathcal{Y}$. Then $G$ is self-adjoint and positive. It is also surjective, because $(x, Gx)_0 = 0$ if and only if

$$ (x, ((P^Y_\mathcal{U})^* P^Y_\mathcal{U} + (P^U_\mathcal{Y})^* P^U_\mathcal{Y}) x)_0 = \|P^Y_\mathcal{U} x\|_0^2 + \|P^U_\mathcal{Y} x\|_0^2 = 0. $$

By the closed graph theorem, $G^{-1}$ is continuous. By Lemma A.1.4, $(x_1, x_2)_1$ is an admissible Hilbert space inner product in $\mathcal{X}$. Clearly, $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal to each other with respect to this inner product. □
A.1.3. Operators in H-spaces. The following two lemmas follow immediately from Definition A.1.6 and the fact that the corresponding results are true for Hilbert spaces.

A.1.13. Lemma. Let $A$ be a linear operator mapping one H-space $X$ into another H-space $Y$, with $\text{dom}(A) = X$. Then the following conditions are equivalent:

(i) $A$ is continuous, i.e., the inverse image of every open set in $Y$ is open in $X$.

(ii) $A$ is bounded, i.e., the image under $A$ of any bounded set in $X$ is bounded in $Y$.

(iii) For each admissible norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ in $X$ respectively there exists a positive constant $M$ such that

$$\|Ax\|_Y \leq M\|x\|_X, \quad x \in X.$$

Proof. This follows immediately from Definition A.1.6 and the fact that the corresponding results are true for Hilbert spaces. □

A.1.14. Notation. If $X$ and $Y$ are Hilbert spaces or H-spaces, then we denote the set of all continuous linear operators from $X$ into $Y$ by $B(X; Y)$, and write $B(X)$ in the case where $X = Y$.

A.1.15. Remark. If $X$ and $Y$ are Hilbert spaces, then $B(X; Y)$ can be interpreted as a Banach space in the standard way where addition and scalar multiplication are defined pointwise and the norm is the standard operator norm. If instead $X$ and $Y$ are H-spaces, then $B(X; Y)$ can be interpreted as a topological vector space whose topology is induced by a Banach space norm (i.e., a complete normable topological vector space).

A.1.16. Lemma (Closed Graph Lemma). Let $A$ be a linear operator mapping one H-space $X$ into another H-space $Y$ with $\text{dom}(A) = X$. Then the following conditions are equivalent:

(i) $A$ is continuous,

(ii) $A$ is closed, i.e., the graph of $A$ is closed in $[Y, X]$.

If, in addition, $A$ is injective and $\text{rng}(A) = Y$, then $A^{-1}$ is continuous from $Y$ into $X$.

A.1.17. Lemma. Let $A \in B(X; Y)$, where $X$ and $Y$ are H-spaces, and let $Z$ be an H-space which is continuously contained in $Y$. Define the operator $B : X \to Z$ by

$$\text{dom}(B) = \{x \in X \mid Ax \in Z\}.$$

Then $B$ is closed.

Proof. The easy proof is left to the reader □

A.1.18. Lemma. Let $A \in B(X; Y)$, where $X$ and $Y$ are H-spaces, and let $Z$ be an H-space which is continuously contained in $Y$. If $\text{rng}(A) \subset Z$, then $A \in B(X; Z)$.

Proof. It follows from Lemma A.1.17 that the operator $A$ is closed as an operator $X \to Z$. In addition, the domain of this operator is $X$. Therefore by Lemma A.1.16 $A \in B(X; Z)$. □
A.1.4. The graph norm and graph topology.

A.1.19. Definition. Let $A$ be a closed linear operator mapping a Hilbert space $\mathcal{X}$ into a Hilbert space $\mathcal{Y}$. By the graph norm induced by $A$ on $\text{dom}(A) \subset \mathcal{X}$ we mean the norm

$$
\|x\|_{\text{dom}(A)} = \left(\|x\|_\mathcal{X}^2 + \|Ax\|_\mathcal{Y}^2\right)^{1/2}.
$$

(A.1.9)

A.1.20. Remark. The reason why the norm in (i) is called the graph norm is the following. When $A$ maps a Hilbert space $\mathcal{X}$ into a Hilbert space $\mathcal{Y}$, then one usually interprets the graph of

$$
gph(A) = \left\{ \begin{bmatrix} y \\ x \end{bmatrix} \in \mathcal{Y} \times \mathcal{X} \mid y = Ax \right\},
$$

as a subspace of the Hilbert space $\mathcal{Y} \oplus \mathcal{U}$ that one gets by equipping the product space $[\mathcal{Y} \times \mathcal{X}]$ with the inner product

$$
\left\langle \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ x_2 \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}} = (y_1, y_2)_\mathcal{Y} + (x_1, x_2)_\mathcal{X},
$$

(A.1.11)

By definition, $gph(A)$ is closed in $[\mathcal{Y} \times \mathcal{X}]$, or equivalently, closed in $\mathcal{Y} \oplus \mathcal{U}$, if and only if $A$ is a closed operator. In this case $gph(A)$ is a Hilbert space with the norm inherited from $\mathcal{Y} \oplus \mathcal{U}$. The operator $x \mapsto [\begin{bmatrix} Ax \\ x \end{bmatrix}]$ maps $\text{dom}(A)$ one-to-one onto $gph(A)$, and its inverse $[\begin{bmatrix} 0 \\ 1 \end{bmatrix}]_{gph(A)}$ maps $gph(A)$ one-to-one onto $\text{dom}(A)$. The norm in (A.1.9) has been chosen in such a way that these two maps become unitary, i.e., $\|A\|_{\text{dom}(A)} = \|[\begin{bmatrix} Ax \\ x \end{bmatrix}]\|_{gph(A)}$.

A.1.21. Lemma. Let $A$ be a closed operator mapping a Hilbert space $\mathcal{X}$ into a Hilbert space $\mathcal{Y}$.

(i) The graph norm defined in (A.1.9) is a Hilbert space norm in $\text{dom}(A)$. The corresponding Hilbert space inner product in $\text{dom}(A)$ is given by

$$
\langle x_1, x_2 \rangle_{\text{dom}(A)} = (x_1, x_2)_\mathcal{X} + (Ax_1, Ax_2)_\mathcal{Y}, \quad x_1, x_2 \in \mathcal{X}.
$$

(A.1.12)

(ii) The graph norm defined in (A.1.9) is the weakest norm in $\text{dom}(A)$ that guarantees the continuity of both the operator $A : \text{dom}(A) \to \mathcal{Y}$ and the embedding map $\text{dom}(A) \hookrightarrow \mathcal{X}$.

Proof. (i) This follows from Remark A.1.20 (an unitary image of a Hilbert space is always a Hilbert space).

(ii) Clearly, both the operator $A : \text{dom}(A) \to \mathcal{Y}$ and the embedding map $\text{dom}(A) \hookrightarrow \mathcal{X}$ are continuous when $\text{dom}(A)$ is equipped with the graph norm. On the other hand, if both the operator $A : \text{dom}(A) \to \mathcal{Y}$ and the embedding map $\text{dom}(A) \hookrightarrow \mathcal{X}$ are continuous with respect to some norm $\| \cdot \|_\mathcal{X}$ on $\text{dom}(A)$, then there exists a positive constant $C$ such that

$$
\|Ax\|^2_\mathcal{Y} + \|x\|^2_\mathcal{X} \leq C^2 \|x\|^2_1,
$$

i.e., $\|x\|_{\text{dom}(A)} \leq C \|x\|_1$. This means that the norm $\| \cdot \|_1$ cannot be weaker than the norm $\| \cdot \|_{\text{dom}(A)}$. □

A.1.22. Lemma. Let $A$ be a closed linear operator mapping an $H$-space $\mathcal{X}$ into a $H$-space $\mathcal{Y}$. If we fix some arbitrary admissible norms $\| \cdot \|_\mathcal{X}$ and $\| \cdot \|_\mathcal{Y}$ $\mathcal{X}$ respectively $\mathcal{Y}$, then the topology in $\text{dom}(A)$ induced by the norm $\| \cdot \|_{\text{dom}(A)}$ defined in (A.1.9) does not depend on the choice of admissible norms in $\mathcal{X}$ and $\mathcal{Y}$.
Proof. This is true because any two admissible norms in $\mathcal{X}$ or $\mathcal{Y}$ are equivalent to each other. □

A.1.23. Definition. Let $A$ be a closed linear operator mapping an $H$-space $\mathcal{X}$ into a $H$-space $\mathcal{Y}$. By the graph topology induced on $\text{dom}(A) \subset \mathcal{X}$ by $A$ we mean the topology induced by (A.1.9), where $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ are arbitrary admissible Hilbert space inner products in $\mathcal{X}$ respectively $\mathcal{Y}$.

A.1.24. Lemma. Let $A$ be a closed operator mapping an $H$-space $\mathcal{X}$ into an $H$-space $\mathcal{Y}$. Then the graph topology in $\text{dom}(A)$ induced by $A$ is the weakest $H$-space topology in $\text{dom}(A)$ that guarantees the continuity of both the operator $A: \text{dom}(A) \rightarrow \mathcal{Y}$ and the embedding map $\text{dom}(A) \hookrightarrow \mathcal{X}$.

Proof. This follows from Lemma [A.1.21] □
A.2. Kreĭn Spaces (Jan 09, 2016)

A.2.1. Introduction to Kreĭn spaces. We expect the reader to be familiar with the theory of Hilbert spaces, but not necessarily with the theory of Kreĭn spaces. We therefore review the main Kreĭn space notions and results that we shall need in this book.

A.2.1. Definition. A Kreĭn space \( \mathcal{W} \) is a vector space with an inner product \([\cdot, \cdot]_\mathcal{W}\) that satisfies all the standard properties required by an inner product, except for the condition \([w, w]_\mathcal{W} \geq 0\). In addition, it is required that the space \( \mathcal{W} \) can be decomposed into a direct sum \( \mathcal{W} = \mathcal{U} \oplus -\mathcal{Y} \), such that the following conditions are satisfied:

(i) \( \mathcal{U} \) and \( -\mathcal{Y} \) are orthogonal to each other with respect to the inner product \([\cdot, \cdot]_\mathcal{W}\), i.e., \([y, u]_\mathcal{W} = 0\) for all \( u \in \mathcal{U} \) and all \( y \in -\mathcal{Y} \).

(ii) \( \mathcal{U} \) is a Hilbert space with the inner product \((u, u')_\mathcal{U} := [u, u']_\mathcal{W}, u, u' \in \mathcal{U}\), inherited from \( \mathcal{W} \).

(iii) \( -\mathcal{Y} \) is an anti-Hilbert space with the inner product \([y, y']_{-\mathcal{Y}} := [y, y']_\mathcal{W}, y, y' \in -\mathcal{Y}\), inherited from \( \mathcal{W} \).

A (definite or indefinite) inner product with the above properties is called a Kreĭn inner product.

Here and later we shall use the notation \( -\mathcal{Y} \) for the so called anti-space of a vector space \( \mathcal{Y} \). In this construction the space \( \mathcal{Y} \) is supposed to be equipped with some Kreĭn inner product \((\cdot, \cdot)_\mathcal{Y}\). The space \( -\mathcal{Y} \) is the same vector space, but it has been equipped with a different inner product, namely

\([y_1, y_2]_{-\mathcal{Y}} := -[y_1, y_2]_\mathcal{Y}, y_1, y_2 \in \mathcal{Y}\).

The condition that \( -\mathcal{Y} \) is an anti-Hilbert space with the inner product inherited from \( \mathcal{W} \) is equivalent to saying that \( \mathcal{Y} \) is a Hilbert space with the inner product

\((y_1, y_2)_\mathcal{Y} := -[y_1, y_2]_{-\mathcal{Y}} = -[y_1, y_2]_\mathcal{W}, y_1, y_2 \in \mathcal{Y}\),

inherited from \( -\mathcal{W} \). Since \( \mathcal{Y} \) and \( \mathcal{U} \) are orthogonal to each other we shall denote the direct sum by \( \mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y} \).

Any decomposition \( \mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y} \) with the properties listed above is called a fundamental decomposition of \( \mathcal{W} \). If the space \( \mathcal{W} \) itself is neither a Hilbert space nor an anti-Hilbert space, then it has infinite many fundamental decompositions. If \( \mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y} \) is a fundamental decomposition of \( \mathcal{W} \), then

\[ [w, w]_\mathcal{W} = \|u\|^2_\mathcal{U} - \|I_{(-\mathcal{Y}, \mathcal{Y})} y\|^2_\mathcal{Y}, w = u + y, u \in \mathcal{U}, y \in -\mathcal{Y}, \]

The dimensions of the positive space \( \mathcal{U} \) and the negative space \( -\mathcal{Y} \) do not depend on the particular decomposition. These dimensions are called the positive respectively negative indices of \( \mathcal{W} \), and they are denoted by \( \text{ind}_+ \mathcal{W} \) and \( \text{ind}_- \mathcal{W} \).

An arbitrary choice of fundamental decomposition \( \mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y} \) determines an admissible norm on \( \mathcal{W} \) by

\[ \|w\|^2_{\mathcal{U} \oplus \mathcal{Y}} = \|u\|^2_\mathcal{U} + \|I_{(-\mathcal{Y}, \mathcal{Y})} y\|^2_\mathcal{Y}, w = u + y, u \in \mathcal{U}, y \in -\mathcal{Y}. \]

Thus, we may interpret \( \mathcal{W} \) as an \( H \)-space for which the above norm is admissible. While the norm \( \|\cdot\|_{\mathcal{U} \oplus \mathcal{Y}} \) defined above depends on the choice of fundamental decomposition \( \mathcal{W} = \mathcal{U} \boxplus -\mathcal{Y} \) for \( \mathcal{W} \), all these norms are equivalent, so they all define the same \( H \)-space.
A subspace \( Z \) of a Kreın space \( W \) with inner product \([\cdot,\cdot]_W\) is called \textit{positive}, \textit{nonnegative}, \textit{negative}, \textit{nonpositive}, or \textit{neutral} if every nonzero vector \( z \in Z \) is positive (i.e., \([z,z]_W > 0\)), nonnegative (i.e., \([z,z]_W \geq 0\)), negative (i.e., \([z,z]_W < 0\)), nonpositive (i.e., \([z,z]_W \leq 0\)), or neutral (i.e., \([z,z]_W = 0\)). A nonnegative or nonpositive subspace \( Z \) is called \textit{maximal nonnegative} or \textit{maximal nonpositive} if it is not properly contained in any other nonnegative respectively nonpositive subspace. Every nonnegative subspace is contained in some maximally nonnegative subspace, and every nonpositive subspace is contained in some maximally nonpositive subspace. Maximal nonnegative or nonpositive subspaces are always closed.

