Frequency Domain Well-Posed Linear Systems

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Outline of Talk

- Time domain well-posed input/state/output systems
- Frequency domain well-posed input/state/output systems
- Intertwinement in time and frequency domain
- Compressions and dilations in time and frequency domain
- Controllability, observability, and minimality
- Work in progress
Outline of Talk

- Time domain well-posed input/state/output systems
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- Work in progress
“Classical” infinite-dimensional i/s/o system

One of the first serious attempts to do infinite-dimensional control theory was to study systems of the type

\[ \Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (1) \]

\( x(t) \in \mathcal{X} \) is the state, \\
\( u(t) \in \mathcal{U} \) is the input, \\
\( y(t) \in \mathcal{Y} \) is the output \\
\( \mathcal{X}, \mathcal{U} \) and \( \mathcal{Y} \) are Hilbert spaces. 

The main operator \( A \) is the generator of a \( C_0 \) semigroup, but 
the control operator \( B \), 
the observation operator \( C \), and 
the feed-through operator \( D \) are all bounded linear operators. 
This class of systems is studied in the book \((CZ95)\).

However, it is not really “good enough” to study boundary control systems.
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“Regular” infinite-dimensional i/s/o systems

One gets a significantly more powerful theory by keeping the same set of equations

\[ \Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \]  

but allowing also \( B \) and \( C \) to be unbounded:

- \( A \) is the generator of a \( C_0 \) semigroup,
- \( C \) maps \( \text{dom}(A) \rightarrow \mathcal{Y} \) (continuous w.r.t. graph norm of \( A \)),
- \( B \) maps \( \mathcal{U} \rightarrow \mathcal{X}_- \), where \( \mathcal{X}_- \) is an “extrapolation space”, which contains \( \mathcal{X} \) as a dense subspace,
- \( D \) maps \( \mathcal{U} \rightarrow \mathcal{Y} \).

This class of systems has been studied in a sequence of papers by George Weiss (the first of these appeared in 1989). (See also \( \text{Sal87} \) and \( \text{Šmu86} \).)

After a small modification (replace “regular” by “compatible”) this becomes a good class for the study of boundary control systems.
One gets a significantly more powerful theory by keeping the same set of equations

$$\Sigma : \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (1)$$

but allowing also $B$ and $C$ to be unbounded:

- $A$ is the generator of a $C_0$ semigroup,
- $C$ maps $\text{dom}(A) \to \mathcal{Y}$ (continuous w.r.t. graph norm of $A$),
- $B$ maps $U \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is an “extrapolation space”, which contains $\mathcal{X}$ as a dense subspace,
- $D$ maps $U \to \mathcal{Y}$.

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This class of systems has been studied in a sequence of papers by George Weiss (the first of these appeared in 1989). (See also (Sal87) and (Šmu86).)

After a small modification (replace “regular” by “compatible”) this becomes a good class for the study of boundary control systems.
In the theory of “regular” and “compatible” systems the definition of the operator feed-through operator $D$ causes some problems. One solution to this problem is to collapse the block matrix operator $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ into one operator, called the system node $S$, and to rewrite (1) in the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (2)$$

A classical trajectory $[x, u]$ of (2) satisfies $x, \dot{x} \in C(\mathbb{R}^+; X)$, $u \in C(\mathbb{R}^+; U)$, and $[x(t), u(t)] \in \text{dom}(S)$ for all $t \in \mathbb{R}^+$. In the regular case the operators $A$, $B$, $C$, and $D$ can be recovered from $S$, but (2) makes sense also without any “regularity” assumptions. Of course, we still need some assumptions on $S$. 
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A classical trajectory $[x \ u]$ of (2) satisfies $x, \dot{x} \in C(\mathbb{R}^+; \mathcal{X})$, $u \in C(\mathbb{R}^+; \mathcal{U})$, and $[x(t) \ u(t)] \in \text{dom} (S)$ for all $t \in \mathbb{R}^+$. In the regular case the operators $A$, $B$, $C$, and $D$ can be recovered from $S$, but (2) makes sense also without any “regularity” assumptions. Of course, we still need some assumptions on $S$. 

Olof Staffans, Åbo Akademi University, Finland Aalto University, Finland Frequency Domain Well-Posed Linear Systems
Definition

By an operator node on a triple of Hilbert spaces \((\mathcal{X}, \mathcal{U}, \mathcal{Y})\) we mean a (possibly unbounded) linear operator \(S: \left[ \mathcal{X} \atop \mathcal{U} \right] \rightarrow \left[ \mathcal{X} \atop \mathcal{Y} \right]\) with the following properties. We denote \( \text{dom} (A) = \{ x \in \mathcal{X} \mid [\check{x}] \in \text{dom} (S) \} \), define \( A: \text{dom} (A) \rightarrow \mathcal{X} \) by \( Ax = P_\mathcal{X} S [\check{x}] \), and require the following conditions to hold:

1. \( S \) is closed as an operator from \( \left[ \mathcal{X} \atop \mathcal{U} \right] \) to \( \left[ \mathcal{X} \atop \mathcal{Y} \right] \) (with domain \( \text{dom} (S) \)).
2. \( P_\mathcal{X} S \) is closed as an operator from \( \left[ \mathcal{X} \atop \mathcal{U} \right] \) to \( \mathcal{X} \) (with domain \( \text{dom} (S) \)).
3. \( \text{dom} (A) \) is dense in \( \mathcal{X} \) and \( \rho (A) \neq \emptyset \).
4. For every \( u \in \mathcal{U} \) there exists a \( x \in \mathcal{X} \) with \( [\check{x}] \in \text{dom} (S) \).

