$H$-Passive Linear Discrete Time Invariant State/Signal Systems

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Summary

- Discrete time-invariant i/s/o systems
- $H$-passivity with different supply rates
- State/signal systems
- $H$-passive s/s systems
- The KYP inequality
- Signal behaviors
- Passive S/S Systems $\leftrightarrow$ Passive Behaviors
- Realization theory
Discrete time-invariant i/s/o systems
Discrete Time-Invariant I/S/O System

Linear discrete-time-invariant systems are typically modeled as i/s/o (input/state/output) systems of the type

\[ x(n + 1) = Ax(n) + Bu(n), \quad n \in \mathbb{Z}^+, \quad x(0) = x_0, \]
\[ y(n) = Cx(n) + Du(n), \quad n \in \mathbb{Z}^+. \]  

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Here \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) and
\( A, B, C, D, \) are bounded operators.
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\( u(n) \in \mathcal{U} = \text{the input space}, \)
\( x(n) \in \mathcal{X} = \text{the state space}, \)
\( y(n) \in \mathcal{Y} = \text{the output space} \) (all Hilbert spaces).
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By a trajectory of this system we mean a triple of sequences $(u, x, y)$ satisfying (1).
$H$-Passive I/S/O System
The system (1) is $H$-passive if all trajectories satisfy the condition

$$E_H(x(n+1)) - E_H(x(n)) \leq j(u(n), y(n)), \quad n \in \mathbb{Z}^+, \quad (2)$$

where $E_H$ is a positive storage function (Lyapunov function)

$$E_H(x) = \langle Hx, x \rangle_X, \quad H > 0,$$

and $j$ is an indefinite quadratic supply rate

$$j(u, y) = \langle [y^T u], J[y^T u] \rangle_{Y \oplus U}$$

determined by a signature operator $J$ ($= J^* = J^{-1}$).
The Three Most Common Supply Rates
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(i) The scattering supply rate \( j_{\text{sca}}(u, y) = -\|y\|^2_Y + \|u\|^2_U \) with signature operator

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J_{\text{sca}} = \begin{bmatrix}
-1 & 0 \\
0 & 1_U
\end{bmatrix}.
\]
The Three Most Common Supply Rates

(i) The scattering supply rate $j_{\text{sca}}(u, y) = -\|y\|_Y^2 + \|u\|_U^2$ with signature operator $J_{\text{sca}} = \begin{bmatrix} -1_Y & 0 \\ 0 & 1_U \end{bmatrix}$.

(ii) The impedance supply rate $j_{\text{imp}}(u, y) = 2\Re\langle y, \Psi u \rangle_U$ with signature operator $J_{\text{imp}} = \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix}$, where $\Psi$ is a unitary operator $U \to Y$. 
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(iii) The transmission supply rate \( j_{\text{tra}}(u, y) = -\langle y, J_Y y \rangle_Y + \langle u, J_U u \rangle_U \) with signature operator \( J_{\text{tra}} = \begin{bmatrix} -J_Y & 0 \\ 0 & J_U \end{bmatrix} \), where \( J_Y \) and \( J_U \) are signature operators in \( Y \) and \( U \), respectively.
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It is possible to combine all these cases into one single setting, called the s/s (state/signal) setting. The idea is to introduce a class of systems which does not distinguish between inputs and outputs.
State/Signal Systems
State/Signal System: Definition

A linear discrete time-invariant s/s system $\Sigma$ is modelled by a system of equations

$$x(n + 1) = F \left[ \begin{array}{c} x(n) \\ w(n) \end{array} \right], \quad n \in \mathbb{Z}^+, \quad x(0) = x_0, \quad (3)$$

Here $F$ is a bounded linear operator with a closed domain $\mathcal{D}(F) \subset [X] \ (\mathbb{Z}^+ = 0, 1, 2, \ldots)$ and certain additional properties.
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In the case of an i/s/o system we take $w = [y \ u]$, $F \begin{bmatrix} x \\ y \end{bmatrix} = Ax + Bu$, and
$\mathcal{D}(F) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y = Cx + Du \right\}$. 
Additional Properties of $F$

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(i) Every $x_0 \in \mathcal{X}$ is the initial state of some trajectory,