The \textit{orthogonal companion} \( Z^\perp \) of an arbitrary subset \( Z \subset W \) with respect to the Kreın inner product \([\cdot,\cdot]_W\) consists of all vectors in \( W \) that are orthogonal to all vectors in \( Z \), i.e.,

\[
Z^\perp = \{ w' \in W | [w',w]_W = 0 \text{ for all } w \in Z \}.
\]

This is always a closed subspace of \( W \), and \( Z = (Z^\perp)^\perp \) if and only if \( Z \) is a closed subspace. If \( W \) is a Hilbert space, then we write \( Z^\perp \) instead of \( Z^\perp \). Note that, by definition, a subspace \( Z \) is neutral if and only if \( Z \subset Z^\perp \). If instead \( Z^\perp \subset Z \), then \( Z \) is called co-neutral. Finally, if \( Z = Z^\perp \), then \( Z \) is called \textit{Lagrangian}.

The fundamental decompositions that we have considered above are a special case of \textit{orthogonal decompositions} \( W = U \oplus Y \) of \( W \), where \( U \) and \( Y \) are orthogonal with respect to \([\cdot,\cdot]_W\), and both \( U \) and \( Y \) are Kreın spaces with the inner products inherited from \( W \) respectively \(-W \). Thus, if \( w = u + y \) with \( u \in U \) and \( y \in Y \), then

\[
[w,w]_W = [u,u]_W + [y,y]_W = [u,u]_U + [I(-Y,Y)y,I(-Y,Y)y]_Y.
\]

This orthogonal decomposition is fundamental if and only if \( U \) and \( Y \) are Hilbert spaces, i.e., if they are both positive.

A direct sum decomposition \( W = U + Y \) of \( W \) where both \( Y \) and \( U \) are neutral is called a \textit{Lagrangian decomposition} of \( W \). The subspaces \( Y \) and \( U \) are automatically Lagrangian in this case. Such a decomposition exists if and only if \( \text{ind}_+ W = \text{ind}_- W \) (this index may be finite or infinite).
A.2.2. Representations of Kreĭn space inner products. The following lemma clarifies the connection between the Kreĭn space inner product in $\mathcal{W}$ and an admissible Hilbert space inner product in $\mathcal{W}$.

A.2.2. Lemma. 

(i) Let $\mathcal{W}$ be a Kreĭn space $\mathcal{W}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, and let $(\cdot, \cdot)_{\mathcal{W}}$ be an admissible Hilbert space inner product in $\mathcal{W}$. Then there exists a unique operator $J \in B(\mathcal{W})$ such that

\begin{align}
[w_1, w_2]_{\mathcal{W}} &= (w_1, Jw_2)_{\mathcal{W}}, \quad w_1, w_2 \in \mathcal{W}, \\
(w_1, w_2)_{\mathcal{W}} &= [w_1, Jw_2]_{\mathcal{W}}, \quad w_1, w_2 \in \mathcal{W}.
\end{align}

The operator $J$ is both unitary and self-adjoint with respect to both $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ and $(\cdot, \cdot)_{\mathcal{W}}$. The fundamental decomposition $\mathcal{W} = U \oplus \mathcal{Y}$ on which the definition of an admissible norm is based is uniquely determined by the admissible norm, which follows from the fact that

\begin{align}
P_U &= \frac{1}{2}(1_{\mathcal{W}} + J), \quad P_{-\mathcal{Y}} = \frac{1}{2}(1_{\mathcal{W}} - J),
\end{align}

where $P_U$ and $P_{-\mathcal{Y}}$ are the complementary projections in $\mathcal{W}$ onto $\mathcal{U}$ respectively $-\mathcal{Y}$.

(ii) Conversely, let $\mathcal{W}$ be a Hilbert space with the inner product $(\cdot, \cdot)_{\mathcal{W}}$, let $J$ be an operator in $\mathcal{W}$ which is both unitary and self-adjoint, and define $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ by (A.2.4). Then $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ is a Kreĭn space inner product in $\mathcal{W}$, $(\cdot, \cdot)_{\mathcal{W}}$ is admissible with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, and the conclusion of (i) holds.

Proof. Proof of (i). Let $\mathcal{W} = U \oplus -\mathcal{Y}$ be the fundamental decomposition of the Kreĭn space $\mathcal{W}$ appearing in the definition of the admissibility of the inner product $(\cdot, \cdot)_{\mathcal{W}}$, and let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $J$ is self-adjoint and unitary both in the Kreĭn space $\mathcal{W}$ and in the Hilbert space $\mathcal{Y} \oplus \mathcal{U}$, and (A.2.4) holds. Clearly, $J$ is determined uniquely by (A.2.4). It is easy to see that $P_U$ and $P_{-\mathcal{Y}}$ given in (A.2.6) are the complementary orthogonal projections onto $\mathcal{U}$ respectively $-\mathcal{Y}$. Formula (A.2.5) follows from (A.2.4) by replacing $w_2$ in (A.2.4) by $J^{-1}w_2 = Jw_2$.

Proof of (ii). Define $P_U$ and $P_{-\mathcal{Y}}$ by (A.2.6). Then $P_U$ and $P_{-\mathcal{Y}}$ are complementary projection, and if we denote there ranges by $\mathcal{U}$ and $-\mathcal{Y}$, then it is easy to see that (A.2.4) defines a Kreĭn space inner product in $\mathcal{W}$ with fundamental decomposition $\mathcal{W} = U \oplus -\mathcal{Y}$. It is also clear that $(\cdot, \cdot)_{\mathcal{W}}$ is the admissible inner product corresponding to this fundamental decomposition. Thus, the assumption of (i) is fulfilled, and hence so is the conclusion of (i). \hfill $\square$

An operator which is both self-adjoint and unitary is usually called a signature operator.

A.2.3. Lemma.

(i) Let $\mathcal{W}$ be a topological vector space which is equipped with two Kreĭn space inner products $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ and $(\cdot, \cdot)_{\mathcal{W}'}$, and suppose that the admissible norms induced by $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ are equivalent to the admissible norms induced by $\langle \cdot, \cdot \rangle_{\mathcal{W}'}$. Then there exists a unique bicontinuous bijection $G$ in $\mathcal{W}$ such that

\begin{align}
[w_1, w_2]_{\mathcal{W}'} &= [w_1, Gw_2]_{\mathcal{W}}, \quad w_1, w_2 \in \mathcal{W}, \\
[w_1, w_2]_{\mathcal{W}} &= [w_1, G^{-1}w_2]_{\mathcal{W}'} \quad w_1, w_2 \in \mathcal{W}.
\end{align}
The operator $G$ is self-adjoint with respect to both $[\cdot, \cdot]_W$ and $[\cdot, \cdot]_{W'}$. More precisely, if we denote $W$ equipped with the inner product $[\cdot, \cdot]_W$ by $W_i$, $i = 1, 2$, and let $I^*_W$ and $I^*_{W'}$ be the identity operators $W \to W'$ respectively $W' \to W$, then $G$, interpreted as an operator $W' \to W$ is given by $G = I^*_W I_W^*$, and $G^{-1}$, interpreted as an operator $W \to W'$ is given by $G^{-1} = I^*_{W'} I_{W'}^*$.

(ii) Conversely, let $W$ be a Krein space with the inner product $[\cdot, \cdot]_W$, let $G$ be a bicontinuous self-adjoint bijection in $W$, and define $[\cdot, \cdot]_{W'}$ by (A.2.7).

Then $[\cdot, \cdot]_{W'}$ is a Krein space inner product in $W$, and the admissible norms induced by $[\cdot, \cdot]_W$ are equivalent to the admissible norms induced by $[\cdot, \cdot]_{W'}$.

**Proof.** Proof of (i): By the definition of the adjoint of an operator, for all $w_1, w_2 \in W$ we have

$$[I_{(W,W')}, I_{(W,W')}^* w_1, I_{(W,W')}^* w_2]_W = [w_1, I_{(W,W')}^* I_{(W,W')}^* w_2]_W.$$  

Thus, if we define $G : W \to W$ to be the operator $G = I_{(W,W')}^* I_{(W,W')}^*$, then (A.2.7) holds.

By the definition of the adjoint of an operator, for all $w = (w_1, w_2), w' = (w'_1, w'_2) \in W$, we have

$$G^* = G^{-1} = I_{(W,W')}^* I_{(W,W')}^*.$$  

This shows that if we denote the adjoint of $G$ with respect to $(\cdot, \cdot)_W$ by $G^*$, then $G^* = G^{-1}$.

Combining this with (A.2.7) we find that

$$[w_1, w_2]_W = (w_1, K w_2)_W, \quad w_1, w_2 \in W,$$

where $K := G_1 G$ is a bicontinuous bijection in $W$. We claim that $K$ is self-adjoint with respect to the inner product $(\cdot, \cdot)_W$. Indeed, $G$ is self-adjoint with respect to the inner product $(\cdot, \cdot)_W$ and $G_1$ is self-adjoint and unitary with respect to both $(\cdot, \cdot)_W$ and $(\cdot, \cdot)_W$, and hence, for all $w_1, w_2 \in W$,

$$(w_1, G w_2)_W = (w_1, K w_2)_W = (G_1 w_1, G w_2)_W = (G_1 G_1 w_1, w_2)_W.$$  

This shows that if we denote the adjoint of $G$ with respect to $(\cdot, \cdot)_W$ by $G^*$, then $G^* = G^{-1}$.

Since the norm induced by $(\cdot, \cdot)_W$ is admissible in $W$, this means that we may without loss of generality replace the original Krein space $W$.
by the Hilbert space that we get by replacing the original inner product in $\mathcal{W}$ by $(\cdot, \cdot)_{\mathcal{W}}$ if we at the same time replace $G$ by $K$.

Let $\mathcal{U}$ be the positive eigenspace of $K$, and let $\mathcal{Y}$ be the negative eigenspace of $K$, and define $K_+ = K|_{\mathcal{U}}$ and $K_- = -K|_{\mathcal{Y}}$. Then $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal with respect to $(\cdot, \cdot)_{\mathcal{W}}$, and both $K_+$ and $K_-$ are uniformly positive (i.e., they are positive and have a bounded inverse). With respect to this decomposition of $\mathcal{W}$ the inner product $[\cdot, \cdot]_{\mathcal{W}}$ can be written in the form

$$\begin{bmatrix} u_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}_{\mathcal{W}} = (u_1, K_+ u_2)_{\mathcal{W}} - (y_1, K_- y_2)_{\mathcal{W}}.$$ 

This formula implies that $[\cdot, \cdot]_{\mathcal{W}}$ defines a Krein space inner product in $\mathcal{W}$, that $\mathcal{W} = \mathcal{U} \oplus -\mathcal{Y}$ is a fundamental decomposition in $\mathcal{W}$ for this inner product, and that the corresponding admissible inner product is given by 

$$(u_1, K_+ u_2)_{\mathcal{W}} + (y_1, K_- y_2)_{\mathcal{W}}.$$ 

Since both $K_+$ and $K_-$ are bounded with bounded inverses, the norm induced by this inner product is equivalent to the norm induced by the inner product $(\cdot, \cdot)_{\mathcal{W}}$. Since all the admissible norms for $[\cdot, \cdot]_{\mathcal{W}}$ are equivalent to each other, they are all equivalent to the norm induced by $(\cdot, \cdot)_{\mathcal{W}}$. \hfill $\Box$

**A.2.4. Lemma.** Let $\mathcal{W}$ and $\mathcal{W}_1$ be two Krein spaces, let $E: \mathcal{W} \to \mathcal{W}_1$ be a bicontinuous linear bijection, and let $\mathcal{Z} \subset \mathcal{W}$. Then $(EZ)^\perp = E^{-*}V^\perp$. In particular, if $E$ is unitary or anti-unitary, i.e., $E^{-1} = \pm E^*$, then $(EV)^\perp = EV^\perp$.

**Proof.** The first claim follows from the following chain of equivalences:

$$x_1 \in (EZ)^\perp$$

$$\iff [x_1, Ex]_{\mathcal{W}_1} = 0 \text{ for all } x \in \mathcal{Z}$$

$$\iff [E^*x_1, x]_{\mathcal{W}} = 0 \text{ for all } x \in \mathcal{Z}$$

$$\iff E^*x_1 \in \mathcal{Z}^\perp$$

$$\iff x_1 \in E^{-*}\mathcal{Z}^\perp.$$ 

The second claim follows from the first. \hfill $\Box$
A.2.3. Sum decompositions of Kreĭn spaces. In this work we shall also need direct sum decompositions $W = \mathcal{Y} + \mathcal{U}$ of $W$ which are not orthogonal with respect to the original inner product $[,]_W$. In this case we shall treat $\mathcal{U}$ and $\mathcal{Y}$ as Hilbert spaces, and require that the inner products in $\mathcal{U}$ and $\mathcal{Y}$ are inherited from some Hilbert space inner product $(,)$ in $W$ which is equivalent to an admissible inner product.

A.2.5. Lemma. Let $W = \mathcal{Y} + \mathcal{U}$ be a direct sum decomposition of the Kreĭn space $W$. Then $W = \mathcal{Y}^{[\perp]} + \mathcal{U}^{[\perp]}$ is another direct sum decomposition of $W$. Moreover,

$$\langle w, w' \rangle_W = \left[ P_{\mathcal{Y}} w, P_{\mathcal{U}^{[\perp]}}^{[\perp]} w' \right]_{\mathcal{Y}} + \left[ P_{\mathcal{U}^{[\perp]}} w, P_{\mathcal{Y}^{[\perp]}}^{[\perp]} w' \right]_{\mathcal{U}} \quad \text{for all } w, w' \in W.$$  

Proof. Since $P_{\mathcal{Y}} + P_{\mathcal{U}} = 1_W$, we find after taking adjoints in the Kreĭn space $W$ that $(P_{\mathcal{Y}}^*)^* + (P_{\mathcal{U}}^*)^* = 1_W$. Here $(P_{\mathcal{Y}}^*)^*$ and $(P_{\mathcal{U}}^*)^*$ are projections onto complementary subspaces of $W$. These subspaces are $\text{rng} ((P_{\mathcal{Y}}^*)^*) = \text{ker} (P_{\mathcal{Y}}^*)^{[\perp]} = \mathcal{Y}^{[\perp]}$ and $\text{rng} ((P_{\mathcal{U}}^*)^*) = \text{ker} (P_{\mathcal{U}}^*)^{[\perp]} = \mathcal{U}^{[\perp]}$. Thus, $(P_{\mathcal{Y}}^*)^* = P_{\mathcal{U}^{[\perp]}}^{[\perp]}$ and $(P_{\mathcal{U}}^*)^* = P_{\mathcal{Y}^{[\perp]}}^{[\perp]}$. We conclude that $W = \mathcal{Y}^{[\perp]} + \mathcal{U}^{[\perp]}$. Moreover, for all $w, w' \in W$,

$$\langle w, w' \rangle_W = \left[ P_{\mathcal{Y}} w, P_{\mathcal{U}^{[\perp]}}^{[\perp]} w' \right]_{\mathcal{Y}} + \left[ P_{\mathcal{U}^{[\perp]}} w, P_{\mathcal{Y}^{[\perp]}}^{[\perp]} w' \right]_{\mathcal{U}} = \left[ P_{\mathcal{Y}} w, P_{\mathcal{U}^{[\perp]}}^{[\perp]} w' \right]_{W}.$$ \hfill $\square$