We call \( S \) a system node if, in addition, \( A \) generates a \( C_0 \) semigroup.
Time domain well-posed i/s/o systems

Definition

An i/s/o system $\Sigma = (S; X, U, Y)$, where $S$ is a “system node”, is time-domain well-posed if there exists a nonnegative function $\eta$ such that all classical trajectories $\begin{bmatrix} \dot{x} \\ y \end{bmatrix}$ of $\Sigma$ on $\mathbb{R}^+$ satisfy

$$\|x(t)\|^2_X + \int_0^t \|y(s)\|^2_Y \, ds \leq \eta(t)^2 \left(\|x(0)\|^2_X + \int_0^t \|u(s)\|^2_U \, ds\right), \quad t \in \mathbb{R}^+. $$

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- **Frequency domain well-posed input/state/output systems**
- Intertwinement in time and frequency domain
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- Work in progress
We can rewrite the i/s/o equation

\[
\begin{bmatrix}
\dot{x}(t) \\
y(t) \\
x(t) \\
u(t)
\end{bmatrix}
= S
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{2}
\]

in graph form to get:

\[
\begin{bmatrix}
\dot{x}(t) \\
y(t) \\
x(t) \\
u(t)
\end{bmatrix}
\in \text{graph } (S), \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{3}
\]

\[
\text{graph } (S) := \left\{ \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \middle| \begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \text{dom } (S), \begin{bmatrix} z \\ y \end{bmatrix} = S \begin{bmatrix} x \\ u \end{bmatrix} \right\}. \tag{4}
\]

In this form it does not matter if $S$ is a (single-valued) operator or a multi-valued operator, i.e., a relation.
We can rewrite the i/s/o equation
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\begin{bmatrix}
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\]
(3)
\[
\text{graph } (S) := \left\{ \begin{bmatrix} z \\
y \\
x \\
u \end{bmatrix} \mid \begin{bmatrix} x \\
y \\
x \\
u \end{bmatrix} \in \text{dom } (S), \quad [y] = S [x] \right\}.
\]
(4)
In this form it does not matter if \( S \) is a (single-valued) operator or a multi-valued operator, i.e., a relation.
If $S$ is a relation, then $S$ maps every pair $[\dot{x} \ u]$ into an affine subspace (which may be empty for some $[\dot{x} \ u]$), and the equation

$$
\begin{bmatrix}
\dot{x}(t) \\
y(t)
\end{bmatrix} = S \begin{bmatrix} x(t) \\
u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0,
$$

must be replaced by the inclusion

$$
\begin{bmatrix}
\dot{x}(t) \\
y(t)
\end{bmatrix} \in S \begin{bmatrix} x(t) \\
u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0.
$$

Can we say anything about equations of this type?

Throughout the rest of the talk I assume that $S$ is a closed relation (i.e., the graph of $S$ is a closed subspace of $[X \ U \ Y]$).

For this class of systems I shall not say anything about time domain well-posedness.

Instead I shall look at frequency domain well-posedness.
If $S$ is a relation, then $S$ maps every pair $[\hat{x} \ u]$ into an affine subspace (which may be empty for some $[\hat{x} \ u]$), and the equation

\[
\begin{bmatrix}
\dot{x}(t) \\
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Can we say anything about equations of this type? Throughout the rest of the talk I assume that $S$ is a closed relation (i.e., the graph of $S$ is a closed subspace of $[\hat{x} \ u \ y]$). For this class of systems I shall not say anything about time domain well-posedness. Instead I shall look at frequency domain well-posedness.
If $S$ is a relation, then $S$ maps every pair $[\dot{x} \ u]$ into an affine subspace (which may be empty for some $[\dot{x} \ u]$), and the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0,$$

must be replaced by the inclusion

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0.$$

Can we say anything about equations of this type? Throughout the rest of the talk I assume that $S$ is a closed relation (i.e., the graph of $S$ is a closed subspace of $[\dot{x} \ u]$).

For this class of systems I shall not say anything about time domain well-posedness.

Instead I shall look at frequency domain well-posedness.
By taking (formal) Laplace transforms in the equation

\[
\begin{bmatrix}
\dot{x}(t) \\
y(t)
\end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{5}
\]

and using the fact that \( S \) is closed we get

\[
\begin{bmatrix}
\lambda \hat{x}(\lambda) - x_0 \\
\hat{y}(\lambda)
\end{bmatrix} \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}. \tag{6}
\]

**Definition**

The system (5) is frequency domain well-posed if there exists at least one \( \lambda \in \mathbb{C} \) such that the equation (6) defines a bounded linear everywhere defined map from \( \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} \) to \( \begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} \).

(In the time-domain well-posed case this condition will be true for all \( \lambda \) in some right half-plane.)
Clearly the condition (6) is (by definition) equivalent to

\[
\begin{bmatrix}
\lambda \hat{x} - x_0 \\
\hat{y} \\
\hat{x} \\
\hat{u}
\end{bmatrix} \in \text{graph } (S).
\]

and this can further be rewritten in another equivalent form, namely

\[
\begin{bmatrix}
x_0 \\
\hat{y} \\
\hat{x} \\
\hat{u}
\end{bmatrix} \in \mathcal{E}(\lambda),
\]

where

\[
\mathcal{E}(\lambda) = \begin{bmatrix}
-1_A & 0 & \lambda & 0 \\
0 & 1_Y & 0 & 0 \\
0 & 0 & 1_A & 0 \\
0 & 0 & 0 & 1_U
\end{bmatrix} \text{graph } (S).
\]

We call \( \mathcal{E} \) the node bundle of the system. It is a subspace-valued analytic function of the complex variable \( \lambda \). If \( \mathcal{U} = \mathcal{Y} = \{0\} \), then \( \mathcal{E}(\lambda) = \text{graph } (\lambda - A) \).
Lemma