(ii) The trajectory $(x, w)$ is determined uniquely by $x_0$ and $w$. 
The Adjoint State/Signal System

Each s/s system $\Sigma$ has an adjoint s/s system $\Sigma^*$ with the same state space $\mathcal{X}$ and the Kreǐn signal space $\mathcal{W}^* = -\mathcal{W}$. 
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This system is determined by the fact that $(x_*(\cdot), w_*(\cdot))$ is a trajectory of $\Sigma_*$ if and only if

$$-\langle x(n + 1), x_*(0) \rangle_{\mathcal{X}} + \langle x(0), x_*(n + 1) \rangle_{\mathcal{X}} + \sum_{k=0}^{n} [w(k), w_*(n - k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$. 
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for all trajectories $(x(\cdot), w(\cdot))$ of $\Sigma$.

The adjoint of $\Sigma_*$ is the original system $\Sigma$. 
Controllability and Observability

A s/s system $\Sigma$ is controllable if the set of all states $x(n)$, $n \geq 1$, which appear in some trajectory $(x(\cdot), w(\cdot))$ of $\Sigma$ with $x(0) = 0$ (i.e., an externally generated trajectory) is dense in $\mathcal{X}$. 
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The system $\Sigma$ is **observable** if there do not exist any nontrivial trajectories $(x(\cdot), w(\cdot))$ where the signal component $w(\cdot)$ is identically zero.
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**Fact:** $\Sigma$ is observable if and only $\Sigma_*$ is controllable.

$\Sigma$ is **minimal** if $\Sigma$ is both controllable and observable.
\( H \)-Passive State/Signal Systems
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Let $H = H^* > 0$.\(^1\) Here $H$ and $H^{-1}$ may be unbounded. A s/s system $\Sigma$ is

\(^1\) $H > 0$ means that $\langle x, Hx \rangle > 0$ for all nonzero $x \in \mathcal{D}(H)$. 
$H$-Passive State/Signal Systems

Let $H = H^* > 0$.\(^1\) Here $H$ and $H^{-1}$ may be unbounded. A s/s system $\Sigma$ is

(i) **forward $H$-passive** if $x(n) \in \mathcal{D}(\sqrt{H})$ and

$$\|\sqrt{H}x(n+1)\|_X^2 - \|\sqrt{H}x(n)\|_X^2 \leq [w(n), w(n)]_W, \quad n \in \mathbb{Z}^+,$$

for every trajectory $(x, w)$ of $\Sigma$ with $x(0) \in \mathcal{D}(\sqrt{H})$.

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(iii) **$H$-passive** if it is both forward $H$-passive and backward $H$-passive.

(iv) **passive** if it is $1_X$-passive ($1_X$ is the identity operator in $X$).

$^1H > 0$ means that $\langle x, Hx \rangle > 0$ for all nonzero $x \in D(H)$.
The S/S KYP Inequality

It is not difficult to see that a s/s system $\Sigma$ whose trajectories are defined by (3) is forward $H$-passive if and only if $H > 0$ is a solution of the generalized s/s KYP (Kalman–Yakubovich–Popov) inequality$^2$

$$\|H^{1/2}F \begin{bmatrix} x \\ w \end{bmatrix}\|_{\mathcal{X}}^2 - \|H^{1/2}x\|_{\mathcal{X}}^2 \leq [w, w]_{\mathcal{W}}, \quad \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F), \quad x \in \mathcal{D}(H^{1/2}). \quad (4)$$

$^2$In particular, in order for the first term in this inequality to be well-defined we require $F$ to map $\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F) \mid x \in \mathcal{D}(H^{1/2}) \}$ into $\mathcal{D}(H^{1/2})$. 


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$$\|H^{1/2} F [\tfrac{x}{w}]\|_X^2 - \|H^{1/2} x\|_X^2 \leq [w, w]_W, \quad [\tfrac{x}{w}] \in D(F), \quad x \in D(H^{1/2}). \quad (4)$$

This inequality is named after Kalman [Kal63], Yakubovich [Yak62], and Popov [Pop61] (who at that time restricted themselves to the finite-dimensional input/state/output case).

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13
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This inequality is named after Kalman [Kal63], Yakubovich [Yak62], and Popov [Pop61] (who at that time restricted themselves to the finite-dimensional input/state/output case).