A.2.6. Remark. Lemma A.2.5 makes it possible to identify the dual of $\mathcal{U}$ with $\mathcal{Y}^{[\perp]}$ and the dual of $\mathcal{Y}$ with $\mathcal{U}^{[\perp]}$. Indeed, it is clear that every $u \in \mathcal{Y}^{[\perp]}$ defines a bounded linear functional $u \mapsto [u, u]_W$ on $\mathcal{U}$. Moreover, if $[u, u]_W = 0$ for every $u \in \mathcal{U}$, then $u \in \mathcal{Y}^{[\perp]} \cap \mathcal{U}^{[\perp]}$, and hence $u = 0$. Thus, the correspondence between the functional $u \mapsto [u, u]_W$ and the vector $u \in \mathcal{Y}^{[\perp]}$ is one-to-one. Conversely, we claim that every bounded linear functional $F$ on $\mathcal{U}$ is of this type. Given any functional $F$ on $\mathcal{U}$ we can extend $F$ to a bounded linear functional $\mathcal{F}$ on $W$, which (since $W$ is a Kreĭn space) must be of the form $\mathcal{F}w = [w, w]_W$ for some $w \in W$. By A.2.9, for all $u \in \mathcal{U}$,

$$\mathcal{F}u = [u, u]_W = [u, P_{\mathcal{U}^{[\perp]}}^{[\perp]} w]_W,$$

where $P_{\mathcal{U}^{[\perp]}}^{[\perp]} w \in \mathcal{Y}^{[\perp]}$. We conclude that $F$ is a bounded linear functional on $\mathcal{U}$ if and only if $F$ is of the form $\mathcal{F}u = [u, u]_W$ for some $u \in \mathcal{Y}^{[\perp]}$, and hence we can identify the dual of $\mathcal{U}$ with $\mathcal{Y}^{[\perp]}$. An analogous argument shows that every bounded linear functional $G$ on $\mathcal{Y}$ is of the form $\mathcal{G}y = [y, y]_W$ for some $y \in \mathcal{U}^{[\perp]}$, and hence we can identify the dual of $\mathcal{Y}$ with $\mathcal{U}^{[\perp]}$.

A.2.7. Lemma. Let $W$ be a Kreĭn space. Given any direct sum decomposition $W = \mathcal{Y} + \mathcal{U}$ there exists a Hilbert space inner product $(,)_W$ in $W$ such that the norm induced by $(,)_W$ is equivalent to an admissible norm, and such that $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal with respect to $(,)_W$.

Proof. We begin by choosing an arbitrary admissible Hilbert space inner product $(,)_1$ in $W$ (without requiring $\mathcal{Y}$ and $\mathcal{U}$ to be orthogonal), and denote the Hilbert space inner products in $\mathcal{Y}$ and $\mathcal{U}$ inherited from $(,)_1$ by $(,)_Y$ respectively $(,)_U$. We then define a new positive inner product $(,)_W$ in $W$ by

$$(w, w)'_W = (P_{\mathcal{Y}} w, P_{\mathcal{U}} w')_Y + (P_{\mathcal{U}} w, P_{\mathcal{Y}} w')_U.$$
Clearly, $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal with respect to $(\cdot, \cdot)_{\mathcal{W}}$. The fact that $\mathcal{W}$ is a direct sum decomposition of $\mathcal{Y}$ and $\mathcal{U}$ implies that the norm induced by $(\cdot, \cdot)_{\mathcal{W}}$ is equivalent to the norm induced by $(\cdot, \cdot)_{\mathcal{Y}}$ (a sequence $w_n$ tends to zero in $\mathcal{W}$ if and only if both $y_n = P_{\mathcal{Y}}^n w_n$ and $u_n = P_{\mathcal{U}}^n w_n$ tend to zero in $\mathcal{Y}$ respectively $\mathcal{U}$). Thus, $(\cdot, \cdot)_{\mathcal{W}}$ satisfies the requirements listed in Lemma A.2.7. □

A.2.8. Lemma. Let $\mathcal{W}$ be a Krejn space with the direct sum decomposition $\mathcal{W} = \mathcal{F} \oplus \mathcal{E}$, and let $(\cdot, \cdot)_{\mathcal{W}}$ be a Hilbert space inner product in $\mathcal{W}$ such that the norm induced by $(\cdot, \cdot)_{\mathcal{W}}$ is equivalent to an admissible norm, and such that $\mathcal{F}$ and $\mathcal{E}$ are orthogonal with respect to $(\cdot, \cdot)_{\mathcal{W}}$ (by Lemma A.2.7 such an inner product always exists). Thus, $(\cdot, \cdot)_{\mathcal{W}} = (\cdot, \cdot)_{\mathcal{F} \oplus \mathcal{E}}$, where we equip $\mathcal{F}$ and $\mathcal{E}$ with the inner products inherited from $(\cdot, \cdot)_{\mathcal{W}}$. Let $J \in \mathcal{B}(\mathcal{W})$ be the operator in A.2.4 obtained from Lemma A.2.2, and decompose $J$ into $J = [J_{11} J_{22}]$ is accordance with the decomposition $\mathcal{W} = \mathcal{F} \oplus \mathcal{E}$ (so that $J_{11} = P_{\mathcal{F}} J |_{\mathcal{F}}$, etc.).

(i) The subspaces $\mathcal{E}$ and $\mathcal{F}$ are both Lagrangian if and only if they are both neutral, and this is true if and only if both $J_{11} = 0$ and $J_{22} = 0$. In this case $J_{12}$ and $J_{21}$ are boundedly invertible, $J_{21} = J_{21}^*$, and for all $[f, e]$, $[f', e'] \in [\mathcal{F}]$ we have

$$
(A.2.10) \quad \left[ \begin{array}{c} f \\ e \\
\end{array} \right] , \left[ \begin{array}{c} f' \\ e' \\
\end{array} \right] _{\mathcal{W}} = (f, \Psi e')_{\mathcal{F}} + (\Psi e, f')_{\mathcal{E}},
$$

where $\Psi = J_{12} \in \mathcal{B}(\mathcal{E}; \mathcal{F})$ has a bounded inverse $\Psi^{-1} \in \mathcal{B}(\mathcal{F}; \mathcal{E})$. Moreover, the formula

$$
(A.2.11) \quad \left[ \begin{array}{c} f \\ e \\
\end{array} \right] , \left[ \begin{array}{c} f' \\ e' \\
\end{array} \right] _{0} = (f, f')_{\mathcal{F}} + (\Psi e, \Psi e')_{\mathcal{E}}
$$

defines an admissible inner product on $\mathcal{W}$, and $\mathcal{F}$ and $\mathcal{E}$ are orthogonal also with respect to this inner product.

(ii) Assuming that the subspaces $\mathcal{E}$ and $\mathcal{F}$ are Lagrangian, the inner product $(\cdot, \cdot)_{\mathcal{W}}$ is itself admissible if and only if the operator $\Psi$ in (A.2.10) and (A.2.11) is unitary.

Proof. It is easy to see that $\mathcal{F}$ is neutral if and only if $J_{11} = 0$, and that $\mathcal{E}$ is neutral if and only if $J_{22} = 0$. If both these conditions are satisfied, then the invertibility of $J$ implies that $J_{12}$ and $J_{21}$ are invertible. The invertibility of $J_{12}$ and $J_{21}$ imply that $\mathcal{F}$ and $\mathcal{E}$ must, in fact, be Lagrangian. The self-adjointness of $J$ with respect to $(\cdot, \cdot)_{\mathcal{W}}$ together with the orthogonality of $\mathcal{F}$ and $\mathcal{E}$ implies that $J_{21} = J_{21}^*$, after which [A.2.4] can be written in the form [A.2.10]. By Lemma A.2.7 $(\cdot, \cdot)_{\mathcal{W}}$ is itself admissible if and only if $J$ is unitary, and this is true if and only if $\Psi$ is unitary.

The inner product $(\cdot, \cdot)_{0}$ defined in A.2.11 is obtained from $(\cdot, \cdot)_{\mathcal{W}}$ by a rescaling of the norm in $\mathcal{E}$ in the sense that for all for all $[f, e]$, $[f', e'] \in [\mathcal{F}]$,

$$
\left[ \begin{array}{c} f \\ e \\
\end{array} \right] , \left[ \begin{array}{c} f' \\ e' \\
\end{array} \right] _{0} = (f, f')_{\mathcal{F}} + (e, \Psi^* \Psi e')_{\mathcal{E}}.
$$

Clearly, $\mathcal{F}$ and $\mathcal{E}$ are still orthogonal with respect to $(\cdot, \cdot)_{0}$, and the norm induced by $(\cdot, \cdot)_{0}$ is equivalent to the norm induced by $(\cdot, \cdot)_{\mathcal{W}}$, hence equivalent to an admissible
norm. Let $E'$ stand for $E$ equipped with the inner product
\[(e,e')_{E'} = (e, \Psi^*\Psi e')_E, \quad e, \ e' \in E\]
induced by $(\cdot,\cdot)_0$. Then $(\cdot,\cdot)_0 = (\cdot,\cdot)_{F\oplus E'}$. With respect to this inner product the operator $\Psi$ is unitary, and hence, by assertion (ii), $(\cdot,\cdot)_0$ is admissible. □

A.2.9. COROLLARY. Let $W$ be a Krein space with the direct sum decomposition $W = F \oplus E$. If both $F$ and $E$ are Lagrangian, then there exists an admissible Hilbert space inner product $(\cdot,\cdot)_W$ in $W$ such that $F$ and $E$ are orthogonal with respect to $(\cdot,\cdot)_W$, and such that (A.2.10) holds for some bounded linear operator $\Psi : E \to F$ which is unitary with respect to the inner products in $F$ and $E$ induced by $(\cdot,\cdot)_W$.

PROOF. The inner product $(\cdot,\cdot)_0$ constructed in Lemma A.2.8 is of this type. □

We shall in the sequel write $W = F \oplus E$ instead of $W = F \oplus E$ whenever $W$ is the direct sum of two Lagrangian subspaces $F$ and $E$ and (2.8) holds for some unitary operator $\Psi : E \to F$. We call this a Lagrangian decomposition of $W$, with impedance weighting operator $\Psi$.

A.2.10. LEMMA. Let $\mathfrak{K}$ be a Krein space with the direct sum decomposition $\mathfrak{K} = U + Y$ and the dual direct sum decomposition $\mathfrak{K} = Y^{[1]} + U^{[1]}$ (cf. Lemma A.2.9), and let $V$ be a subspace of $\mathfrak{K}$. Then

(i) \( (P_U^Y V)^{[1]} = V^{[1]} \cap Y^{[1]} + U^{[1]} \). In particular, if $P_U^Y V$ is closed, then
\[ P_U V = (V^{[1]} \cap Y^{[1]} + U^{[1]})^{[1]} \]

(ii) If $V$ is closed, then $P_U^Y V$ is closed if and only if $P_U^{Y^{[1]}} V^{[1]}$ is closed.

PROOF. Proof of (i). That \( (P_U^Y V)^{[1]} = V^{[1]} \cap Y^{[1]} + U^{[1]} \) follows from the following chain of equivalences:
\[
\begin{align*}
&z_1 \in (P_U^Y V)^{[1]} \\
\iff &[z_1, P_U^Y z]_{\mathfrak{K}} = 0 \text{ for all } z \in V \\
\iff &([P_U^Y]^* z_1, z]_{\mathfrak{K}} = 0 \text{ for all } z \in V \\
\iff &P_U^{Y^{[1]}} z_1, z]_{\mathfrak{K}} = 0 \text{ for all } z \in V \\
\iff &P_U^{Y^{[1]}} z_1 \in V^{[1]}, \\
\iff &z_1 = P_U^{Y^{[1]}} z_1 + P_U^{Y^{[1]}} z_1 \in V^{[1]} \cap Y^{[1]} + U^{[1]}.
\end{align*}
\]
This implies that \( P_U^Y V = (V^{[1]} \cap Y^{[1]} + U^{[1]})^{[1]} \).

Proof of (ii). It suffices to prove this in one direction only, since \( (V^{[1]})^{[1]} = V \), \( (U^{[1]})^{[1]} = U \) and \( (Y^{[1]})^{[1]} = Y \).

Suppose that $P_U^Y V$ is closed. Then so is $P_U^Y V + Y$ (since $W = U + Y$). Since $P_U^Y V + Y = \{u + y \mid y \in Y, \ u \in U \text{ and } u + y' \in V \text{ for some } y' \in Y\}$
\[= \{u + y' + y \mid y \in Y, \ u \in U \text{ and } u + y' \in V \text{ for some } y' \in Y\} = V + Y,\]
we find that $V + Y$ is closed. Since $\mathfrak{K}$ is a Krein space, we may interpret $\mathfrak{K}$ as a Banach space and identify the dual of $\mathfrak{K}$ with $\mathfrak{K}$ itself. This makes it possible to apply Kato [1980, Theorem 4.8, p. 221] to show that $V^{[1]} + Y^{[1]}$ is closed. But,
arguing as above, we find that this set coincides with $P_{U^{[\perp]}}^V V^{[\perp]} + Y^{[\perp]}$, and hence $P_{U^{[\perp]}}^V V^{[\perp]}$ is closed. □

Often when we use the above lemma we take $\mathcal{K}$ to be the node space of a s/s system $\Sigma$, and let $V$ be the generating subspace of $\Sigma$. In this connection the following lemma is useful.

A.2.11. Lemma. Let $X$ be a Hilbert space, let $\mathcal{W}$ be a Krein space, let $\mathcal{W} = U \oplus Y$ be a direct sum decomposition of $\mathcal{W}$, let $K$ be the space $X \otimes X$ equipped with the indefinite inner product (10.1.4), and let $V$ be a subspace of $\mathcal{K}$. Then

(A.2.12) $\left[ \begin{array}{c} X \\ Y \end{array} \right]^{[\perp]} + U = \left[ \begin{array}{c} X \\ Y \end{array} \right]^{[\perp]} + U + Y$.

(A.2.13) $\left[ \begin{array}{c} X \\ Y \end{array} \right]^{[\perp]} + U = \left[ \begin{array}{c} X \\ Y \end{array} \right]^{[\perp]} + U + Y$.

(A.2.14) $\left[ \begin{array}{c} X \\ Y \end{array} \right]^{[\perp]} + U = \left[ \begin{array}{c} X \\ Y \end{array} \right]^{[\perp]} + U + Y$.

Proof. The proof of (A.2.12) is straightforward, (A.2.13) follows from part (i) of Lemma A.2.10, and (A.2.14) follows from (A.2.12), (A.2.13), and part (ii) of Lemma A.2.10. □

A.2.12. Lemma. Let $\mathcal{K}$ be a Krein space with the orthogonal decomposition $\mathcal{K} = U \oplus Y$, and let $V$ be a subspace of $\mathcal{K}$. Then

(i) $(P_Y V)^{[\perp]} = V^{[\perp]} \cap U + Y$. In particular, if $P_Y V$ is closed, then $P_Y V = (V^{[\perp]} \cap U + Y)^{[\perp]}$.