The system (3) is frequency domain well-posed if and only if there exists at least one \( \lambda \in \mathbb{C} \) such that \( \mathcal{E}(\lambda) \) has the graph representation

\[
\mathcal{E}(\lambda) = \text{im} \left( \begin{bmatrix} 1 & 0 \\ \hat{C}(\lambda) & \hat{D}(\lambda) \\ \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_u \end{bmatrix} \right) \tag{9}
\]

for some bounded linear operator

\[
\mathcal{S}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}.
\]

The proof is trivial.
Another graph representation

Of course, the same condition can be written directly in terms of graph \( \mathcal{S} \) without using the node bundle \( \mathcal{E}(\lambda) \):

**Lemma**

The system (3) is frequency domain well-posed if and only if there exists at least one \( \lambda \in \mathbb{C} \) such that \( V \) has the graph representation

\[
\text{graph } (\mathcal{S}) = \text{im} \left( \begin{bmatrix}
\lambda \hat{A}(\lambda) - 1_x & \lambda \hat{B}(\lambda) \\
\hat{C}(\lambda) & \hat{D}(\lambda)
\end{bmatrix} \right)
\]

for some operator \( \hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \end{bmatrix} \in \mathcal{L} \left( \begin{bmatrix} x \\ u \end{bmatrix} ; \begin{bmatrix} x \\ y \end{bmatrix} \right) \).

The proof is still trivial.
The i/s/o resolvent matrix

Definition

1. The i/s/o resolvent set $\rho_{iso}(S)$ of a closed relation $S: \begin{bmatrix} X \end{bmatrix} \rightarrow \begin{bmatrix} Y \end{bmatrix}$ consists of those point $\lambda \in \mathbb{C}$ for which $\text{graph}(S)$ has a representation of the type

$$\text{graph}(S) = \text{im} \left( \begin{bmatrix} \lambda \hat{A}(\lambda) & \lambda \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \\ \hat{A}(\lambda) & \hat{B}(\lambda) \\ 0 & 1_U \end{bmatrix} \right)$$

for some operator $\hat{S}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} \in \mathcal{L}(\begin{bmatrix} X \\ U \end{bmatrix}; \begin{bmatrix} X \\ Y \end{bmatrix})$.

2. The i/s/o resolvent matrix of $S$ is the operator-valued function $\hat{S}(\lambda)$ above defined for all $\lambda \in \text{dom}(\hat{S}(\lambda)) := \rho_{iso}(S)$. 

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Frequency Domain Well-Posed Linear Systems
The i/s/o resolvent matrix

**Definition**

1. The i/s/o resolvent set $\rho_{iso}(S)$ of a closed relation $S : \left[ \begin{array}{c} X \\ U \end{array} \right] \rightarrow \left[ \begin{array}{c} \dot{X} \\ \dot{Y} \end{array} \right]$ consists of those point $\lambda \in \mathbb{C}$ for which the following identity is valid

$$\begin{bmatrix} -1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{graph} (S) = \text{im} \left( \begin{bmatrix} 1 & 0 \\ \hat{\mathcal{E}}(\lambda) & \hat{\mathcal{D}}(\lambda) \\ \hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\ 0 & 1 \end{bmatrix} \right), \quad (10)$$

for some operator $\hat{\mathcal{G}}(\lambda) = \begin{bmatrix} \hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\ \hat{\mathcal{E}}(\lambda) & \hat{\mathcal{D}}(\lambda) \end{bmatrix} \in \mathcal{L} \left( \left[ \begin{array}{c} \dot{X} \\ U \end{array} \right] ; \left[ \begin{array}{c} \dot{X} \\ \dot{Y} \end{array} \right] \right)$.

2. The i/s/o resolvent matrix of $S$ is the operator-valued function $\hat{\mathcal{G}}(\lambda)$ above defined for all $\lambda \in \text{dom} \left( \hat{\mathcal{G}}(\lambda) \right) := \rho_{iso}(S)$. 

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Frequency Domain Well-Posed Linear Systems
By taking $\mathcal{U} = \mathcal{V} = \{0\}$ our i/s/o resolvent becomes the “standard” resolvent of a closed relation:

**Definition**

1. The resolvent set $\rho(A)$ of a closed relation $A: \mathcal{X} \to \mathcal{X}$ consists of those point $\lambda \in \mathbb{C}$ for which
   \[
   \text{graph} \ (\lambda - A) := \left\{ \begin{bmatrix} \lambda x - y \\ x \end{bmatrix} \mid x \in \text{dom}(A), \ y \in Ax \right\}
   \]
   has a representation of the type
   \[
   \text{graph} \ (\lambda - A) = \begin{bmatrix} -1 \ x \\ 0 \ 1 \ x \end{bmatrix} \text{graph} \ (A) = \text{im} \left( \begin{bmatrix} 1 \ x \\ \hat{A}(\lambda) \end{bmatrix} \right) \quad (11)
   \]
   for some operator $\hat{A}(\lambda) \in \mathcal{L}(\mathcal{X})$.

2. The resolvent of $A$ is the operator-valued function $\hat{G}(\lambda)$ above defined for all $\lambda \in \text{dom} \left( \hat{G}(\lambda) \right) := \rho(A)$.