There is a rich literature on this version of the KYP inequality and the corresponding equality; see, e.g., [PAJ91], [IW93], and [LR95], and the references mentioned there.

\footnote{In particular, in order for the first term in this inequality to be well-defined we require $F$ to map $\left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F) \mid x \in \mathcal{D}(H^{1/2}) \right\}$ into $\mathcal{D}(H^{1/2})$.}
In the seventies the classical results on the i/s/o KYP inequalities were extended to systems with \( \dim \mathcal{X} = \infty \) by Yakubovich and his students and collaborators (see [Yak74, Yak75, LY76] and the references listed there).
Infinite-Dimensional I/S/O KYP Inequality: History

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There is now a rich literature also on this subject; see, e.g., the discussion in [Pan99] and the references cited there.
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There is now a rich literature also on this subject; see, e.g., the discussion in [Pan99] and the references cited there.

However, it is (almost) always assumed that \( H \) or \( H^{-1} \) is bounded. The only exception is the article [AKP05] by Arov, Kaashoek and Pik.
Signal behaviors
Behaviors: Definition

By a behavior on the signal space $\mathcal{W}$ we mean a closed right-shift invariant subspace of the Fréchet space $\mathcal{W}^{\mathbb{Z}^+}$. 
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Thus, in particular, the set $\mathcal{W}$ of all sequences $w$ that are the signal part of some externally generated trajectory $(x, w)$ of a s/s system $\Sigma$ is a behavior.
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Thus, in particular, the set $\mathcal{M}$ of all sequences $w$ that are the signal part of some externally generated trajectory $(x, w)$ of a s/s system $\Sigma$ is a behavior.

We call this the behavior induced by $\Sigma$, and refer to $\Sigma$ as a s/s realization of $\mathcal{M}$, or, in the case where $\Sigma$ is minimal, as a minimal s/s realization of $\mathcal{M}$. 
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A behavior is realizable if it has a s/s realization.
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We call this the behavior induced by $\Sigma$, and refer to $\Sigma$ as a s/s realization of $\mathcal{W}$, or, in the case where $\Sigma$ is minimal, as a minimal s/s realization of $\mathcal{W}$.

A behavior is realizable if it has a s/s realization.

Two s/s systems $\Sigma_1$ and $\Sigma_2$ with the same signal space are externally equivalent if they induce the same behavior.
Pseudo-Similarity

Two s/s systems $\Sigma$ and $\Sigma_1$ with the same signal space $\mathcal{W}$ and state spaces $\mathcal{X}$ and $\mathcal{X}_1$, respectively, are called pseudo-similar if there exists an injective densely defined closed linear operator $R: \mathcal{X} \rightarrow \mathcal{X}_1$ with dense range such that the following conditions hold:

$$(x(\cdot), w(\cdot)) \text{ is a trajectory of } \Sigma \iff (Rx(\cdot), w(\cdot)) \text{ is a trajectory of } \Sigma_1.$$

In particular, if $\Sigma_1$ and $\Sigma_2$ are pseudo-similar, then they are externally equivalent.
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Conversely, if $\Sigma_1$ and $\Sigma_2$ are minimal and externally equivalent, then they are necessarily pseudo-similar.
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Conversely, if $\Sigma_1$ and $\Sigma_2$ are minimal and externally equivalent, then they are necessarily pseudo-similar.

A realizable behavior $\mathcal{W}$ on the signal space $\mathcal{W}$ has a minimal s/s realization, which is determined by $\mathcal{W}$ up to pseudo-similarity. (See [AS05, Section 7] for details.)
The Adjoint Behavior

The adjoint of the behavior $\mathcal{W}$ on $\mathcal{W}$ is a behavior $\mathcal{W}_*$ on $\mathcal{W}_*$ defined as the set of sequences $w_*$ satisfying

$$\sum_{k=0}^{n}[w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all $w \in \mathcal{W}$. 
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for all \( w \in \mathcal{W} \).

If \( \mathcal{W} \) is induced by \( \Sigma \), then \( \mathcal{W}_* \) is (realizable and) induced by \( \Sigma_* \),
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for all $w \in \mathcal{W}$.