(ii) If $V$ is closed, then $P_Y V^{[\perp]}$ is closed if and only if $P_Y V^{[\perp]}$ is closed.

Proof. This follows from Lemma A.2.10 since we now have $U^{[\perp]} = Y$ and $Y^{[\perp]} = U$. □
A.2.4. Maximal nonnegative and maximal nonpositive subspaces. If we fix a fundamental decomposition $\mathcal{W} = \mathcal{U} \oplus -\mathcal{Y}$, then we may view elements of $\mathcal{W}$ as consisting of column vectors

$$w = \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix},$$

where we view $\mathcal{U}$ and $\mathcal{Y}$ as Hilbert spaces, and the Kreĭn space inner product on $\mathcal{W}$ is given by

$$(A.2.15) \quad \left[ \begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} u' \\ y' \end{bmatrix} \right]_{\mathcal{W}} = \left( \begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u' \\ y' \end{bmatrix} \right)_{\mathcal{U} \oplus \mathcal{Y}} = (u, u')_{\mathcal{U}} - (y, y')_{\mathcal{Y}}.$$

In this representation, nonnegative, neutral, nonpositive, and Lagrangian subspaces are characterized as follows.

A.2.13. Proposition. Let $\mathcal{W}$ be a Kreĭn space represented in the form $\mathcal{W} = \begin{bmatrix} \mathcal{U} \\ -\mathcal{Y} \end{bmatrix}$ with Kreĭn space inner product equal to the quadratic form $\left[ \cdot, \cdot \right]_J$ induced by the operator $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in the Hilbert space inner product of $\begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$ as explained above, and let $Z$ be a subspace of $\mathcal{W}$. Then the following claims are true:

(i) $Z$ is nonnegative if and only if there is a linear Hilbert space contraction $K_+ : \mathcal{D}_+ \mapsto \mathcal{Y}$ from some domain $\mathcal{D}_+ \subset \mathcal{U}$ into $\mathcal{Y}$ (where both $K_+$ and $\mathcal{D}_+$ are uniquely determined by $Z$) such that

$$(A.2.16) \quad Z = K_+ \begin{bmatrix} 1 \\ K_+ \end{bmatrix} \mathcal{D}_+ = \left\{ \begin{bmatrix} d_+ \\ K_+ d_+ \end{bmatrix} \bigg| d_+ \in \mathcal{D}_+ \right\}.$$

$Z$ is maximal nonnegative if and only if, in addition, $\mathcal{D}_+ = \mathcal{U}$.

(ii) $Z$ is nonpositive if and only if there is a linear contraction $K_- : \mathcal{D}_- \mapsto \mathcal{U}$ from some domain $\mathcal{D}_- \subset \mathcal{Y}$ into $\mathcal{U}$ (where both $K_-$ and $\mathcal{D}_-$ are uniquely determined by $Z$) such that

$$(A.2.17) \quad Z = \begin{bmatrix} K_- \\ 1 \end{bmatrix} \mathcal{D}_- = \left\{ \begin{bmatrix} K_- d_- \\ d_- \end{bmatrix} \bigg| d_- \in \mathcal{D}_- \right\}.$$

$Z$ is maximal nonnegative if and only if, in addition, $\mathcal{D}_- = \mathcal{Y}$.

(iii) $Z$ is neutral if and only if there is an isometry $K_+$ mapping a subspace $\mathcal{D}_+$ of $\mathcal{U}$ isometrically onto a subspace $\mathcal{D}_- \subset \mathcal{Y}$, or equivalently, an isometry $K_-$ mapping $\mathcal{D}_- \subset \mathcal{Y}$ isometrically onto $\mathcal{D}_+ \subset \mathcal{U}$ (where $K_+, \mathcal{D}_+, K_-$ and $\mathcal{D}_-$ are all uniquely determined by $Z$), such that

$$(A.2.19) \quad Z = K_+ \begin{bmatrix} 1 \\ K_+ \end{bmatrix} \mathcal{D}_+ = \begin{bmatrix} K_- \\ 1 \end{bmatrix} \mathcal{D}_-.$$

$Z$ is Lagrangian if and only if, in addition, $\mathcal{D}_+ = \mathcal{U}$ and $\mathcal{D}_- = \mathcal{Y}$.
(v) \( \mathcal{Z} \) is Lagrangian if and only if \( \mathcal{Z} \) is both maximal nonnegative and maximal nonpositive.

(vi) \( \mathcal{Z} \) is maximal nonnegative if and only if \( \mathcal{Z} \) is closed and nonnegative and \( \mathcal{Z}^{\perp} \) is nonpositive.

(vii) Every nonnegative subspace is contained in some maximally nonnegative subspace, and every nonpositive subspace is contained in some maximally nonpositive subspace.

**Proof.** Proof of (i): If \( \mathcal{Z} \) has a representation of the type \([A.2.16]\), then every \( W = \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{Z} \) satisfies

\[
[w, w]_W = \|u\|^2_W - \|K_+ u\|^2_Y \geq (1 - \|K\|^2_+) \|u\|^2_W \\
\geq 0,
\]

so \( \mathcal{Z} \) is nonnegative. Conversely, suppose that \( \mathcal{Z} \) is nonnegative. Define \( \mathcal{D}_+ = P_+ \mathcal{Z} \). Since \( \mathcal{Z} \) is nonnegative, if \( y \in \mathcal{Z} \cap \mathcal{Y} \), then on one hand \( (y, y)_W \geq 0 \) since \( y \in \mathcal{Z} \), and on the other hand \( (y, y)_Y = -\|y\|^2_Y \leq 0 \) since \( y \in \mathcal{Y} \). This shows that \( \mathcal{Z} \cap \mathcal{Y} = \{0\} \), and hence \( \mathcal{Z} \) has a graph representation of the type \([A.2.16]\). Both \( \mathcal{D}_+ \) and \( K_+ \) in \([A.2.16]\) are uniquely determined by \( \mathcal{Z} \) since an operator and its domain are uniquely determined by its graph. That \( K_+ \) in \([A.2.16]\) must be a contraction follows from the negativity of \( \mathcal{Z} \), which gives for all \( w = \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{Z} \),

\[
0 \leq (w, w)_W = \|u\|^2_W - \|K_+ u\|^2_Y.
\]

It follows from the representation \([A.2.16]\) that \( \mathcal{Z} \) has a proper nonnegative extension if and only if \( K_+ \) can be extended to a contraction \( \tilde{K}_+ \) defined on a strictly larger domain \( \mathcal{D}_+ \), and this is true if and only if \( \mathcal{D}_+ \neq \mathcal{U} \).

Proof of (ii): The proof of (ii) is analogous to the proof of (i).

Proof of (iii): Suppose that \( \mathcal{Z} \) is maximal nonnegative. Then \( \mathcal{Z} \) is closed, because otherwise the closure of \( \mathcal{Z} \) is a proper nonnegative extension of \( \mathcal{Z} \). Moreover, \( \mathcal{Z} \) has the representation \([A.2.16]\) with \( \mathcal{D} = \mathcal{U} \). A vector \( w^\dagger = \begin{bmatrix} u^\dagger \\ y^\dagger \end{bmatrix} \) belongs to \( \mathcal{Z}^{\perp} \) if and only if it is true for all \( u \in \mathcal{U} \) that

\[
0 = \begin{bmatrix} u^\dagger \\ y^\dagger \end{bmatrix} \cdot \begin{bmatrix} u \\ K_+ u \end{bmatrix} = (u^\dagger, u)_U - (y^\dagger, K_+ u)_Y = (u^\dagger, u)_U - (K_+ y^\dagger, u)_Y,
\]

or equivalently, \( u^\dagger = K_+ y^\dagger \). This means that \( \mathcal{Z}^{\perp} \) has a representation of the type \([A.2.17]\) with \( K_- = K_+^* \) and \( \mathcal{D}_- = \mathcal{Y} \), and hence \( \mathcal{Z}^* \) is maximal nonnegative.

Conversely, suppose that \( \mathcal{Z} \) is closed and \( \mathcal{Z}^{\perp} \) is maximal nonpositive. Since \( \mathcal{Z} \) is closed we have \( \mathcal{Z} = (\mathcal{Z}^{\perp})^* \). An argument analogous to the one above shows that the maximal nonpositivity of \( \mathcal{Z}^{\perp} \) implies that \( (\mathcal{Z}^{\perp})^{\perp} \) is maximal nonpositive, and hence \( \mathcal{Z} \) is maximal nonpositive.

Proof of (iv): If \( \mathcal{Z} \) is neutral, then \( \mathcal{Z} \) is both nonnegative and nonpositive, so both \([A.2.16]\) and \([A.2.17]\) hold for some contractions \( K_+ \) and \( K_- \). It follow from \([A.2.19]\) that \( K_- = K_+^* \), and hence both \( K_+ \) and \( K_- \) must be contractions. Conversely, if \([A.2.19]\) holds for some isometries \( K_+ \) and \( K_- \), then by (i) and (ii), \( \mathcal{Z} \) is both nonnegative and nonpositive, and hence neutral.

If \( \mathcal{D}_+ = \mathcal{U} \) and \( \mathcal{D}_- = \mathcal{Y} \), then it follows from \([A.2.19]\) that \( K_+ \) and \( K_- \) are unitary, with \( K_- = K_+^{-1} = K_+^* \). This together with \([A.2.18]\) implies that \( \mathcal{Z}^{\perp} = \mathcal{Z} \), i.e., \( \mathcal{Z} \) is Lagrangian. Conversely, suppose that \( \mathcal{Z} \) is Lagrangian. Then \( \mathcal{Z} \) is closed, and consequently both \( \mathcal{D}_+ \) and \( \mathcal{D}_- \) are closed. If, for example, \( \mathcal{D}_+ \neq \mathcal{U} \), then there exists some nonzero vector \( u_0 \in (\mathcal{D}_+)^\perp \). It follows from \([A.2.19]\) that the vector \( \begin{bmatrix} u_0 \\ 0 \end{bmatrix} \in \mathcal{Z}^{\perp} \), but \( \begin{bmatrix} u_0 \\ 0 \end{bmatrix} \notin \mathcal{Z} \), so it is not true that \( \mathcal{Z}^{\perp} \subseteq \mathcal{Z} \), and therefore \( \mathcal{Z} \) cannot
be Lagrangian. An analogous argument shows that if \( D_+ \neq \mathcal{Y} \), then it will again not be true that \( \mathcal{Z}^{(1)} \subset \mathcal{Z} \). Therefore, both the conditions \( D_+ = \mathcal{U} \) and \( D_- = \mathcal{Y} \) are necessary for \( \mathcal{Z} \) to be Lagrangian.

**Proof of (v):** It follows from (i), (ii), and (iv) that if \( \mathcal{Z} \) is Lagrangian, then \( \mathcal{Z} \) is both maximal nonnegative and maximal nonpositive. Conversely, if \( \mathcal{Z} \) is both maximal nonnegative and maximal nonpositive, then \( \mathcal{Z} \) has the representation (A.2.19) since \( \mathcal{Z} \) is neutral. In this representation \( D_+ = \mathcal{U} \) since \( \mathcal{Z} \) is maximal nonnegative, and \( D_- = \mathcal{Y} \) since \( \mathcal{Z} \) is maximal nonpositive, and hence by (iv), \( \mathcal{Z} \) is Lagrangian.

**Proof of (vi):** We know from (iii) that if \( \mathcal{Z} \) is maximal nonnegative, then \( \mathcal{Z} \) is closed and \( \mathcal{Z}^{[\perp]} \) is (even maximal) nonnegative. Conversely, suppose that \( \mathcal{Z} \) is closed and nonnegative, and that \( \mathcal{Z}^{[\perp]} \) is nonnegative. Since \( \mathcal{Z} \) is closed, the domain \( D_+ \) in (A.2.16) is closed. If \( D_+ \neq \mathcal{U} \), then, by arguing as in the proof of (iv) above, we can find a nonzero vector \( \mathcal{u} \in \mathcal{Z}^{[\perp]} \). This vector is positive, which contradicts the assumption that \( \mathcal{Z}^{[\perp]} \) is nonpositive. Thus, the nonpositivity of \( \mathcal{Z}^{[\perp]} \) implies that \( D_+ = \mathcal{U} \), and therefore by (i), \( \mathcal{Z} \) is maximal nonnegative.

**Proof of (vi):** If \( \mathcal{Z} \) is nonnegative, then we can extend the contraction \( K_+ \) in (i) to a contraction defined on all of \( \mathcal{U} \), after which the right-hand side of (A.2.16) with \( D_+ \) replaced by \( \mathcal{U} \) defines a maximal nonnegative subspace \( \mathcal{Z}_{\text{max}} \) which contains \( \mathcal{Z} \). The proof of the case where \( \mathcal{Z} \) is nonpositive is analogous, with (A.2.16) replaced by (A.2.17). □

**A.2.14. LEMMA.** Let \( \mathcal{Z} \) be either a maximal nonnegative or maximal nonpositive subspace of a Kreın space \( \mathcal{K} \). Then a neutral vector \( \mathcal{z} \) belongs to \( \mathcal{Z} \) if and only if it belongs to \( \mathcal{Z}^{[\perp]} \). Thus, \( \mathcal{Z}_0 := \mathcal{Z} \cap \mathcal{Z}^{[\perp]} \) is the maximal neutral subspace contained in \( \mathcal{Z} \), and \( \mathcal{Z}_0 \) is also the maximal neutral subspace contained in \( \mathcal{Z}^{[\perp]} \).

**PROOF.** It suffices to prove the case where \( \mathcal{Z} \) is maximal nonnegative, because we then get also the case where \( \mathcal{Z} \) is maximal nonpositive simply by interchanging \( \mathcal{Z} \) and \( \mathcal{Z}^{[\perp]} \).

If \( \mathcal{z}_0 \in \mathcal{Z} \) is neutral, then it follows from the nonnegativity of \( \mathcal{Z} \) and the Schwartz inequality (which is valid in any semi-inner product space) that \( [\mathcal{z}_0, \mathcal{z}]_{\mathcal{K}} = 0 \) for all \( \mathcal{z} \in \mathcal{Z} \). This implies that \( \mathcal{z}_0 \in \mathcal{Z}^{[\perp]} \). □

The next lemma will be used later to find out if certain subspaces of a Kreın space with a special orthogonal decomposition are maximal nonnegative, or maximal nonpositive, or Lagrangian.

**A.2.15. LEMMA.** Let \( \mathcal{X} \) and \( \mathcal{Z} \) be two Hilbert spaces and \( \mathcal{W} \) a Kreın space, and let \( \mathcal{K} \) be the Kreın space \( \mathcal{K} = \begin{bmatrix} \mathcal{X} & \mathcal{Z} \end{bmatrix} \).