In this case the “node bundle” is simply the graph of $\lambda - A$. 
We call:

- $\hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \hat{\mathcal{A}}(\lambda) & \hat{\mathcal{B}}(\lambda) \\ \hat{\mathcal{C}}(\lambda) & \hat{\mathcal{D}}(\lambda) \end{bmatrix}$ is the i/s/o resolvent matrix,
- $\hat{\mathcal{A}}(\lambda)$ is the s/s resolvent function,
- $\hat{\mathcal{B}}(\lambda)$ is the i/s resolvent function (= the “incoming wave function” or the “Gamma field”),
- $\hat{\mathcal{C}}(\lambda)$ is the s/o resolvent function (= the “outgoing wave function”),
- $\hat{\mathcal{D}}(\lambda)$ is the i/o resolvent function (= the “transfer function” or the “Weyl function”).
The i/s/o resolvent identities

**Theorem**

The i/s/o resolvent matrix \( \hat{\mathcal{S}} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \) satisfies the following i/s/o resolvent identities for all \( \lambda, \mu \in \text{dom}(\hat{\mathcal{S}}) \):

\[
\hat{\mathcal{S}}(\lambda) = \hat{\mathcal{S}}(\mu) + (\mu - \lambda) \begin{bmatrix} \hat{A}(\mu) \\ \hat{C}(\mu) \end{bmatrix} \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \end{bmatrix}.
\] (12)

or equivalently,

\[
\hat{A}(\lambda) - \hat{A}(\mu) = (\mu - \lambda)\hat{A}(\mu)\hat{A}(\lambda) = (\mu - \lambda)\hat{A}(\lambda)\hat{A}(\mu),
\]

\[
\hat{B}(\lambda) - \hat{B}(\mu) = (\mu - \lambda)\hat{A}(\mu)\hat{B}(\lambda) = (\mu - \lambda)\hat{A}(\lambda)\hat{B}(\mu),
\]

\[
\hat{C}(\lambda) - \hat{C}(\mu) = (\mu - \lambda)\hat{C}(\mu)\hat{A}(\lambda) = (\mu - \lambda)\hat{C}(\lambda)\hat{A}(\mu),
\]

\[
\hat{D}(\lambda) - \hat{D}(\mu) = (\mu - \lambda)\hat{C}(\mu)\hat{B}(\lambda) = (\mu - \lambda)\hat{C}(\lambda)\hat{B}(\mu).
\] (13)

These identities imply, among others, that \( \hat{\mathcal{S}} \) must be analytic.
In (Opm06) Mark Opmeer uses the above i/s/o resolvent identities to define what he calls a resolvent linear system. It consists of a quadruple of operator-valued functions \( \hat{G} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \) which satisfy the i/s/o resolvent identities on some open connected subset \( \Omega \) of the complex plane.

By adding the condition that \( \Omega \) contains some right half-plane and that the above functions are polynomially bounded on that half plane he gets a class of dynamical systems, which he calls integrated resolvent linear systems.

He also defines a slightly larger class of dynamical systems that he calls distributional resolvent linear systems.
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\[ \hat{\mathcal{S}} = \left[ \hat{A} \hat{B} \hat{C} \hat{D} \right] \]
which satisfy the i/s/o resolvent identities on some open connected subset \( \Omega \) of the complex plane.

By adding the condition that \( \Omega \) contains some right half-plane and that the above functions are polynomially bounded on that half plane he gets a class of dynamical systems, which he calls integrated resolvent linear systems.

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Mark Opmeer’s “Resolvent Linear Systems

- In (Opm06) Mark Opmeer uses the above i/s/o resolvent identities to define what he calls a resolvent linear system. It consists of a quadruple of operator-valued functions \( \mathcal{G} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \) which satisfy the i/s/o resolvent identities on some open connected subset \( \Omega \) of the complex plane.

- By adding the condition that \( \Omega \) contains some right half-plane and that the above functions are polynomially bounded on that half plane he gets a class of dynamical systems, which he calls integrated resolvent linear systems.

- He also defines a slightly larger class of dynamical systems that he calls distributional resolvent linear systems.
**Definition**

1. A $\mathcal{L}(\mathcal{X})$-valued function $\hat{A}$ defined on some open set $\Omega \in \mathbb{C}$ is called a **pseudo-resolvent** if it satisfies

   $$
   \hat{A}(\lambda) - \hat{A}(\mu) = (\mu - \lambda)\hat{A}(\mu)\hat{A}(\lambda)
   $$

   (14)

   for all $\lambda, \mu \in \Omega$.

2. A $\mathcal{L} \left( \left[ \begin{array}{c} \mathcal{X} \\ \mathcal{U} \end{array} \right] ; \left[ \begin{array}{c} \mathcal{X} \\ \mathcal{Y} \end{array} \right] \right)$-valued function $\hat{S} = \left[ \begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array} \right]$ defined on some open set $\Omega \in \mathbb{C}$ is called an **i/s/o pseudo-resolvent matrix** if it satisfies the i/s/o resolvent identity

   $$
   \hat{S}(\lambda) - \hat{S}(\mu) = (\mu - \lambda) \left[ \begin{array}{c} \hat{A}(\mu) \\ \hat{C}(\mu) \end{array} \right] \left[ \begin{array}{cc} \hat{A}(\lambda) & \hat{B}(\lambda) \end{array} \right]
   $$

   (12)

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A $\mathcal{L}(\mathcal{X})$-valued function $\hat{A}$ defined on some open set $\Omega \in \mathbb{C}$ is called a pseudo-resolvent if it satisfies

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A $\mathcal{L} ([\mathcal{X}, \mathcal{U}] ; [\mathcal{X}, \mathcal{Y}])$-valued function $\hat{S} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$ defined on some open set $\Omega \in \mathbb{C}$ is called an i/s/o pseudo-resolvent matrix if it satisfies the i/s/o resolvent identity

$$\hat{S}(\lambda) - \hat{S}(\mu) = (\mu - \lambda) \begin{bmatrix} \hat{A}(\mu) \\ \hat{C}(\mu) \end{bmatrix} \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \end{bmatrix} \quad (12)$$

for all $\lambda, \mu \in \Omega$. 
Pseudo-resolvents are resolvents!

