$H$-Passive Linear Discrete Time Invariant State/Signal Systems

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Summary

- Discrete time-invariant i/s/o systems

- $H$-passivity with different supply rates

- State/signal systems

- $H$-passive s/s systems

- The KYP inequality

- Signal behaviors

- Passive S/S Systems $\leftrightarrow$ Passive Behaviors

- Realization theory
Discrete time-invariant i/s/o systems
Discrete Time-Invariant I/S/O System

Linear discrete-time-invariant systems are typically modeled as i/s/o (input/state/output) systems of the type

\[ x(n + 1) = Ax(n) + Bu(n), \quad n \in \mathbb{Z}^+, \quad x(0) = x_0, \]

\[ y(n) = Cx(n) + Du(n), \quad n \in \mathbb{Z}^+. \]

(1)

Here \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) and \( A, B, C, D, \) are bounded operators.
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By a trajectory of this system we mean a triple of sequences \((u, x, y)\) satisfying (1).

We denote this system by \( \Sigma_{i/s/o} = ([A \ B \ C \ D]; \mathcal{X}, \mathcal{U}, \mathcal{Y}). \)
Forward $H$-Passive I/S/O System
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The system (1) is forward $H$-passive if all trajectories satisfy the condition

$$\|\sqrt{H}x(n+1)\|_X^2 - \|\sqrt{H}x(n)\|_X^2 \leq \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{Y \oplus U} , \ n \in \mathbb{Z}^+, \quad (2)$$

where $H > 0$ and $J$ is a given signature operator ($J = J^* = J^{-1}$).
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The positive quadratic form

$$E_H(x) = \|\sqrt{H}x\|_X^2 = \langle x, Hx \rangle_X$$

is called the storage function (Lyapunov function), and the indefinite bilinear form

$$j(u, y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix} , J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{Y \oplus U} .$$

is called the supply rate.
The Three Most Common Supply Rates
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(i) The scattering supply rate \( j_{sca}(u, y) = \|u\|_U^2 - \|y\|_Y^2 \) with signature operator \( J_{sca} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ \end{bmatrix} \).
The Three Most Common Supply Rates

(i) The scattering supply rate $j_{sca}(u, y) = \|u\|_U^2 - \|y\|_Y^2$ with signature operator $J_{sca} = \begin{bmatrix} -\frac{1}{\gamma} & 0 \\ 0 & 1_U \end{bmatrix}$.

(ii) The impedance supply rate $j_{imp}(u, y) = 2\Re\langle \Psi u, y \rangle_U$ with signature operator $J_{imp} = \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix}$, where $\Psi$ is a unitary operator $U \to Y$. 
The Three Most Common Supply Rates

(i) The **scattering** supply rate \( j_{\text{sca}}(u, y) = \|u\|_U^2 - \|y\|_Y^2 \) with signature operator 
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J_{\text{sca}} = \begin{bmatrix}
-1_Y & 0 \\
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(ii) The **impedance** supply rate \( j_{\text{imp}}(u, y) = 2\Re\langle \Psi u, y \rangle_U \) with signature operator 
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J_{\text{imp}} = \begin{bmatrix}
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where \( \Psi \) is a unitary operator \( U \to Y \).

(iii) The **transmission** supply rate \( j_{\text{tra}}(u, y) = \langle u, J_U u \rangle_U - \langle y, J_Y y \rangle_Y \) with signature operator 
\[
J_{\text{tra}} = \begin{bmatrix}
-J_Y & 0 \\
0 & J_U
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where \( J_Y \) and \( J_U \) are signature operators in \( Y \) and \( U \), respectively.
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It is possible to combine all these cases into one single setting, called the **s/s** (state/signal) setting. The idea is to introduce a class of systems which does not distinguish between inputs and outputs.
State/Signal Systems
The Signal Space
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We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one signal space $\mathcal{W} = [\mathcal{Y} \mathcal{U}]$. This signal space has a natural Kreĭn space inner product obtained from the signature operator $J$ in the supply rate $j$, namely

$$\left[ \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \right]_{\mathcal{W}} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}}.$$
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\]