(i) A nonnegative subspace \( \mathcal{V} \) of \( \mathcal{K} \) is maximal nonnegative if and only if conditions (a) and (b) below hold:

(a) For each \( \mathcal{x} \in \mathcal{X} \) there exists some \( \mathcal{z} \in \mathcal{Z} \) and \( \mathcal{w} \in \mathcal{W} \) such that \( \begin{bmatrix} \mathcal{z} \\ \mathcal{w} \end{bmatrix} \in \mathcal{V} \);

(b) The set of all \( \mathcal{w} \in \mathcal{W} \) for which there exists some \( \mathcal{z} \in \mathcal{Z} \) such that \( \begin{bmatrix} \mathcal{z} \\ \mathcal{w} \end{bmatrix} \in \mathcal{V} \) is maximal nonnegative in \( \mathcal{W} \).

(ii) A nonpositive subspace \( \mathcal{V} \) of \( \mathcal{K} \) is maximal nonpositive if and only if conditions (c) and (d) below hold:
(c) For each \( z \in Z \) there exists some \( x \in \mathcal{X} \) and \( w \in W \) such that \( \begin{bmatrix} z \\ w \end{bmatrix} \in V \).

(d) The set of all \( w \in W \) for which there exists some \( x \in \mathcal{X} \) such that \( \begin{bmatrix} 0 \\ w \end{bmatrix} \in V \) is maximal nonpositive in \( W \).

(iii) A neutral subspace \( V \) of \( \mathfrak{K} \) is Lagrangian if and only if conditions (a)–(d) above hold.

Proof of (iii). Assume first that (a) and (b) hold. Let \( W = -\mathcal{Y} \oplus \mathcal{U} \) be a fundamental decomposition of \( W \). Then \( -\begin{bmatrix} \hat{\alpha} \\ \hat{\mathcal{Y}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \hat{\mathcal{U}} \end{bmatrix} \) is a fundamental decomposition of \( \mathfrak{K} \). By assertion (i) in Proposition A.2.13, \( V \) has a representation

\[
V = \left\{ \begin{bmatrix} K_1 [x_{z_0}] \\ x_{z_0} \\ K_2 [x_{z_0}] + u_0 \end{bmatrix} \right| [x_{z_0}] \in D_+ \},
\]

where \([-K_1 K_2]\) is a contraction defined on some subspace \( D_+ \) of \( [\mathcal{X} \mathcal{U}] \) with values in \( [\hat{\mathcal{Y}} \hat{\mathcal{U}}] \). By Proposition A.2.13, in order to show that \( V \) is maximal nonnegative it suffices to show that \( D_+ = [\hat{\mathcal{Y}} \hat{\mathcal{U}}] \).

Let \( x \) be an arbitrary vector in \( \mathcal{X} \). Then by (a), there exist \( z_1 \in Z \) and \( w \in W \) such that \( \begin{bmatrix} z_1 \\ x \\ w \end{bmatrix} \in V \). Since the set in (b) is maximal nonnegative it follows that for any \( u \in \mathcal{U} \) there exist \( z_2 \in Z \) and \( \bar{w} \in W \) such that \( P_{\mathcal{U}} \bar{w} = u - P_{\mathcal{U}} w \) and \( \begin{bmatrix} z_2 \\ 0 \\ \bar{w} \end{bmatrix} \in V \).

Since \( V \) is a subspace, also

\[
\begin{bmatrix} z_1 \\ x \\ w \end{bmatrix} + \begin{bmatrix} z_2 \\ 0 \\ \bar{w} \end{bmatrix} = \begin{bmatrix} z_1 + z_2 \\ x + \bar{w} \end{bmatrix} \in V,
\]

with \( P_{\mathcal{U}} (w + \bar{w}) = u \). Thus \( [\hat{x}_{z}] \in D_+ \), \( z_1 + z_2 = K_1 [\hat{x}_{z}] \), and \( P_{\mathcal{Y}} (w + \bar{w}) = K_2 [\hat{x}_{z}] \).

Since \( x \in \mathcal{X} \) and \( u \in \mathcal{U} \) are arbitrary we find that \( D_+ = [\hat{\mathcal{Y}} \hat{\mathcal{U}}] \). This proves that \( V \) is maximal nonnegative.

Conversely, suppose that \( V \) maximal nonnegative. By Proposition A.2.13, \( V \) has a representation of the form (A.2.20) for some contraction \([-K_1 K_2] : [\mathcal{X} \mathcal{U}] \to [\hat{\mathcal{Y}} \hat{\mathcal{U}}] \).

Clearly this implies that (a) holds. Moreover, the set in (b) is given by \( \{ u_0 + K_2 [z_0] \mid u_0 \in \mathcal{U} \} \), and by Proposition A.2.13 it is maximal nonnegative. Thus also (b) holds.

Proof of (ii). The proof of (ii) is analogous to the proof of (i).

Proof of (iii). This follows from (i) and (ii) together Proposition A.2.13(v). □

A.2.16. Lemma. Let \( \mathfrak{K} \) be a Krein space with the orthogonal decomposition \( \mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \), and let \( Z \) be a maximal nonnegative subspace of \( \mathfrak{K} \). Then the following conditions are equivalent:

(i) \( P_{\mathfrak{K}_2} Z \) is a nonnegative subspace of \( \mathfrak{K}_2 \);
(ii) \( P_{\mathfrak{K}_2} Z \) is a maximal nonnegative subspace of \( \mathfrak{K}_2 \);
(iii) \( Z \cap \mathfrak{K}_1 \) is a maximal nonnegative subspace of \( \mathfrak{K}_1 \);
(iv) \( P_{\mathfrak{K}_1} | Z \) is a contraction \( Z \to \mathfrak{K}_1 \), i.e.,

\[
[z, z]_\mathfrak{K} \leq [P_{\mathfrak{K}_1} z, P_{\mathfrak{K}_1} z]_{\mathfrak{K}_1} \text{ for all } z \in Z.
\]

(v) \( P_{\mathfrak{K}_1} Z^{(1)} \) is a nonpositive subspace of \( \mathfrak{K}_1 \);
(vi) \( P_{\mathfrak{K}_1} Z^{(1)} \) is a maximal nonpositive subspace of \( \mathfrak{K}_1 \);
(vii) \( Z^{(1)} \cap \mathfrak{K}_2 \) is a maximal nonpositive subspace of \( \mathfrak{K}_2 \);
(viii) $P_{\mathcal{R}_2}|_{\mathcal{R}_1} \mathcal{Z}^{[1]} \to \mathcal{R}_2$, i.e.,

$$[z^t, z^t]|_{\mathcal{R}_1} \geq [P_{\mathcal{R}_2} z^t, P_{\mathcal{R}_2} z^t]|_{\mathcal{R}_2}$$

for all $z^t \in \mathcal{Z}^{[1]}$.

When these equivalent conditions holds we have $(\mathcal{Z} \cap \mathcal{R}_1)^{[1]} = P_{\mathcal{R}_1} \mathcal{Z}^{[1]}$ and $P_{\mathcal{R}_2} \mathcal{Z} = (\mathcal{Z}^{[1]} \cap \mathcal{R}_2)^{[1]}$, where the orthogonal companions on the right-hand sides are computed in $\mathcal{R}_1$ respectively $\mathcal{R}_2$.

**Proof.** We first show that (i), (ii), (iv), and (vii) are equivalent to each other, and that (analogously) (v), (vi), (viii), and (iii) are equivalent to each other, and then complete the proof of the equivalence of the conditions (i)–(viii) by showing that (iii) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (v). The final claim follows from Lemma A.2.12.

(i) $\Leftrightarrow$ (ii): Trivially (ii) $\Rightarrow$ (i). If $P_{\mathcal{R}_2} \mathcal{Z}$ is not maximal nonnegative in $\mathcal{R}_2$, then $P_{\mathcal{R}_1} \mathcal{Z}$ is properly contained in some nonnegative subspace $\mathcal{Z}_2$ of $\mathcal{R}_2$. This implies that $\mathcal{Z}$ is properly contained in the nonnegative subspace $\mathcal{Z} \vee \{0\}$ of $\mathcal{R}$, and hence $\mathcal{Z}$ cannot be maximal. Thus (i) $\Rightarrow$ (ii).

(i) $\Leftrightarrow$ (iv): Since $\mathcal{R} = \mathcal{R}_1 \boxplus \mathcal{R}_2$ we have

$$[z, z]|_{\mathcal{R}} = [P_{\mathcal{R}_1} z, P_{\mathcal{R}_1} z]|_{\mathcal{R}_1} + [P_{\mathcal{R}_2} z, P_{\mathcal{R}_2} z]|_{\mathcal{R}_2}$$

for all $z \in \mathcal{Z}$. Thus, (i) $\Leftrightarrow$ (iv).

(ii) $\Leftrightarrow$ (vii): This follows from Proposition A.2.13(iii) and Lemma A.2.12

(v) $\Leftrightarrow$ (vi): This follows from the equivalence (i) $\Leftrightarrow$ (ii) if we replace $\mathcal{R}$ by its anti-space $-\mathcal{R}$, interchange $\mathcal{R}_1$ and $\mathcal{R}_2$, and also interchange $\mathcal{Z}$ and $\mathcal{Z}^{[1]}$.

(v) $\Leftrightarrow$ (viii): This follows from the equivalence (i) $\Leftrightarrow$ (iv) if we replace $\mathcal{R}$ by its anti-space $-\mathcal{R}$, interchange $\mathcal{R}_1$ and $\mathcal{R}_2$, and also interchange $\mathcal{Z}$ and $\mathcal{Z}^{[1]}$.

(vi) $\Leftrightarrow$ (iii): This follows from Proposition A.2.13(iii) and Lemma A.2.12

(iii) $\Rightarrow$ (i): Suppose that (i) does not hold. Then there exists a vector $z_0 \in \mathcal{Z}$ such that $[P_{\mathcal{R}_2} z_0, P_{\mathcal{R}_2} z_0]|_{\mathcal{R}_2} < 0$. In particular, since $\mathcal{Z}$ is nonnegative, this implies that $P_{\mathcal{R}_2} z_0 \notin \mathcal{Z}$, and consequently, $P_{\mathcal{R}_1} z_0 = z_0 - P_{\mathcal{R}_2} z_0 \notin \mathcal{Z}$. Thus, $\mathcal{Z} \cap \mathcal{R}_1$ is a proper subset of $P_{\mathcal{R}_1} \mathcal{Z} \vee (\mathcal{Z} \cap \mathcal{R}_1)$. We claim that this subspace is nonnegative. This is true because for all $z \in \mathcal{Z}$ and all $\lambda \in \mathbb{C}$, we have $\lambda z_0 + z \in \mathcal{Z}$, and hence

$$[\lambda P_{\mathcal{R}_1} z_0 + z, \lambda P_{\mathcal{R}_1} z_0 + z]|_{\mathcal{R}} = [\lambda z_0 + z, \lambda z_0 + z]|_{\mathcal{R}} - |\lambda|^2 [P_{\mathcal{R}_2} z_0, P_{\mathcal{R}_2} z_0]|_{\mathcal{R}_2} \geq 0.$$

Thus, if (i) is false, then so is (iii).

(vii) $\Rightarrow$ (v): This follows from the implication (iii) $\Rightarrow$ (i) if we replace $\mathcal{R}$ by its anti-space $-\mathcal{R}$, interchange $\mathcal{R}_1$ and $\mathcal{R}_2$, and also interchange $\mathcal{Z}$ and $\mathcal{Z}^{[1]}$. □
A.3. Operators in Kreĭn Spaces (Jan 09, 2016)

A.3.1. Some basic results from operator theory.

A.3.1. Lemma. Let \( A \in \mathcal{ML}(\mathcal{X}) \) be a multi-valued linear operator in an H-space \( \mathcal{X} \). Then the following conditions are equivalent:

(i) \( A \in \mathcal{B}(\mathcal{X}) \);
(ii) \( A \) is closed and densely defined and \( (\lambda - A) \) is analytic at infinity (in particular, \( \rho(A) \) contains a neighborhood of infinity).
(iii) \( (\lambda - A) \) is analytic at infinity (in particular, \( \rho(A) \) contains a neighborhood of infinity), \( \lim_{\lambda \to \infty} (\lambda - A)^{-1} = 0 \), and \( \lim_{\lambda \to \infty} \lambda (\lambda - A)^{-1} = 1_{\mathcal{X}} \).

Proof. (i) \( \Rightarrow \) (ii): This follows from Lemma 3.1.2.
(ii) \( \Rightarrow \) (iii): Let \( x \in \text{dom}(A) \) and \( \lambda, \mu \in \rho(A), \lambda \neq 0 \neq \mu \). Then by the resolvent identity (5.2.4),
\[
(\lambda - A)^{-1} x = (\lambda - A)^{-1}(\mu - A)^{-1}(\mu - A)x = \frac{1}{\mu - \lambda}((\lambda - A)^{-1} - (\mu - A)^{-1})(\mu - A)x.
\]
Here the factor \( \frac{1}{\mu - \lambda} \) tends to zero and the rest stays bounded as \( \lambda \to \infty \), and therefore the right-hand side tends to zero as \( \lambda \to \infty \). Since this is true for all \( x \in \text{dom}(A) \) and since \( \text{dom}(A) \) is dense in \( \mathcal{X} \) (and the limit \( \lim_{\lambda \to \infty} (\lambda - A)^{-1} \) exists in \( \mathcal{B}(\mathcal{X}) \) due to the analyticity assumption) this implies that \( \lim_{\lambda \to \infty} (\lambda - A)^{-1} = 0 \). By multiplying the same identity with \( \lambda \) and letting \( \lambda \to \infty \) we get for all \( x \in \text{dom}(A) \) and \( \mu \in \rho(\lambda) \)
\[
\lim_{\lambda \to \infty} \lambda (\lambda - A)^{-1} x = (\mu - A)^{-1}(\mu - A)x = x.
\]
As above this implies that \( \lim_{\lambda \to \infty} \lambda (\lambda - A)^{-1} = 1_{\mathcal{X}} \) (by the analyticity assumption this limit exists in \( \mathcal{B}(\mathcal{X}) \)).