**Lemma**

1. If $\hat{A}$ is the resolvent of a closed relation $A: X \to X$, then $\hat{A}$ satisfies the resolvent identity (14) for all $\lambda, \mu \in \rho(A)$.

2. Conversely, if $\hat{A}$ is a pseudo-resolvent defined on some open set $\Omega \subset \mathbb{C}$, then $\hat{A}$ is the restriction to $\Omega$ of the resolvent of some closed relation $A: X \to X$.

3. $A$ is single-valued if and only if $\hat{A}(\lambda)$ is injective for some (and hence for all) $\lambda \in \Omega$.

4. $\text{dom}(A)$ is dense in $X$ if and only if $\text{im}(\hat{A}(\lambda))$ is dense in $X$ for some (and hence for all) $\lambda \in \Omega$.

5. $\hat{A}$ is an analytic function of $\lambda$ on $\Omega$.

This was proved in (DdS87).
Lemma

1. If \( \hat{\mathbf{A}} \) is the resolvent of a closed relation \( \mathbf{A} : \mathcal{X} \to \mathcal{X} \), then \( \hat{\mathbf{A}} \) satisfies the resolvent identity (14) for all \( \lambda, \mu \in \rho(\mathbf{A}) \).

2. Conversely, if \( \hat{\mathbf{A}} \) is a pseudo-resolvent defined on some open set \( \Omega \subset \mathbb{C} \), then \( \hat{\mathbf{A}} \) is the restriction to \( \Omega \) of the resolvent of some closed relation \( \mathbf{A} : \mathcal{X} \to \mathcal{X} \).

3. \( \mathbf{A} \) is single-valued if and only if \( \hat{\mathbf{A}}(\lambda) \) is injective for some (and hence for all) \( \lambda \in \Omega \).

4. \( \text{dom}(\mathbf{A}) \) is dense in \( \mathcal{X} \) if and only if \( \text{im}(\hat{\mathbf{A}}(\lambda)) \) is dense in \( \mathcal{X} \) for some (and hence for all) \( \lambda \in \Omega \).

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3. \( A \) is single-valued if and only if \( \hat{A}(\lambda) \) is injective for some (and hence for all) \( \lambda \in \Omega \).

4. \( \text{dom} \, (A) \) is dense in \( \mathcal{X} \) if and only if \( \text{im} \left( \hat{A}(\lambda) \right) \) is dense in \( \mathcal{X} \) for some (and hence for all) \( \lambda \in \Omega \).

5. \( \hat{A} \) is an analytic function of \( \lambda \) on \( \Omega \).

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Pseudo-resolvents are resolvents!

Lemma

1. If $\hat{\mathcal{A}}$ is the resolvent of a closed relation $\mathcal{A}: \mathcal{X} \to \mathcal{X}$, then $\hat{\mathcal{A}}$ satisfies the resolvent identity (14) for all $\lambda, \mu \in \rho(\mathcal{A})$.

2. Conversely, if $\hat{\mathcal{A}}$ is a pseudo-resolvent defined on some open set $\Omega \subset \mathbb{C}$, then $\hat{\mathcal{A}}$ is the restriction to $\Omega$ of the resolvent of some closed relation $\mathcal{A}: \mathcal{X} \to \mathcal{X}$.

3. $\mathcal{A}$ is single-valued if and only if $\hat{\mathcal{A}}(\lambda)$ is injective for some (and hence for all) $\lambda \in \Omega$.

4. $\text{dom} \ (\mathcal{A})$ is dense in $\mathcal{X}$ if and only if $\text{im} \ (\hat{\mathcal{A}}(\lambda))$ is dense in $\mathcal{X}$ for some (and hence for all) $\lambda \in \Omega$.

5. $\hat{\mathcal{A}}$ is an analytic function of $\lambda$ on $\Omega$.

This was proved in (DdS87).
I/s/o pseudo-resolvents are i/s/o resolvents!

**Theorem**

1. **Recall:** If $\hat{G}$ is the i/s/o resolvent of a closed relation $S : [X] \rightarrow [X]$, then $\hat{G}$ satisfies the resolvent identity (12) for all $\lambda, \mu \in \rho_{iso}(S)$.

2. **Conversely,** if $\hat{G}$ is an i/s/o pseudo-resolvent matrix defined on some open set $\Omega \subset \mathbb{C}$, then $\hat{G}$ is the restriction to $\Omega$ of the i/s/o resolvent matrix of some closed relation $S : [X] \rightarrow [X]$.

3. $S$ is single-valued if and only if the s/s resolvent function $\hat{A}(\lambda)$ is injective for some (and hence for all) $\lambda \in \Omega$.

4. dom $(S)$ is dense in $[X]$ if and only if im $(\hat{A}(\lambda))$ is dense in $X$ for some (and hence for all) $\lambda \in \Omega$.

5. $\hat{G}$ is an analytic function of $\lambda$ on $\Omega$.

This result will be found in the work (AS16).
Theorem

1 Recall: If $\hat{\mathbf{G}}$ is the i/s/o resolvent of a closed relation $S: \begin{bmatrix} X \\ U \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \end{bmatrix}$, then $\hat{\mathbf{G}}$ satisfies the resolvent identity (12) for all $\lambda, \mu \in \rho_{\text{iso}}(S)$.

2 Conversely, if $\hat{\mathbf{G}}$ is an i/s/o pseudo-resolvent matrix defined on some open set $\Omega \subset \mathbb{C}$, then $\hat{\mathbf{G}}$ is the restriction to $\Omega$ of the i/s/o resolvent matrix of some closed relation $S: \begin{bmatrix} X \\ U \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \end{bmatrix}$.

3 $S$ is single-valued if and only if the s/s resolvent function $\hat{\mathbf{A}}(\lambda)$ is injective for some (and hence for all) $\lambda \in \Omega$.