The (forward) \( H \)-passivity-inequality (2) now becomes (with \( w(k) = \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \))

\[
\| \sqrt{H} x(k + 1) \|_{\mathcal{X}}^2 - \| \sqrt{H} x(k) \|_{\mathcal{X}}^2 \leq [w(k), w(k)]_{\mathcal{W}}, \quad k \in \mathbb{Z}^+.
\]
State/Signal System: Definition

A linear discrete time-invariant s/s system $\Sigma$ is modelled by a system of equations

$$x(n + 1) = F \left[ \begin{bmatrix} x(n) \\ w(n) \end{bmatrix} \right], \quad n \in \mathbb{Z}^+, \quad x(0) = x_0,$$

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Here $F$ is a bounded linear operator with a closed domain $\mathcal{D}(F) \subset [X, W]$ ($\mathbb{Z}^+ = 0, 1, 2, \ldots$) and a certain additional property.
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In the case of an i/s/o system we take $w = \begin{bmatrix} y \\ u \end{bmatrix}$, $F \begin{bmatrix} x \\ y \end{bmatrix} = Ax + Bu$, and

$\mathcal{D}(F) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y = Cx + Du \right\}$. 
Additional Property of $F$

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(ii) A trajectory $(x, w)$ is uniquely determined by the initial state $x_0$ and the signal part $w$.

(iii) The trajectory $(x, w)$ depends continuously on the initial state $x_0$ and the signal part $w$. 
The Adjoint State/Signal System

Each state/signal system $\Sigma$ has an adjoint state/signal system $\Sigma_*$ with the same state space $\mathcal{X}$ and the Kreĭn signal space $\mathcal{W}_* = -\mathcal{W}$. 
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Each state/signal system $\Sigma$ has an adjoint state/signal system $\Sigma^*$ with the same state space $\mathcal{X}$ and the Kreǐn signal space $\mathcal{W}^* = -\mathcal{W}$.

This system is determined by the fact that $(x^*(\cdot), w^*(\cdot))$ is a trajectory of $\Sigma^*$ if and only if

$$-\langle x(n+1), x^*(0) \rangle_{\mathcal{X}} + \langle x(0), x^*(n+1) \rangle_{\mathcal{X}} + \sum_{k=0}^{n} [w(k), w^*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$. 
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Each state/signal system $\Sigma$ has an adjoint state/signal system $\Sigma_*$ with the same state space $\mathcal{X}$ and the Kreǐn signal space $\mathcal{W}_* = -\mathcal{W}$.

This system is determined by the fact that $(x_*(\cdot), w_*(\cdot))$ is a trajectory of $\Sigma_*$ if and only if

$$-\langle x(n+1), x_*(0) \rangle_{\mathcal{X}} + \langle x(0), x_*(n+1) \rangle_{\mathcal{X}} + \sum_{k=0}^{n} [w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$.

The adjoint of $\Sigma_*$ is the original system $\Sigma$. 
Controllability and Observability

A state/signal system $\Sigma$ is
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- **controllable** if the set of all states $x(n), \, n \geq 1$, which appear in some trajectory $(x(\cdot), w(\cdot))$ of $\Sigma$ with $x(0) = 0$ (i.e., an externally generated trajectory) is dense in $\mathcal{X}$. 

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- **observable** if there do not exist any nontrivial trajectories $(x(\cdot), w(\cdot))$ where the signal component $w(\cdot)$ is identically zero.
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**Fact**: $\Sigma$ is observable if and only $\Sigma_*$ is controllable.
$H$-Passive State/Signal Systems
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Let $H = H^* > 0$.\footnote{\textit{H} > 0 \text{ means that } \langle x, H x \rangle > 0 \text{ for all nonzero } x \in \mathcal{D}(H).}$ Here $H$ and $H^{-1}$ may be unbounded. A s/s system $\Sigma$ is
**$H$-Passive State/Signal Systems**