(ii) \( \Rightarrow \) (iii): Suppose that (iii) holds. Define \( F(z) = (1/z - A)^{-1} \) for \( z \neq 0 \), \( 1/z \in \rho(A) \), and \( F(0) = 0 \). Then it follows from (ii) that \( F \) is analytic at zero. Since \( F(0) = 0 \), also the function \( G \) defined by \( G(z) = 1/z F(z) = (1_{\mathcal{X}} - z A)^{-1} \), \( z \neq 0, 1/z \in \rho(A) \), and \( G(0) = 1_{\mathcal{X}} \) is analytic at zero. In particular, \( G \) is continuous at zero, and \( G(z) \to 1_{\mathcal{X}} \) as \( z \to 0 \). This implies that for all \( z \) sufficiently close to zero the operator \( G(z) \) has a bounded inverse. Consequently, \( \text{ker}(G(z)) = \{0\} \) and \( \text{rng}(G(z)) = \mathcal{X} \) for all sufficiently large \( |z| \). If we here replace \( z \) by \( 1/\lambda \) we find that \( \text{ker}((\lambda - A)^{-1}) = \{0\} \) and \( \text{rng}((\lambda - A)^{-1}) = \mathcal{X} \) for all sufficiently large \( |\lambda| \). By [5.2.3], \( \text{mul}(A) = \text{ker}((\lambda - A)^{-1}) = \{0\} \) and \( \text{dom}(A) = \text{rng}((\lambda - A)^{-1}) = \mathcal{X} \). Thus \( A \) is a closed operator with domain \( \mathcal{X} \). By the closed graph theorem \( A \) is bounded. \( \square \)
A.3.2. Analytic vector bundles and analytic operator-valued functions.

A.3.2. Definition. Let $X$ be an $H$-space.

(i) By a vector bundle in $X$ we mean a family of subspaces $\mathcal{G} = \{G(\lambda)\}_{\lambda \in \text{dom}(\mathcal{G})}$ of $X$ parameterized by a complex parameter $\lambda \in \text{dom}(\mathcal{G}) \subset \mathbb{C} \cup \{\infty\}$.

(ii) For each $\lambda \in \text{dom}(\mathcal{G})$, the subspace $G(\lambda)$ of $X$ is called the fiber of $\mathcal{G}$ at $\lambda$.

(iii) The vector bundle $\mathcal{G}$ is analytic at a point $\lambda_0 \in \text{dom}(\mathcal{G}) \cap \mathbb{C}$ if there exists a Hilbert space $Y$ and an analytic $B(U, X)$-valued function $F$ with an analytic $B(X, U)$-valued left-inverse, both of which are defined in some neighborhood $O(\lambda_0)$ of $\lambda_0$, such that

\begin{equation}
\mathcal{G}(\lambda) = \text{rng}(F(\lambda)), \quad \lambda \in O(\lambda_0).
\end{equation}

(iv) The vector bundle $\mathcal{G}$ is analytic at infinity if it is possible to define $\mathcal{G}(\infty)$ in such a way that the bundle $\lambda \mapsto \mathcal{G}(1/\lambda)$ is analytic at zero.

(v) The vector bundle $\mathcal{G}$ is analytic if dom $(\mathcal{G})$ is open and $\mathcal{G}$ is analytic at every point in dom $(\mathcal{G})$.

A.3.3. Lemma. For each vector bundle $\mathcal{G}$ in an $H$-space $X$ and for each $\lambda_0 \in \text{dom}(\mathcal{G}) \cap \mathbb{C}$ the following conditions are equivalent:

(i) $\mathcal{G}$ is analytic at $\lambda_0$.

(ii) $\mathcal{G}(\lambda_0)$ is closed in $X$, and for each direct complement $Z$ to $\mathcal{G}(\lambda_0)$ there exists an analytic $B(\mathcal{G}(\lambda_0); Z)$-valued function $H$ defined in some neighborhood $O(\lambda_0)$ of $\lambda_0$ with $H(\lambda_0) = 0$ such that $G(\lambda)$ is the graph of $H(\lambda)$ for all $\lambda \in O(\lambda_0)$, i.e.,

\begin{equation}
\mathcal{G}(\lambda) = \left\{ k \in X \mid P^Z_{\mathcal{G}(\lambda_0)}k = H(\lambda)P^Z_{\mathcal{G}(\lambda_0)}k \right\}.
\end{equation}

(iii) There exists a Hilbert space $Y$ and an analytic $B(X, Y)$-valued function $K$ with an analytic $B(Y, X)$-valued right-inverse, both of which are defined in some neighborhood $O(\lambda_0)$ of $\lambda_0$, such that

\begin{equation}
\mathcal{G}(\lambda) = \ker(K(\lambda)), \quad \lambda \in O(\lambda_0).
\end{equation}

Proof. (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii): Suppose that (ii) holds. Then both $\mathcal{G}(\lambda_0)$ and $Z$ are vector spaces with Hilbert space topologies (since they are closed in $X$), and by fixing some Hilbert space inner products in $\mathcal{G}(\lambda_0)$ and $Z$ we may interpret them as Hilbert spaces. Denote the imbedding operators $\mathcal{G}(\lambda_0) \hookrightarrow X$ and $Z \hookrightarrow X$ by $I_{\mathcal{G}(\lambda_0)}$ and $I_Z$. For each $\lambda \in O(\lambda_0)$ we may define $F(\lambda) \in B(\mathcal{G}(\lambda_0); X)$ and $K(\lambda) \in B(X; Z)$ by

\begin{equation}
F(\lambda) := I_{\mathcal{G}(\lambda_0)} + I_Z H(\lambda),
\end{equation}

\begin{equation}
K(\lambda) := P^Z_{\mathcal{G}(\lambda_0)} - H(\lambda)P^Z_{\mathcal{G}(\lambda_0)}, \quad \lambda \in O(\lambda_0).
\end{equation}

Then $G(\lambda) = \text{rng}(F(\lambda)) = \ker(K(\lambda))$ for all $\lambda \in O(\lambda_0)$. The operator $F(\lambda)$ has the constant (hence analytic) left-inverse $P^Z_{\mathcal{G}(\lambda_0)}$, and the operator $K(\lambda)$ has the constant (hence analytic) right-inverse $I_Z$. Thus (i) implies both (i) and (iii).

(i) $\Rightarrow$ (ii): Assume that $G(\lambda)$ has a representation of the type described in part (iii) of Definition A.3.2. The operator $F(\lambda_0)$ maps $U$ one-to-one onto $\mathcal{G}(\lambda_0)$, and it follows from part (iii) of Definition A.3.2 that this operator has a bounded inverse,
which we may denote by $F(\lambda_0)^{-1}$. In particular, this implies that $\mathcal{G}(\lambda_0)$ is closed in $\mathcal{X}$.

For each $\lambda \in \mathcal{O}(\lambda_0)$ we may write $F(\lambda)$ in the form $F(\lambda) = \mathcal{I}_{\mathcal{G}(\lambda_0)} F_1(\lambda) + \mathcal{I}_Z F_2(\lambda)$, where $F_1(\lambda) = P_{\mathcal{G}(\lambda_0)}^Z F(\lambda)$ maps $\mathcal{U}$ into $\mathcal{G}(\lambda_0)$ and $F_2(\lambda) = P_Z^2 F(\lambda)$ maps $\mathcal{U}$ into $\mathcal{Z}$. Since $F_1(\lambda_0) = F(\lambda_0)$ has a continuous inverse and $F_1(\lambda)$ depends continuously on $\lambda$, it follows that for all $\lambda$ in possibly some smaller neighborhood $\mathcal{O}'(\lambda_0)$ of $\lambda_0$ the function $F_1(\lambda)$ has a bounded inverse $F_1(\lambda)^{-1}$: $\mathcal{G}(\lambda_0) \to \mathcal{U}$ which is an analytic function of $\lambda$. For all $\lambda \in \mathcal{O}'(\lambda_0)$ we have

$$\mathcal{G}(\lambda) = \ker (F(\lambda)) = \ker \left( \mathcal{I}_{\mathcal{G}(\lambda_0)} F_1(\lambda) + \mathcal{I}_Z F_2(\lambda) \right)$$

This is a graph representation of the type described in (ii) with $H(\lambda) = F_2(\lambda) F_1(\lambda)^{-1}$.

(iii) $\Rightarrow$ (ii): Assume that $\mathcal{G}(\lambda)$ has a representation of the type described in (iii). Since $\ker (K(\lambda_0)) = \mathcal{G}(\lambda_0)$ it follows that $\mathcal{G}(\lambda_0)$ is a closed subspace of $\mathcal{X}$ and that $K(\lambda_0)$ maps $\mathcal{Z}$ one-to-one onto $\mathcal{Y}$, and hence $K(\lambda_0)|_\mathcal{Z}$ has a bounded inverse $(K(\lambda_0)|_\mathcal{Z})^{-1}$ mapping $\mathcal{Y}$ one-to-one onto $\mathcal{Z}$.

The operator $K(\lambda)$ may be split into $K(\lambda) = K_1(\lambda) P_{\mathcal{G}(\lambda_0)}^Z + K_2(\lambda) P_Z \mathcal{G}(\lambda_0)$, where $K_1(\lambda) = K(\lambda)|_{\mathcal{G}(\lambda_0)}$ and $K_2(\lambda) = K(\lambda)|_\mathcal{Z}$. Since $K_2(\lambda_0) = K(\lambda_0)|_\mathcal{Z}$ has a continuous inverse and $K_2(\lambda)$ depends continuously on $\lambda$, it follows that for all $\lambda$ in possibly some smaller neighborhood $\mathcal{O}'(\lambda_0)$ of $\lambda_0$ the function $K_2(\lambda)$ has a bounded inverse $K_2(\lambda)^{-1}$: $\mathcal{Y} \to \mathcal{Z}$ which is an analytic function of $\lambda$. For all $\lambda \in \mathcal{O}'(\lambda_0)$ we have

$$\mathcal{G}(\lambda) = \ker (K(\lambda)) = \ker \left( K_1(\lambda) P_{\mathcal{G}(\lambda_0)}^Z + K_2(\lambda) P_Z \mathcal{G}(\lambda_0) \right)$$

This is a graph representation of the type described in (ii) with $H(\lambda) = -K_2(\lambda)^{-1} K_1(\lambda)$.

\begin{proof}
(i) Each fiber of an analytic vector bundle is closed.
(ii) Let $V$ be a subspace of $\mathcal{X}$, and let $\mathcal{G}$ be the constant bundle $\mathcal{G}(\lambda) = V$, $\lambda \in \mathbb{C} \cup \{\infty\}$. Then $\mathcal{G}$ is analytic in $\mathbb{C} \cup \{\infty\}$ if and only if $V$ is closed.

(iii) The graph of an analytic $\mathcal{B}(\mathcal{X}; \mathcal{Y})$-valued function is an analytic vector bundle in $\{ \mathcal{Y} \}_{\mathcal{X}}$.

By the kernel representation (A.3.3) (the kernel of a bounded operator is closed).

If $G$ is an analytic bundle, then $V$ must be closed according to (i). Conversely, if $V$ is closed, then we may choose the function $F$ in the range representation (A.3.1) to be the identity function on $V$.

(iii) The graph of an $\mathcal{B}(\mathcal{X}; \mathcal{Y})$-valued function $F$ the analytic range representation $\text{gph}(F(\lambda)) = \text{rng} \left( \begin{bmatrix} F(\lambda) \\ 1_{\mathcal{Y}} \end{bmatrix} \right)$, where $\begin{bmatrix} F(\lambda) \\ 1_{\mathcal{X}} \end{bmatrix}$ as the left-inverse $\begin{bmatrix} 0 & 1_{\mathcal{X}} \end{bmatrix}$. \hfill \Box

Note, however, that neither the kernel nor the range of an analytic function need be an analytic bundle. This is clear already from the scalar case, since the dimension of the kernel and the range of an analytic function is not constant in general.
A.3.5. Lemma. For each analytic vector bundle $\mathfrak{G}$ in a Krein space $\mathcal{X}$ the bundle $\lambda \mapsto \mathfrak{G}(\lambda)^*$ is another analytic vector bundle in $\mathcal{X}$.

**Proof.** The easy proof is left to the reader. □

A.3.6. Lemma. Let $G$ be an analytic $\mathcal{B}(\mathcal{X}; \mathcal{Z})$-valued function defined on some open connected set $\Omega \subset \mathbb{C}$, let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\Omega$, and let $\lambda_0 \in \Omega$. Then

\[
\bigcap_{\lambda \in \Omega} \ker (G(\lambda)) = \bigcap_{\lambda \in \Omega'} \ker (G(\lambda)) = \bigcap_{n \in \mathbb{Z}^+} \ker \left( G^{(n)}(\lambda_0) \right),
\]

\[
\bigvee_{\lambda \in \Omega} \text{rng} \,(G(\lambda)) = \bigvee_{\lambda \in \Omega'} \text{rng} \,(G(\lambda)) = \bigvee_{n \in \mathbb{Z}^+} \text{rng} \,(G^{(n)}(\lambda_0)).
\]

**Proof.** For each $x \in \mathcal{X}$, the function $\lambda \mapsto G(\lambda)x$ is analytic, so it vanishes everywhere in $\Omega$ if and only if it vanishes at every point in $\Omega'$, or equivalently, it vanishes together with all its derivatives at the point $\lambda_0$. This proves (A.3.5).

Formula (A.3.6) follows from (A.3.5) by duality: After fixing some Hilbert space inner products in $\mathcal{X}$ and $\mathcal{Z}$ we may apply (A.3.5) to the analytic function $\lambda \mapsto G(\lambda)^*$ defined on $\Omega^* = \{ \lambda \in \mathbb{C} \mid \overline{\lambda} \in \Omega \}$ and use the fact that

\[
\bigvee_{\lambda \in \Omega} \text{rng} \,(G(\lambda)) = \left( \bigcap_{\lambda \in \Omega} \ker (G(\lambda)^*) \right)^\perp = \left( \bigcap_{\lambda \in \Omega^*} \ker (G(\lambda)^*) \right)^\perp.
\]

□

A.3.7. Lemma. Let $F$ and $G$ be two $\mathcal{B}(\mathcal{U}; \mathcal{X})$-valued respectively $\mathcal{B}(\mathcal{X}; \mathcal{Y})$-valued analytic functions, let $\Omega$ be an open connected subset of $\text{dom}(F) \cap \text{dom}(G)$, and let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\Omega$. Then

\[
\text{rng} \,(F(\lambda)) \subset \ker (G(\lambda)) \text{ for all } \lambda \in \Omega \text{ if and only if } \text{rng} \,(F(\lambda)) \subset \ker (G(\lambda)) \text{ for all } \lambda \in \Omega'.
\]

**Proof.** The “only if” part of (i) is obvious. For the converse part we notice that for each $\lambda \in \Omega$ we have $\text{rng} \,(F(\lambda)) \subset \ker (G(\lambda))$ if and only if $G(\lambda)F(\lambda) = 0$. Consequently, if $\text{rng} \,(F(\lambda)) \subset \ker (G(\lambda))$ for all $\lambda \in \Omega'$, then $G(\lambda)F(\lambda) = 0$ for all $\lambda \in \Omega'$. Since the function $\lambda \mapsto G(\lambda)F(\lambda)$ is analytic, this implies that $G(\lambda)F(\lambda) = 0$ for all $\lambda \in \Omega$, i.e., $\text{rng} \,(F(\lambda)) \subset \ker (G(\lambda))$ for all $\lambda \in \Omega$. □

A.3.8. Lemma. Let $\mathfrak{G}$ be an analytic vector bundle in $\mathcal{X}$, let $F$ and $G$ be two $\mathcal{B}(\mathcal{U}; \mathcal{X})$-valued respectively $\mathcal{B}(\mathcal{X}; \mathcal{Y})$-valued analytic functions, let $\Omega$ be an open connected subset of $\text{dom}(\mathfrak{G}) \cap \text{dom}(F) \cap \text{dom}(G)$, and let $\Omega'$ be an arbitrary subset of $\Omega$ which has a cluster point in $\Omega$. Then

(i) $\mathfrak{G}(\lambda) \subset \ker (G(\lambda))$ for all $\lambda \in \Omega$ if and only if $\mathfrak{G}(\lambda) \subset \ker (G(\lambda))$ for all $\lambda \in \Omega'$, and

(ii) $\text{rng} \,(F(\lambda)) \subset \mathfrak{G}(\lambda)$ for all $\lambda \in \Omega$ if and only if $\text{rng} \,(F(\lambda)) \subset \mathfrak{G}(\lambda)$ for all $\lambda \in \Omega'$.