4 $\text{dom}(S)$ is dense in $\begin{bmatrix} X \\ U \end{bmatrix}$ if and only if $\text{im}(\hat{\mathbf{A}}(\lambda))$ is dense in $X$ for some (and hence for all) $\lambda \in \Omega$.

5 $\hat{\mathbf{G}}$ is an analytic function of $\lambda$ on $\Omega$.

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Theorem

1. Recall: If \( \hat{\mathcal{G}} \) is the i/s/o resolvent of a closed relation \( S : [\chi \atop u] \to [\chi \atop y] \), then \( \hat{\mathcal{G}} \) satisfies the resolvent identity (12) for all \( \lambda, \mu \in \rho_{\text{iso}}(S) \).

2. Conversely, if \( \hat{\mathcal{G}} \) is an i/s/o pseudo-resolvent matrix defined on some open set \( \Omega \subset \mathbb{C} \), then \( \hat{\mathcal{G}} \) is the restriction to \( \Omega \) of the i/s/o resolvent matrix of some closed relation \( S : [\chi \atop u] \to [\chi \atop y] \).

3. \( S \) is single-valued if and only if the s/s resolvent function \( \hat{\mathcal{A}}(\lambda) \) is injective for some (and hence for all) \( \lambda \in \Omega \).

4. \( \text{dom} \,(S) \) is dense in \( [\chi \atop u] \) if and only if \( \text{im} \,(\hat{\mathcal{A}}(\lambda)) \) is dense in \( \chi \) for some (and hence for all) \( \lambda \in \Omega \).

5. \( \hat{\mathcal{G}} \) is an analytic function of \( \lambda \) on \( \Omega \).

This result will be found in the work (AS16).
Theorem

1. Recall: If \( \widehat{\mathcal{G}} \) is the i/s/o resolvent of a closed relation \( S: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \), then \( \widehat{\mathcal{G}} \) satisfies the resolvent identity (12) for all \( \lambda, \mu \in \rho_{iso}(S) \).

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3. \( S \) is single-valued if and only if the s/s resolvent function \( \widehat{\mathcal{A}}(\lambda) \) is injective for some (and hence for all) \( \lambda \in \Omega \).

4. \( \text{dom}(S) \) is dense in \( \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \) if and only if \( \text{im}\left(\widehat{\mathcal{A}}(\lambda)\right) \) is dense in \( \mathcal{X} \) for some (and hence for all) \( \lambda \in \Omega \).

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Recall: If \( \hat{\mathcal{G}} \) is the i/s/o resolvent of a closed relation \( S: [\mathcal{X}] \rightarrow [\mathcal{Y}] \), then \( \hat{\mathcal{G}} \) satisfies the resolvent identity (12) for all \( \lambda, \mu \in \rho_{iso}(S) \).

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\( \text{dom}(S) \) is dense in \( [\mathcal{X}] \) if and only if \( \text{im}(\hat{\mathcal{A}}(\lambda)) \) is dense in \( \mathcal{X} \) for some (and hence for all) \( \lambda \in \Omega \).

\( \hat{\mathcal{G}} \) is an analytic function of \( \lambda \) on \( \Omega \).

This result will be found in the work (AS16).
How is all this related to “operator nodes”?

Definition

Recall: By an operator node on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a (possibly unbounded) linear operator $S: \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with the following properties. We denote $\text{dom}(A) = \{ x \in \mathcal{X} | [\hat{x}] \in \text{dom}(S) \}$, define $A: \text{dom}(A) \rightarrow \mathcal{X}$ by $Ax = P_\mathcal{X} S [\hat{x}]$, and require the following conditions to hold:

1. $S$ is closed as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ (with domain $\text{dom}(S)$).
2. $P_\mathcal{X} S$ is closed as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to $\mathcal{X}$ (with domain $\text{dom}(S)$).
3. $\text{dom}(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $[\hat{x}] \in \text{dom}(S)$.

We call $S$ a system node if, in addition, $A$ generates a $C_0$ semigroup.
Theorem

A linear (single-valued) operator \( S : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \) is an operator node if and only if \( \rho_{iso}(S) \neq \emptyset \), i.e., the if and only if the system

\[
\begin{bmatrix}
\dot{x}(t) \\
y(t)
\end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0.
\]

is frequency domain well-posed.

- In particular, every time domain well-posed i/s/o system is automatically frequency domain well-posed. The converse is not true.
- The system (5) can be frequency domain well-posed even in the case where \( S \) is a relation.
A linear (single-valued) operator $S : \begin{bmatrix} \chi \\ u \end{bmatrix} \to \begin{bmatrix} \chi \\ y \end{bmatrix}$ is an operator node if and only if $\rho_{iso}(S) \neq \emptyset$, i.e., the if and only if the system

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Example: the differentiator

Take $\mathcal{X} = \mathcal{U} = \mathcal{Y} = \mathbb{C}$,
$A = 0$, $B = 1$, $C = 1$, $D = 0$, $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$$\Sigma : \begin{cases} \dot{x}(t) = u(t), \\ y(t) = x(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0.$$ 

This is a integrator: $y(t) = x_0 + \int_0^t u(s) \, ds$, $t \in \mathbb{R}^+$, and the i/s/o resolvent matrix of this system is

$$\hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} = \begin{bmatrix} 1/\lambda & 1/\lambda \\ 1/\lambda & 1/\lambda \end{bmatrix}.$$ 

Let us in this system change the meaning of $u$ and $y$, so that $y$ becomes the input, and $u$ the output. This inverted system will then be a differentiator, and it will be a system of the type

$$\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0,$$

for a suitable relation $S$. 