Let $H = H^* > 0$.\(^1\) Here $H$ and $H^{-1}$ may be unbounded. A s/s system $\Sigma$ is

(i) **forward $H$-passive** if every trajectory $(x, w)$ of $\Sigma$ with $x(0) \in D(\sqrt{H})$ satisfies $x(n) \in D(\sqrt{H})$ and

$$\|\sqrt{H}x(n + 1)\|_X^2 - \|\sqrt{H}x(n)\|_X^2 \leq [w(n), w(n)]_W, \quad n \in \mathbb{Z}^+. \quad (1)$$

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(ii) \textbf{backward $H$-passive} if $\Sigma_*$ is forward $H^{-1}$-passive,

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(i) **forward \( H \)-passive** if every trajectory \( (x, w) \) of \( \Sigma \) with \( x(0) \in \mathcal{D}(\sqrt{H}) \) satisfies
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x(n) \in \mathcal{D}(\sqrt{H}) \quad \text{and} \\
\|\sqrt{H}x(n + 1)\|_\mathcal{X}^2 - \|\sqrt{H}x(n)\|_\mathcal{X}^2 \leq [w(n), w(n)]_\mathcal{W}, \quad n \in \mathbb{Z}^+.
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(ii) **backward \( H \)-passive** if \( \Sigma^* \) is forward \( H^{-1} \)-passive,

(iii) **\( H \)-passive** if it is both forward \( H \)-passive and backward \( H \)-passive.

(iv) **passive** if it is \( 1_\mathcal{X} \)-passive (\( 1_\mathcal{X} \) is the identity operator in \( \mathcal{X} \)).

\(^1\)\( H > 0 \) means that \( \langle x, Hx \rangle > 0 \) for all nonzero \( x \in \mathcal{D}(H) \).
The S/S KYP Inequality

It is not difficult to see that a s/s system $\Sigma$ whose trajectories are defined by (3) is forward $H$-passive if and only if $H > 0$ is a solution of the generalized s/s KYP (Kalman–Yakubovich–Popov) inequality$^2$

$$\|H^{1/2} F [\begin{array}{c} x \\ w \end{array}]\|_{\mathcal{X}}^2 - \|H^{1/2} x\|_{\mathcal{X}}^2 \leq [w, w]_\mathcal{W}, \quad \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F), \quad x \in \mathcal{D}(H^{1/2}).$$

$^2$In particular, in order for the first term in this inequality to be well-defined we require $F$ to map $\{\begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F) \mid x \in \mathcal{D}(H^{1/2})\}$ into $\mathcal{D}(H^{1/2})$. 
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This inequality is named after Kalman [Kal63], Yakubovich [Yak62], and Popov [Pop61] (who at that time restricted themselves to the finite-dimensional input/state/output case).

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14
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There is a rich literature on this version of the KYP inequality and the corresponding equality; see, e.g., [PAJ91], [IW93], and [LR95], and the references mentioned there.

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In the seventies the classical results on the i/s/o KYP inequalities were extended to systems with $\dim X = \infty$ by Yakubovich and his students and collaborators (see [Yak74, Yak75, LY76] and the references listed there).
Infinite-Dimensional I/S/O KYP Inequality: History

In the seventies the classical results on the i/s/o KYP inequalities were extended to systems with \( \dim \mathcal{X} = \infty \) by Yakubovich and his students and collaborators (see [Yak74, Yak75, LY76] and the references listed there).

There is now a rich literature also on this subject; see, e.g., the discussion in [Pan99] and the references cited there.
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However, it is (almost) always assumed that $H$ or $H^{-1}$ is bounded. The only exception is the article [AKP06] by Arov, Kaashoek and Pik.
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However, it is (almost) always assumed that \( H \) or \( H^{-1} \) is bounded. The only exception is the article [AKP06] by Arov, Kaashoek and Pik.