**Proof.** (i) The “only if” part of (i) is obvious. In order to prove the “if” part of (i) we first prove the following claim:

(A) Every point $\lambda_0$ in $\Omega$ contains a neighborhood $\Omega(\lambda_0) \subset \Omega$ with the following property: If $\Omega$ is an arbitrary subset of $\Omega(\lambda_0)$ which has a cluster point in $\Omega(\lambda_0)$, then $\mathfrak{G}(\lambda) \subset \ker (G(\lambda))$ for all $\lambda \in \Omega$ if and only if $\mathfrak{G}(\lambda) \subset \ker (G(\lambda))$ for all $\lambda \in \Omega'$. 

...
That this claim is true can be seen as follows: For each \( \lambda_0 \in \Omega \) we may choose some range representation of \( \mathfrak{G} \) of the type \( \text{[A.3.1]} \), i.e., we let \( \mathcal{U}(\lambda_0) \subset \Omega \) be some connected neighborhood of \( \lambda_0 \) such that \( \mathfrak{G}(\lambda) = \text{rng}(H(\lambda)) \) for all \( \lambda \in \mathcal{U}(\lambda_0) \) where \( H \) is some \( \mathcal{B}(\mathbb{Z}; \mathcal{X}) \)-valued analytic function on \( \mathcal{U}(\lambda_0) \). Then by Lemma \( \text{[A.3.7]} \) if \( \mathfrak{G}(\lambda) \subset \ker(G(\lambda)) \) for all \( \lambda \in \Omega' \), or equivalently, if \( \text{rng}(H(\lambda)) \subset \ker(G(\lambda)) \) for all \( \lambda \in \Omega' \), then \( \text{rng}(H(\lambda)) \subset \ker(G(\lambda)) \) for all \( \lambda \in \Omega \), or equivalently, \( \mathfrak{G}(\lambda) \subset \ker(G(\lambda)) \) for all \( \lambda \in \Omega' \). Thus claim (A) holds.

Let \( \Omega^0 \) be the set of all points \( \lambda \in \Omega_0 \) for which it is true that \( \mathfrak{G}(\lambda) \subset \ker(G(\lambda)) \) for all \( \lambda \) in some neighborhood \( \mathcal{U}(\lambda_0) \) of \( \lambda_0 \). This set is nonempty, since \( \Omega' \) contains a cluster point, and we can apply claim (A) with \( \lambda_0 \) equal to this cluster point. The set \( \Omega^0 \) is obviously open. We claim that it is also closed in \( \Omega \). Let \( \lambda_n \in \Omega^0 \) tend to \( \lambda_0 \) in \( \Omega \) as \( n \to \infty \). Then \( \mathfrak{G}(\lambda_n) \subset \ker(G(\lambda_n)) \) for all \( \lambda_n \), and it follows from claim (A) that \( \lambda_0 \in \Omega^0 \). Thus, \( \Omega^0 \) is nonempty and both open and closed in \( \Omega \), and since \( \Omega \) is connected, this implies that \( \Omega^0 = \Omega \).

(ii) The “only if” part of (ii) is obvious. In order to prove the “if” part of (ii) we first prove the following analogue of claim (A):

\( \text{(B) Every point } \lambda_0 \text{ in } \Omega \text{ contains a neighborhood } \Omega(\lambda_0) \subset \Omega \text{ with the following property: If } \Omega' \text{ is an arbitrary subset of } \Omega(\lambda_0) \text{ which has a cluster point in } \Omega(\lambda_0), \text{ then } \text{rng}(F(\lambda)) \subset \mathfrak{G}(\lambda) \text{ for all } \lambda \in \Omega \text{ if and only if } \text{rng}(F(\lambda)) \subset \mathfrak{G}(\lambda) \text{ for all } \lambda \in \Omega'.} \)

The proof of claim (B) is analogous to the proof of claim (A) above, with the range representation of \( \mathfrak{G} \) replaced by a kernel representation of \( \mathfrak{G} \) of the type \( \text{[A.3.3]} \). Once this claim has been established the proof can be completed as in the proof of (i) above. \( \square \)

A.3.9. Lemma. Let \( \mathfrak{F} \) and \( \mathfrak{G} \) be two analytic vector bundles in \( \mathcal{X} \), let \( \Omega \) be an open connected subset of \( \text{dom}(\mathfrak{F}) \cap \text{dom}(\mathfrak{G}) \), and let \( \Omega' \) be an arbitrary subset of \( \Omega \) which has a cluster point in \( \Omega \). Then

(i) \( \mathfrak{F}(\lambda) \subset \mathfrak{G}(\lambda) \text{ for all } \lambda \in \Omega \text{ if and only if } \mathfrak{F}(\lambda) \subset \mathfrak{G}(\lambda) \text{ for all } \lambda \in \Omega', \) and

(ii) \( \mathfrak{F}(\lambda) = \mathfrak{G}(\lambda) \text{ for all } \lambda \in \Omega \text{ if and only if } \mathfrak{F}(\lambda) = \mathfrak{G}(\lambda) \text{ for all } \lambda \in \Omega'. \)

Proof. (i) The “only if” part of (i) is obvious. In order to prove the converse claim we first prove the following claim:

(A) Every point \( \lambda_0 \) in \( \Omega \) contains a neighborhood \( \Omega(\lambda_0) \subset \Omega \) with the following property: If \( \Omega' \) is an arbitrary subset of \( \Omega(\lambda_0) \) which has a cluster point in \( \Omega(\lambda_0) \), then \( \mathfrak{F}(\lambda) \subset \mathfrak{G}(\lambda) \) for all \( \lambda \in \Omega \) if and only if \( \mathfrak{F}(\lambda) \subset \mathfrak{G}(\lambda) \) for all \( \lambda \in \Omega' \).

To see that this is true we may choose some range representation of \( \mathfrak{F} \) of the type \( \text{[A.3.1]} \), i.e., we let \( \mathcal{U}(\lambda_0) \subset \Omega \) be some connected neighborhood of \( \lambda_0 \) such that \( \mathfrak{F}(\lambda) = \text{rng}(F(\lambda)) \) for all \( \lambda \in \mathcal{U}(\lambda_0) \) where \( F \) is some analytic function on \( \mathcal{U}(\lambda_0) \). Then by Lemma \( \text{[A.3.8]} \) if \( \mathfrak{F}(\lambda) \subset \mathfrak{G}(\lambda) \) for all \( \lambda \in \Omega' \), or equivalently, if \( \text{rng}(F(\lambda)) \subset \mathfrak{G}(\lambda) \) for all \( \lambda \in \Omega' \), then \( \text{rng}(F(\lambda)) \subset \mathfrak{G}(\lambda) \) for all \( \lambda \in \Omega \), or equivalently, \( \mathfrak{F}(\lambda) \subset \mathfrak{G}(\lambda) \) for all \( \lambda \in \Omega' \). Thus claim (A) holds.

Since \( \Omega \) is connected, it follows from (A) that (i) holds. (ii) The claim (ii) follows immediately from (i). \( \square \)

A.3.10. Lemma. Let \( \mathfrak{G} \) be an analytic bundle in \( \mathcal{X} \), where \( \mathcal{X} = \mathbb{Z} + \mathcal{Y} \), let \( \Omega \) be an open connected subset of \( \text{dom}(\mathfrak{G}) \), and let \( \Omega' \) be an arbitrary subset of \( \Omega \) which
contains a cluster point in $\Omega$. Then
\begin{equation}
\bigcap_{\lambda \in \Omega} \mathcal{G}(\lambda) \cap Z = \bigcap_{\lambda \in \Omega'} \mathcal{G}(\lambda) \cap Z,
\end{equation}
\begin{equation}
\bigvee_{\lambda \in \Omega} P_\mathcal{G}(\lambda) = \bigvee_{\lambda \in \Omega'} P_\mathcal{G}(\lambda).
\end{equation}

**Proof.** We begin with the proof of (A.3.7). Clearly $\bigcap_{\lambda \in \Omega} \mathcal{G}(\lambda) \cap Z \subseteq \bigcap_{\lambda \in \Omega'} \mathcal{G}(\lambda) \cap Z$. To prove the opposite inclusion we let $Z_0 := \bigcap_{\lambda \in \Omega} \mathcal{G}(\lambda) \cap Z$. Then $Z_0$ is closed, $Z_0 \subseteq Z$, and $Z_0 \subseteq \bigcap_{\lambda \in \Omega} \mathcal{G}(\lambda) \cap Z$ if and only if $Z_0 \subseteq \mathcal{G}(\lambda)$ for all $\lambda \in \Omega$. By the definition of $Z_0$, $Z_0 \subseteq \mathcal{G}(\lambda)$ for all $\lambda \in \Omega'$, and therefore by Lemma A.3.9 with $\mathcal{F}$ equal to the constant bundle $\mathcal{F}(\lambda) := Z_0$, $\lambda \in \Omega$ (which is analytic since $Z_0$ is closed), we have $Z_0 \subseteq \mathcal{G}(\lambda)$ for all $\lambda \in \Omega$. This completes the proof of (A.3.7).

We next turn to the proof of (A.3.8). It is clear that $\bigvee_{\lambda \in \Omega'} P_\mathcal{G}(\lambda) \subseteq \bigvee_{\lambda \in \Omega} P_\mathcal{G}(\lambda)$. To prove the converse inclusion we start by defining $\mathcal{Y}_0 := \bigvee_{\lambda \in \Omega'} P_\mathcal{G}(\lambda)$. Then $\mathcal{Y}_0 \subseteq \mathcal{Y}$, and $\bigvee_{\lambda \in \Omega} P_\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0$ if and only if $P_\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0$ for all $\lambda \in \Omega$. Let $\mathcal{Y}_1$ be a direct complement to $\mathcal{Y}_0$ in $\mathcal{Y}$. Then $P_\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0$ if and only if \{0\} = $P_{\mathcal{Y}_1} P_\mathcal{G}(\lambda) = P_{\mathcal{Y}_1} + \mathcal{Z}(\lambda)$, or equivalently, $\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0 + \mathcal{Z}$. By the definition of $\mathcal{Y}_0$, $\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0 + \mathcal{Z}$ for all $\lambda \in \Omega'$, and therefore by Lemma A.3.9 with $\mathcal{G}$ equal to the constant bundle $\mathcal{G}(\lambda) := \mathcal{Y}_0 + \mathcal{Z}$, $\lambda \in \Omega$ (which is analytic since $\mathcal{Y}_0 + \mathcal{Z}$ is closed), we have $\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0 + \mathcal{Z}$ for all $\lambda \in \Omega$, or equivalently, $P_\mathcal{G}(\lambda) \subseteq \mathcal{Y}_0$. This completes the proof of (A.3.7). □
A.4. Linear Multi-Valued Operators in $H$-Spaces (Jan 09, 2016)

A.4.1. Basic definitions. Let $X$ and $Y$ be two $H$-spaces. The class of linear multi-valued operators from $X$ to $Y$ can be most easily described in terms of their graphs. More precisely, there is a one-to-one correspondence between the set of all (linear) subspaces $V$ of the product space $X \times Y$ and the set of all linear multi-valued operators $A : X \to Y$. If $V$ is such a subspace, then we define the corresponding linear multi-valued operator $A$ to be the multi-valued mapping from $X$ to $Y$ whose domain is given by

$$\text{dom}(A) = \{ x \in X \mid [y] \in V \text{ for some } y \in Y \},$$

and which maps a vector $x \in \text{dom}(A)$ into the non-empty affine subspace

$$Ax := \{ y \in Y \mid [y] \in V \}.$$

If $V$ and $A$ are related in this way, then we call $V$ the graph of $A$ and denote it by $\text{gph}(A)$.\footnote{We remark that many authors prefer to replace the notion “linear multi-valued operator from $X$ to $Y$” by the notion “relation from $X$ to $Y$”. Such a relation is simply the graph of a multivalued operator $A : X \to Y$ (i.e., a relation from $X$ to $Y$ can be identified with a subspace $V$ of $X \times Y$). Thus, there is a one-to-one correspondence of all relations $X \to Y$ and all linear multi-valued operators $X \to Y$.}

Thus,

$$\text{gph}(A) = \{ [y] \mid y \in Ax \}.$$  

We shall throughout use the notation “$y \in Ax$” as a synonym to the notation $[y] \in \text{gph}(A)$, so that in particular, “$y \in Ax$” implies that $x \in \text{dom}(A)$. In particular, we can write

$$\text{dom}(A) = \{ x \in X \mid [y] \in \text{gph}(A) \text{ for some } y \in Y \} = \{ x \in X \mid y \in Ax \text{ for some } y \in Y \}.$$  

Since we shall not encounter non-linear multi-valued operators in this book we shall often drop the attribute “linear” and simply call this class of operator “multi-valued operators from $X$ to $Y$.

The kernel $\ker(A)$, range $\text{rng}(A)$, and multi-valued part $\text{mul}(A)$ of $A$ are given by

$$\ker(A) := \{ x \in X \mid [0] \in \text{gph}(A) \} = \{ x \in X \mid 0 \in Ax \},$$

$$\text{rng}(A) := \{ y \in Y \mid [y] \in \text{gph}(A) \text{ for some } x \in X \},$$

$$\text{mul}(A) := \{ y \in Y \mid y \in Ax \text{ for some } x \in X \},$$

$$\text{mul}(A) := \{ y \in Y \mid [y] \in \text{gph}(A) \} = \text{ker}.$$  

It is easy to see that for each $x \in \text{dom}(A)$ the set $Ax$ is a nonempty affine subspace of the form $Ax = y_1 + \text{mul}(A) = \{ y_1 + y_0 \mid y_0 \in \text{mul}(A) \}$, where $y_1$ is an arbitrary vector in $Ax$. We call $A$ single-valued if $\text{mul}(A) = \{ 0 \}$, and in this case we may identify $A$ with an ordinary linear operator $X \to X$. We call $A$ injective if $\ker(A) = \{ 0 \}$, i.e., if $Ax_1 \neq Ax_2$ whenever $x_1 \neq x_2$. The graph of $A$ can be recovered from $A$ as

$$\text{gph}(A) = \text{rng} \left( \begin{bmatrix} A \\ 1_x \end{bmatrix} \right) = \ker \left( \begin{bmatrix} -1 & A \end{bmatrix} \right),$$
where \( \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \) is interpreted as a multi-valued operator \( \mathcal{X} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{Y} \end{bmatrix} \) with domain \( \text{dom}(A) \) and \( \begin{bmatrix} -1_y \\ A \end{bmatrix} \) is interpreted as a multi-valued operator \( \begin{bmatrix} \mathcal{Y} \\ \mathcal{X} \end{bmatrix} \to \mathcal{Y} \) with domain \( \begin{bmatrix} \mathcal{Y} \\ \text{dom}(A) \end{bmatrix} \).