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\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t \in \mathbb{R}^+ , \quad x(0) = x_0, \quad (5)
$$

for a suitable relation $S$. 
Example: the differentiator

It turns out that $S$ is the purely multi-valued relation whose graph is

$$\text{graph}(S) = \text{im} \left( \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Thus,

$$\text{dom}(S) = \{ [x] | x \in \mathbb{C} \}, \quad \text{mul}(S) = \text{im}(S) = \{ [u] | u \in \mathbb{C} \}.$$

If $\begin{bmatrix} x(t) \\ y(t) \\ u(t) \end{bmatrix}$ is a trajectory of this system, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \text{dom}(S)$, or equivalently, $x(t) = y(t)$, and $\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in \text{im}(S)$, i.e., $\dot{x}(t) = u(t)$. Thus, $u(t) = \dot{y}(t)$. The i/s/o resolvent matrix of this system is

$$\hat{\mathcal{S}}(\lambda) = \begin{bmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}. $$
Outline of Talk

- Time domain well-posed input/state/output systems
- Frequency domain well-posed input/state/output systems
- **Intertwinement in time and frequency domain**
- Compressions and dilations in time and frequency domain
- Controllability, observability, and minimality
- Work in progress
### Definition

Let $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ and $\Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be two time domain well-posed i/s/o systems (with the same input and output spaces), and let $R$ be a linear relation $\mathcal{X}_1 \rightarrow \mathcal{X}_2$. We say that $\Sigma_1$ and $\Sigma_2$ are intertwined by $R$ if the following condition holds:

If $\begin{bmatrix} x_1 \\ y_1 \\ u \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ u \end{bmatrix}$ are trajectories of $\Sigma_1$ and $\Sigma_2$ on $\mathbb{R}^+$, respectively (with the same input function $u$), and if $x_2(0) \in Rx_1(0)$, then $y_1 = y_2$ and $x_2(t) \in Rx_1(t)$ for all $t \in \mathbb{R}^+$. 

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Notation for well-posed systems:

- $\mathcal{A}^t$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the final state $x(t) \in \mathcal{X}$ at time $t \geq 0$ when the input is zero.

- $\mathcal{B}$ is the map from an input $u \in L^2(\mathbb{R}^-; \mathcal{U})$ with compact support into the final state $x(0) \in \mathcal{X}$ at time zero, when we take the initial state to be zero for large negative time.

- $\mathcal{C}$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the output $y \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{Y})$ when the input is zero.

- $\mathcal{D}$ is the map from an input $u \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{U})$ whose support is bounded to the left to the output $y \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{Y})$, when we take the initial state to be zero for large negative time.
Intertwinement in terms of characteristic operators

Theorem

The two time domain well-posed i/s/o systems $\Sigma_1$ and $\Sigma_2$ are intertwined by the closed relation $R$ if and only if the characteristic time domain operators of these systems satisfy:

1. $A^t_2 x_2 \in RA^t_1 x_1$ for all $x_2 \subset Rx_1$ and all $t \in \mathbb{R}^+$.
2. For all $u \in L^2(\mathbb{R}^-; U)$ with compact support we have $B_2 u \in RB_1 u$.
3. $C_2 x_2 = C_1 x_1$ for all $x_2 \subset Rx_1$
4. $D_2 = D_1$.

Theorem

The two time domain well-posed i/s/o systems $\Sigma_1$ and $\Sigma_2$ are intertwined by some closed relation $R$ if and only if they have the same i/o map.
Intertwinement in terms of characteristic operators

**Theorem**

The two time domain well-posed i/s/o systems $\Sigma_1$ and $\Sigma_2$ are intertwined by the closed relation $R$ if and only if the characteristic time domain operators of these systems satisfy:

1. $\mathcal{A}_2^t x_2 \in R \mathcal{A}_1^t x_1$ for all $x_2 \subset Rx_1$ and all $t \in \mathbb{R}^+$.
2. For all $u \in L^2(\mathbb{R}^-; \mathcal{U})$ with compact support we have $\mathcal{B}_2 u \in R \mathcal{B}_1 u$.
3. $\mathcal{C}_2 x_2 = \mathcal{C}_1 x_1$ for all $x_2 \subset Rx_1$
4. $\mathcal{D}_2 = \mathcal{D}_1$.

**Theorem**

The two time domain well-posed i/s/o systems $\Sigma_1$ and $\Sigma_2$ are intertwined by some closed relation $R$ if and only if they have the same i/o map.
Theorem

Let $\Sigma_1$ and $\Sigma_2$ be two time domain well-posed linear systems, with growth rates $\omega_1$ and $\omega_2$, respectively, let $\omega = \max\{\omega_1, \omega_2\}$, and denote $\mathbb{C}_\omega^+ = \{\lambda \in \mathbb{C} | \Re \lambda > \omega\}$. Then $\Sigma_1$ and $\Sigma_2$ are intertwined by the closed relation $R$ if and only if the following frequency domain conditions hold:

1. $\hat{A}_2(\lambda)x_2 \in R\hat{A}_1(\lambda)x_1$ for all $x_2 \subset Rx_1$ and all $\lambda \in \mathbb{C}_\omega^+$.
2. $\hat{B}_2(\lambda)u_0 \in R\hat{B}_1(\lambda)u_0$ for all $u_0 \in \mathcal{U}$ and $\lambda \in \mathbb{C}_\omega^+$.
3. $\hat{C}_2(\lambda)x_2 = \hat{C}_1(\lambda)x_1$ for all $x_2 \subset Rx_1$ and all $\lambda \in \mathbb{C}_\omega^+$.
4. $\hat{D}_2(\lambda) = \hat{D}_1(\lambda)$ for all $\lambda \in \mathbb{C}_\omega^+$.
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Compressions and dilations in time domain

**Definition**

Let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}_2$, and let $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ and $\Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be two time domain well-posed i/s/s systems. We call $\Sigma_1$ the (orthogonal) **compression** of $\Sigma_2$ onto $\mathcal{X}_1$, and we call $\Sigma_2$ an (orthogonal) **dilation** of $\Sigma_1$, if the following condition holds:

- For each $x_0 \in \mathcal{X}$ and each $u \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U})$, if we denote the (generalized) future trajectories of $\Sigma_1$ and $\Sigma_2$ with initial state $x_0$ and input function $u$ by $\begin{bmatrix} x_1 \\ y_1 \\ u \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ u \end{bmatrix}$, respectively, then $y_1 = y_2$ and $x_1(t) = P_{\mathcal{X}_1} x_2(t)$ for all $t \in \mathbb{R}^+$. 