An continuous-time example is given in [AS06c] where both \( H \) and \( H^{-1} \) are unbounded for every generalized solution of the i/s/o KYP inequality. The same example can be converted to discrete time and to also to a s/s setting.
Signal Behaviors

(The time domain counterpart of the frequency domain subspace

\[
\begin{bmatrix}
\hat{y}(z) \\
\hat{u}(z)
\end{bmatrix} \mid \hat{y}(z) = \mathcal{D}(z)\hat{u}(z)
\]
The Behavior Induced by a State/Signal System

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The behavior is the set of all possible signal sequences $w$ which are the signal part of some externally generated trajectory $(x, w)$. (Externally generated means that $x_0 = 0$, so that $x$ is uniquely determined by $w$).
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An alternative to working with transfer functions is to study the relationships between “input” and “output” signals directly in the time domain instead of going to the frequency domain.

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Easy: $\mathcal{W}$ is a closed and right-shift invariant subspace of the Fréchet space $\mathcal{W}^{Z^+}$. 
Behavior: Definition

By a (general) behavior\(^3\) on the signal space \(\mathcal{W}\) we mean a closed right-shift invariant subspace of the Fréchet space \(\mathcal{W}^{\mathbb{Z}^+}\).

\(^3\)Our behaviors are what Polderman and Willems call linear time-invariant mainfest behaviors in [PW98, Definitions 1.3.4, 1.4.1, and 1.4.2].
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Thus, in particular, the set \(\mathcal{W}\) of all sequences \(w\) that are the signal part of some externally generated trajectory \((x, w)\) of a given s/s system \(\Sigma\) is a behavior.

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Pseudo-Similarity

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(ii) $(x(\cdot), w(\cdot))$ is a trajectory of $\Sigma \Leftrightarrow (Rx(\cdot), w(\cdot))$ is a trajectory of $\Sigma_1$. 
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A realizable behavior $\mathcal{W}$ on the signal space $\mathcal{W}$ has a minimal s/s realization, which is determined by $\mathcal{W}$ up to pseudo-similarity. (See [AS05, Section 7] for details.)
The Adjoint Behavior

Recall the “orthogonality” between a s/s system $\Sigma$ and its adjoint $\Sigma^*$:

$$-\langle x(n+1), x^*(0) \rangle_{\mathcal{X}} + \langle x(0), x^*(n+1) \rangle_{\mathcal{X}} + \sum_{k=0}^{n} [w(k), w^*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$
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For external trajectories we have $x(0) = 0$ and $x_*(0) = 0$, and hence

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In general we define the adjoint of the behavior $\mathcal{W}$ on $\mathcal{W}$ to be the behavior $\mathcal{W}_*$ on $\mathcal{W}_*$ which consists of all the sequences $w_*$ that satisfy (5) for all $w \in \mathcal{W}$. 
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In general we define the adjoint of the behavior $\mathcal{W}$ on $\mathcal{V}$ to be the behavior $\mathcal{W}^*$ on $\mathcal{V}^*$ which consists of all the sequences $w^*$ that satisfy (5) for all $w \in \mathcal{W}$.

If $\mathcal{W}$ is induced by $\Sigma$, then $\mathcal{W}^*$ is (realizable and) induced by $\Sigma^*$, and the adjoint of $\mathcal{W}^*$ is the original behavior $\mathcal{W}$.

\[\text{4}^*\text{Is this statement true or false if } \mathcal{W}\text{ is not realizable?}\]
Passivity Inequality for Behaviors
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The forward $H$-passivity inequality says

$$\| \sqrt{H} x(k+1) \|_{\mathcal{X}}^2 - \| \sqrt{H} x(k) \|_{\mathcal{X}}^2 \leq [w(k), w(k)] \mathcal{W}, \quad k \in \mathbb{Z}^+. $$
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Sum over $k = 0, 1, 2, \ldots, n$ and take $x(0) = 0$. This gives

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$$\sum_{k=0}^{n} [w(k), w(k)]_W \geq \|\sqrt{H}x(n + 1)\|_{X'}^2.$$  

In particular, every $w$ in the behavior $\mathcal{W}$ induced by $\Sigma$ satisfies

$$\sum_{k=0}^{n} [w(k), w(k)]_W \geq 0, \quad w \in \mathcal{W}, \quad n \in \mathbb{Z}^+.$$
Passive Behaviors

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(iii) **passive** if it is realizable\(^5\) and both forward and backward passive.