The multi-valued operator \( A \) is called \textit{closed} if its graph \( \text{gph}(A) \) is closed. If \( A \) is closed, then both \( \text{ker}(A) \) and \( \text{mul}(A) \) are closed, but neither \( \text{dom}(A) \) nor \( \text{rng}(A) \) need be closed. The \textit{closure} of a the multi-valued operator \( A : \mathcal{X} \to \mathcal{Y} \) is the multi-valued operator \( \overline{A} : \mathcal{X} \to \mathcal{Y} \) whose graph is the closure of \( \text{gph}(A) \). Note that if \( A \) is a single-valued linear operator, then \( A \) is closable in the standard operator theory sense if an only if the (multi-valued) closure of \( A \) is single-valued.

The \textit{inverse} of a the multi-valued operator \( A : \mathcal{X} \to \mathcal{Y} \) is the multi-valued operator \( A^{-1} : \mathcal{Y} \to \mathcal{X} \) whose graph is given by

\[
\text{gph}(A^{-1}) := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \mid y \in \text{gph}(A) \right\} = \begin{bmatrix} 0 & 1_y \\ 1_y & 0 \end{bmatrix} \text{gph}(A).
\]

Equivalently, \( x \in A^{-1}y \) if and only if \( y \in Ax \). Clearly,

\[
\text{dom}(A^{-1}) = \text{rng}(A), \quad \text{rng}(A^{-1}) = \text{dom}(A),
\]

\[
\text{ker}(A^{-1}) = \text{mul}(A), \quad \text{mul}(A^{-1}) = \text{ker}(A).
\]

For each \( \lambda \in \mathbb{C} \) we let \( \lambda A \) be the multi-valued operator whose graph is

\[
\text{gph}(\lambda A) := \left\{ \begin{bmatrix} \lambda y \\ x \end{bmatrix} \mid y \in Ax \right\} = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \text{gph}(A).
\]

The \textit{sum} \( A_1 + A_2 \) of two multi-valued operator \( A_1, A_2 : \mathcal{X} \to \mathcal{Y} \) is the multi-valued operator whose graph is

\[
\text{gph}(A_1 + A_2) := \left\{ \begin{bmatrix} y_1 + y_2 \\ x \end{bmatrix} \mid y_1 \in \text{gph}(A_1) \text{ and } y_2 \in \text{gph}(A_2) \right\}.
\]

In particular, \( \text{dom}(A_1 + A_2) = \text{dom}(A_1) \cap \text{dom}(A_2) \).

The \textit{composition} of two multi-valued operators \( A_1 : \mathcal{X} \to \mathcal{Y} \) and \( A_2 : \mathcal{Y} \to \mathcal{Z} \) is the multi-valued operators \( A_2A_1 \) whose graph is given by

\[
\text{gph}(A_2A_1) := \left\{ \begin{bmatrix} z \\ x \end{bmatrix} \mid z \in A_2y \text{ for some } y \in A_1x \right\} = \left\{ \begin{bmatrix} A_2y \\ x \end{bmatrix} \mid y \in A_1x \right\}.
\]

Note that \( \text{dom}(A_2A_1) = \left\{ x \in \text{dom}(A_1) \mid A_1x \cap \text{dom}(A_2) \neq \emptyset \right\} \). In particular, if \( \text{dom}(A_2) = \mathcal{Y} \), then \( \text{dom}(A_2A_1) = \text{dom}(A_1) \). It is easy to see that \( (A_1A_2)^{-1} = A_2^{-1}A_1^{-1} \). The composition \( A_2A_1 \) need not be closed even if both \( A_1 \) and \( A_2 \) is closed.

In the multi-valued case it is not always true that \( A^{-1}A \) is the identity on \( \text{dom}(A) \). Instead \( y \in A^{-1}Ax \) if and only if \( x \in \text{dom}(A) \) and \( y - x \in \text{ker}(A) \). Thus \( \text{ker}(A^{-1}A) = \text{mul}(A^{-1}A) = \text{ker}(A) = \text{mul}(A^{-1}) \). Observe that \( AA^{-1} = AA^{-1}A^{-1} = A^{-1}A \), and that \( x \in A^{-1}Ax \) for all \( x \in \text{dom}(A) \) and \( y \in AA^{-1}y \) for all \( y \in \text{rng}(A) \).

If \( A : \mathcal{X} \to \mathcal{Y} \) is a multi-valued operator and \( \mathcal{X}_0 \) is a subspace of \( \mathcal{X} \), then the \textit{restriction} \( A|_{\mathcal{X}_0} \) of \( A \) to \( \mathcal{X}_0 \) is the operator whose graph is

\[
\text{gph}(A|_{\mathcal{X}_0}) = \left\{ \begin{bmatrix} y \\ x \end{bmatrix} \in \text{gph}(A) \mid x \in \mathcal{X}_0 \right\} = \text{gph}(A) \cap \begin{bmatrix} \mathcal{Y} \\ \mathcal{X}_0 \end{bmatrix}.
\]

Note that \( \text{dom}(A|_{\mathcal{X}_0}) = \text{dom}(A) \cap \mathcal{X}_0 \). More generally, we call \( A_1 \) a \textit{restriction} of \( A_2 \) if \( \text{gph}(A_1) \subset \text{gph}(A_2) \). In this case we also call \( A_2 \) and \textit{extension} of \( A_1 \).
The common part of $A_1$ and $A_2$, denoted by $A_1 \cap A_2$, is the multi-valued operator whose graph is $\text{gph}(A_1 \cap A_2) = \text{gph}(A_1) \cap \text{gph}(A_2)$ with $\text{dom} (A_1 \cap A_2) = \text{dom} (A_1) \cap \text{dom} (A_2)$.

If $A$ is a multi-valued operator $X \to X$, i.e., if the domain and range space os $A$ are the same space $X$, then we say that $A$ is a multi-valued operator in $X$. In this case we denote by $\lambda - A$ the operator whose graph is given by

$$\text{gph}(\lambda - A) := \left\{ \begin{bmatrix} \lambda x - y \\ x \end{bmatrix} \Bigg| x \in \text{dom} (A) \text{ and } y \in Ax \right\} = \begin{bmatrix} -1 & \lambda \\ 0 & 1 \end{bmatrix} \text{gph}(A).$$

The resolvent set $\rho(A)$ of a multi-valued operator $A$ in $X$ consists of those points $\lambda \in \mathbb{C}$ for which the inverse of the multi-valued operator $(\lambda - A)^{-1}$ is single-valued, continuous, and $\text{dom} ((\lambda - A)^{-1}) = X$. The operator-valued function $\lambda \mapsto (\lambda - A)^{-1}$ defined for all $\lambda \in \rho(A)$ is called the resolvent of $A$. By the closed graph theorem, the above definition of $\rho(A)$ is equivalent to the following characterization: $A$ must be closed in order for $\rho(A)$ to be nonempty, and in this case

$$\lambda \in \rho(A) \text{ if and only if } \ker (\lambda - A) = \{0\} \text{ and } \text{rng} (\lambda - A) = X.$$ 

The spectrum $\sigma(A)$ is the complement of $\rho(A)$. 

A.4.2. The single-valued and injective parts of a multi-valued operator. Let $A : \mathcal{X} \to \mathcal{Y}$ be a closed multi-valued operator. Then $\ker(A)$ is closed and hence complemented in $\mathcal{X}$ and $\mul(A)$ is closed and complemented in $\mathcal{Y}$ (recall that we assume $\mathcal{X}$ and $\mathcal{Y}$ to be $H$-spaces). In this case $A$ has the following simple structure. Denote $\mathcal{X}_0 = \ker(A)$ and $\mathcal{Y}_0 = \mul(A)$. Then $\mathcal{X}_0$ and $\mathcal{Y}_0$ are closed in $\mathcal{X}$ respectively $\mathcal{Y}$. Let $\mathcal{X}_1$ and $\mathcal{Y}_1$ be direct complements to $\mathcal{X}_0$ and $\mathcal{Y}_0$ in $\mathcal{X}$ respectively $\mathcal{Y}$, so that

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_1 = \ker(A) + \mathcal{X}_1, \quad \mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}_1 = \mul(A) + \mathcal{Y}_1.$$  

If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then we may, for example, choose $\mathcal{X}_1 = \mathcal{X}_0^\perp$ and $\mathcal{Y}_1 = \mathcal{Y}_0^\perp$. Define

$$A_{\sing} = P_{\mathcal{Y}_1}^\perp A, \quad A_{\inj} = A|_{\mathcal{X}_1}, \quad A_{\op} := P_{\mathcal{Y}_0}^\perp A|_{\mathcal{X}_1}.$$  

It is easy to see that all of these multi-valued operators are closed, that $A_{\sing}$ and $A_{\op}$ are single-valued, and that $A_{\inj}$ and $A_{\op}$ are injective, with

$$\dom(A_{\sing}) = \dom(A), \quad \ker(A_{\sing}) = \ker(A),$$  

$$\dom(A_{\inj}) = \dom(A_{\op}) = \dom(A) \cap \mathcal{X}_1 = P_{\mathcal{X}_0}^\perp \dom(A),$$  

$$\dom(A) = \dom(A_{\inj}) + \mathcal{X}_0 = \dom(A_{\op}) + \mathcal{X}_0,$$

$$\rng(A_{\inj}) = \rng(A), \quad \mul(A_{\inj}) = \mul(A),$$  

$$\rng(A_{\sing}) = \rng(A_{\op}) = \rng(A) \cap \mathcal{Y}_1 = P_{\mathcal{Y}_0}^\perp \rng(A),$$  

$$\rng(A) = \rng(A_{\sing}) + \mathcal{Y}_0 = \rng(A_{\op}) + \mathcal{Y}_0.$$  

We call $A_{\sing}$ the single-valued part of $A$, $A_{\inj}$ the injective part of $A$, and $A_{\op}$ the single-valued injective part of $A$ with respect to the above decompositions of $\mathcal{X}$ and $\mathcal{Y}$. The graphs of $A_{\sing}$, $A_{\inj}$, and $A$ can be decomposed into

$$\gph(A_{\sing}) = \begin{bmatrix} 0 \\ \mathcal{X}_0^\perp \end{bmatrix} + \gph(A_{\op}),$$

$$\gph(A_{\inj}) = \gph(A_{\op}) + \begin{bmatrix} \mathcal{Y}_0 \\ 0 \end{bmatrix},$$

$$\gph(A) = \begin{bmatrix} 0 \\ \mathcal{X}_0^\perp \end{bmatrix} + \gph(A_{\op}) + \begin{bmatrix} \mathcal{Y}_0 \\ 0 \end{bmatrix}.$$  

If $\dom(A)$ is dense in $\mathcal{X}$ then $\dom(A_{\inj}) = \dom(A_{\op})$ is dense in $\mathcal{X}_1$, and if $\rng(A)$ is dense in $\mathcal{Y}$ then $\rng(A_{\sing}) = \rng(A_{\op})$ is dense in $\mathcal{Y}_1$. If $\dom(A) = \mathcal{X}$, then $\dom(A_{\inj}) = \dom(A_{\op}) = \mathcal{X}_1$, and if furthermore $\mathcal{X}$ and $\mathcal{Y}$ are Fréchet spaces, then $A_{\inj}$ and $A_{\op}$ are continuous (single-valued) linear operators $\mathcal{X}_1 \to \mathcal{Y}$ respectively $\mathcal{X}_1 \to \mathcal{Y}_0$. Since $A_{\op}$ is single-valued and injective, also $A_{\op}^{-1}$ is single-valued and injective, and the graph of $A^{-1}$ is given by

$$\gph(A)^{-1} = \begin{bmatrix} 0 \\ \mathcal{Y}_0 \end{bmatrix} + \gph(A_{\op}^{-1}) + \begin{bmatrix} \mathcal{X}_0 \\ 0 \end{bmatrix}.$$  

Moreover, the inverse of $A_{\sing}$ is the injective part of $A^{-1}$, and the inverse of $A_{\inj}$ is the single-valued part of $A^{-1}$. 
A.4.3. **The adjoint of a multi-valued operator.** Let $A: \mathcal{X} \to \mathcal{Y}$ be a multi-valued operator from the Kreïn space $\mathcal{X}$ to the Kreïn space $\mathcal{Y}$. Then the adjoint of $A$ is the multi-valued operator $A^*: \mathcal{Y} \to \mathcal{X}$ whose graph is given by

$$
\text{gph}(A^*) := \left\{ \begin{bmatrix} -x^\dagger \\ y^\dagger \end{bmatrix} \mid \begin{bmatrix} y^\dagger \\ x^\dagger \end{bmatrix} \in \text{gph}(A) \right\} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{gph}(A).
$$

Equivalently,

$$
x^\dagger \in A^* y^\dagger \text{ if and only if } \begin{bmatrix} x^\dagger \\ y^\dagger \end{bmatrix} \in \text{dom}(A) \quad \text{for all } y^\dagger \in Ax.
$$

In particular,

$$
\text{if } x^\dagger \in A^* y^\dagger \text{ and } y \in Ax \text{ then } [x^\dagger, x]_\mathcal{X} = [y^\dagger, y].
$$

Note that the adjoint is always closed and that $(A^*)^* = A$ whenever $A$ is closed (otherwise $(A^*)^*$ is the closure of $A$). It is easy to check that $\text{mul}(A^*) = \text{dom}(A)^\perp$ and $\text{ker}(A^*) = \text{rng}(A)^\perp$, and if $A$ is closed (so that $(A^*)^* = A$) we also have $\text{mul}(A) = \text{dom}(A^*)^\perp$ and $\text{ker}(A) = \text{rng}(A^*)^\perp$. Moreover, $(A^*)^{-1} = (A^{-1})^*$. We denote $A^{**} = (A^*)^* = (A^{-1})^*$. If $\mathcal{Y} = \mathcal{X}$, then there is a simple connection between the spectra of $A$ and $A^*$. Since $((\lambda - A)^{-1})^* = (\overline{\lambda} - A^*)^{-1}$, we find that

$$
\rho(A^*) = \{ \overline{\lambda} \mid \lambda \in \rho(A) \}, \quad \sigma(A^*) = \{ \overline{\lambda} \mid \lambda \in \sigma(A) \}.
$$
A.5. Notes and Comments (Jan 09, 2016)
Bibliography


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