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Frequency Domain Well-Posed Linear Systems
Compressions in terms of characteristic operators

**Theorem**

The time domain well-posed i/s/o system $\Sigma$ is the compression onto $\mathcal{X}$ of the time domain well-posed i/s/o system $\Sigma_1$ (i.e., $\Sigma_1$ is a dilation of $\Sigma$) if and only if the characteristic time domain operators of these systems satisfy:

1. $\mathcal{A}_1^t = P_{\mathcal{X}_1} \mathcal{A}_2^t|_{\mathcal{X}_1}$ for all $t \in \mathbb{R}^+$.  
2. $\mathcal{B}_1 = P_{\mathcal{X}_1} \mathcal{B}_2$.  
3. $\mathcal{C}_1 = \mathcal{C}_2|_{\mathcal{X}_1}$.  
4. $\mathcal{D}_1 = \mathcal{D}_2$.  

**Theorem**

Every dilation (and compression) can be interpreted as a special case of an intertwinement (for a suitable bounded single-valued intertwining operator $R$ with closed domain).
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Every dilation (and compression) can be interpreted as a special case of an intertwinement (for a suitable bounded single-valued intertwining operator $R$ with closed domain).
Theorem

Let $\Sigma_1$ and $\Sigma_2$ be two time domain well-posed linear systems, with growth rates $\omega_1$ and $\omega_2$, respectively, let $\omega = \max\{\omega_1, \omega_2\}$, and denote $\mathbb{C}_1^+ = \{\lambda \in \mathbb{C} \mid \Re \lambda > \omega\}$. Then $\Sigma_1$ is the projection of $\Sigma_2$ onto $X_1$ if and only if the following frequency domain conditions hold:

1. $\hat{A}_1(\lambda) = P_{X_1} \hat{A}_2(\lambda)|_{X_1}$ for all $\lambda \in \mathbb{C}_1^+$.
2. $\hat{B}_1(\lambda) = \hat{B}_2(\lambda)$ for all $\lambda \in \mathbb{C}_1^+$ (in particular, $\text{im} \left( \hat{B}_2(\lambda) \right) \subset X_1$).
3. $\hat{C}_1(\lambda) = \hat{C}_2(\lambda)|_{X_1}$ for all $\lambda \in \mathbb{C}_1^+$.
4. $\hat{D}_2(\lambda) = \hat{D}_1(\lambda)$ for all $\lambda \in \mathbb{C}_1^+$.
Outline of Talk

- Time domain well-posed input/state/output systems
- Frequency domain well-posed input/state/output systems
- Intertwinement in time and frequency domain
- Compressions and dilations in time and frequency domain
- **Controllability, observability, and minimality**
- Work in progress
Controllable and observable systems

**Definition**

Let $\Sigma = (X; X, U, Y)$ be a time domain well-posed i/s/o system.

- $\Sigma$ is **controllable** if $\text{im}(\mathcal{B})$ is dense in $X$.
- $\Sigma$ is **observable** if $\ker(\mathcal{C}) = \{0\}$.

**Theorem**

Let $\Sigma = (X; X, U, Y)$ be a time domain well-posed i/s/o system with growth bound $\omega(\Sigma)$.

- $\Sigma$ is **controllable** if and only if $\forall \lambda \in \mathbb{C}^+_{\omega(\Sigma)} \ \text{im} \left( \hat{\mathcal{B}}(\lambda) \right) = X$.
- $\Sigma$ is **observable** if and only if $\cap \lambda \in \mathbb{C}^+_{\omega(\Sigma)} \ \ker \left( \hat{\mathcal{C}}(\lambda) \right) = \{0\}$. 
Definition

A time domain well-posed i/s/o system $\Sigma$ is minimal if it does not have any nontrivial compressions (i.e., it is not a nontrivial dilation of any other well-posed i/s/o system).

Theorem

A time domain well-posed i/s/o system $\Sigma$ is minimal if and only if it is both controllable and observable.

Theorem

Every non-minimal time domain well-posed i/s/o system $\Sigma$ can be compressed into a minimal time domain well-posed i/s/o system.
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Work in progress
Above I defined the basic notions of intertwinements, dilations, compressions, controllability, observability, and minimality in the time domain, assuming time domain well-posedness, and then gave frequency domain interpretations of these notions.

If a system is not time-domain well-posed, then the above time domain definitions are no longer valid.

However, nothing prevents us from using the frequency domain characterizations of intertwinements, dilations, compressions, controllability, observability, and minimality as definitions of these notions. Such definitions make sense as soon as the system is frequency domain well-posed.

This seems to work well even when the generating operator $S$ is allowed to be multi-valued (as long as the system is frequency domain well-posed).
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We seem to be able to prove more or less the same results in this frequency domain setting as in the standard time domain well-posed setting.

So far we have encountered only one major problem: We can still compress every nonminimal system into a minimal one, but we have not been able to prove that the compressed generating operator is always single-valued whenever the original generating operator $S$ is single-valued.

This is the main reason why we started to look at multi-valued generating operators $S$ in the first place!
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Observations

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