---

\(^5\)We do not know if the realizability assumption is redundant or not.
Passive S/S Systems $\leftrightarrow$ Passive Behaviors

**Proposition 1.** Let $\mathcal{W}$ be the behavior induced by the s/s system $\Sigma$. 
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(iv) If $\Sigma$ is forward $H_1$ passive for some $H_1 > 0$ and backward $H_2$ passive for some $H_2 > 0$, then $\Sigma$ is both $H_1$-passive and $H_2$-passive, and $\mathcal{W}$ is passive.


**Passive S/S Systems ↔ Passive Behaviors**

**Proposition 1.** Let \( \mathcal{M} \) be the behavior induced by the s/s system \( \Sigma \).

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(iv) If \( \Sigma \) is forward \( H_1 \) passive for some \( H_1 > 0 \) and backward \( H_2 \) passive for some \( H_2 > 0 \), then \( \Sigma \) is both \( H_1 \)-passive and \( H_2 \)-passive, and \( \mathcal{M} \) is passive.

Thus, if \( \Sigma \) is backward \( H_2 \)-passive for at least one \( H_2 \), then forward \( H \)-passivity implies backward \( H \)-passivity for all \( H > 0 \).
**$H$-Passive Realizations**

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**H-Passive Realizations**

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(iii) Every minimal realization of \( \mathcal{W} \) is \( H \)-passive for some \( H > 0 \). Moreover, it is possible to choose \( H \) in such a way that the system \( \Sigma_H \) in (ii) is minimal.
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(ii) says: We can make $\Sigma$ passive by replacing the original norm in $\mathcal{X}$ by the new norm $\|x\|_H = \|\sqrt{H}x\|_{\mathcal{X}}$.

(iii) says: It is possible to make the resulting system both passive and minimal.
Ordering of Solutions of KYP Inequality

We denote the set of all solutions $H = H^* > 0$ of the KYP inequality by $M_\Sigma$, and we let $M_{\Sigma}^{\text{min}}$ be the set of $H \in M_\Sigma$ for which the system $\Sigma_H$ in assertion (ii) of Theorem 2 is minimal by $L_{\Sigma}^{\text{min}}$. 
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**Theorem 3.** Let $\Sigma$ be a minimal s/s system with a passive behavior. Then $M_\Sigma^{\min} \neq \emptyset$ and $M_\Sigma^{\min}$ contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e., $H_\circ \preceq H \preceq H_\bullet$ for every $H \in M_\Sigma^{\min}$.

$$H_1 \preceq H_2 \iff D(\sqrt{H_2}) \subset D(\sqrt{H_1}) \text{ and } \|\sqrt{H_1}x\| \leq \|\sqrt{H_2}x\| \forall x \in D(\sqrt{H_2}).$$
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\[
H_1 \preceq H_2 \iff \mathcal{D}(\sqrt{H_2}) \subset \mathcal{D}(\sqrt{H_1}) \quad \text{and} \quad \|\sqrt{H_1}x\| \leq \|\sqrt{H_2}x\| \quad \forall x \in \mathcal{D}(\sqrt{H_2}).
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\( E_{H_\circ}(\cdot) \) is the available storage, and \( E_{H_\bullet}(\cdot) \) is the required supply (Willems).
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$H_\circ$ is the optimal and $H_\bullet$ is the $\ast$-optimal solution of the KYP inequality (Arov).
Further Extensions

Instead of working with energy inequalities we can also work with energy balance equations. In this case the system will be forward conservative or even conservative.
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Analogous results also hold for the quadratic cost minimization problem and its dual. The advantage with this approach is that we get rid of the finite cost condition. This is current joint work with Mark Opmeer.
References


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