If $\mathcal{W}$ is induced by $\Sigma$, then $\mathcal{W}_*$ is (realizable and) induced by $\Sigma_*$, and the adjoint of $\mathcal{W}_*$ is the original behavior $\mathcal{W}$.

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3 Is this statement true or false if $\mathcal{W}$ is not realizable?
Passive Behaviors

A behavior $W$ on $\mathcal{W}$ is
Passive Behaviors

A behavior $\mathcal{W}$ on $\mathcal{V}$ is

(i) **forward passive** if

$$\sum_{k=0}^{n}[w(k), w(k)]_{\mathcal{V}} \geq 0, \quad w \in \mathcal{W}, \quad n \in \mathbb{Z}^+,$$
Passive Behaviors

A behavior $\mathcal{W}$ on $\mathcal{V}$ is

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Passive Behaviors

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(i) **forward passive** if

$$\sum_{k=0}^{n} [w(k), w(k)]_{\mathcal{V}} \geq 0, \quad w \in \mathcal{W}, \quad n \in \mathbb{Z}^+,$$

(ii) **backward passive** if $\mathcal{W}_*$ is forward passive,

(iii) **passive** if it is realizable\(^4\) and both forward and backward passive.

\(^4\)We do not know if the realizability assumption is redundant or not.
Passive S/S Systems $\leftrightarrow$ Passive Behaviors

**Proposition 1.** Let $\mathcal{W}$ be the behavior induced by a s/s system $\Sigma$. 
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(iii) If $\Sigma$ is forward $H_1$ passive for some $H_1 > 0$ and backward $H_2$ passive for some $H_2 > 0$, then $\Sigma$ is both $H_1$-passive and $H_2$-passive, and $\mathcal{W}$ is passive.
**Proposition 1.** Let $Ψ$ be the behavior induced by a s/s system $Σ$.

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(iii) If $Σ$ is **forward $H_1$ passive** for some $H_1 > 0$ and **backward $H_2$ passive** for some $H_2 > 0$, then $Σ$ is both $H_1$-passive and $H_2$-passive, and $Ψ$ is **passive**.

Thus, if $Σ$ is **backward $H_2$-passive** for at least one $H_2$, then **forward $H$-passivity implies backward $H$-passivity** for all $H > 0$. 
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(ii) Every $H$-passive realization $\Sigma$ of $\mathcal{W}$ is **pseudo-similar** to a passive realization $\Sigma_H$ with pseudo-similarity operator $\sqrt{H}$. The system $\Sigma_H$ is determined uniquely by $\Sigma$ and $H$. 
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**Theorem 2.** Let \( \mathcal{W} \) be a passive behavior on \( \mathcal{W} \). Then

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(ii) says: We can make \( \Sigma \) passive by replacing the original norm in \( X \) by the new norm \( \|x\|_H = \|\sqrt{H}x\|_X \).

(iii) says: It is possible to make the resulting system both passive and minimal.
Ordering of Solutions of KYP Inequality

We denote the set of all solutions $H = H^* > 0$ of the KYP inequality by $M_\Sigma$, and we let $M^\text{min}_\Sigma$ be the set of $H \in M_\Sigma$ for which the system $\Sigma_H$ in assertion (ii) of Theorem 2 is minimal by $L^\text{min}_\Sigma$. 
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**Theorem 3.** Let $\Sigma$ be a minimal s/s system with a passive behavior. Then $M_\Sigma^\text{min} \neq \emptyset$ and $M_\Sigma^\text{min}$ contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e., $H_\circ \preceq H \preceq H_\bullet$ for every $H \in M_\Sigma^\text{min}$.

$H_1 \preceq H_2 \iff \mathcal{D}(\sqrt{H_2}) \subset \mathcal{D}(\sqrt{H_1})$ and $\|\sqrt{H_1}x\| \leq \|\sqrt{H_2}x\|$ $\forall x \in \mathcal{D}(\sqrt{H_2})$. 
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$E_{H_\circ}(\cdot)$ is the available storage, and $E_{H_\bullet}(\cdot)$ is the required supply (Willems).
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$H_\circ$ is the optimal and $H_\bullet$ is the $\ast$-optimal solution of the KYP inequality (Arov).
Further Extensions

Instead of working with energy inequalities we can also work with energy balance equations. In this case the system will be forward conservative or even conservative.
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Analogous results also hold for the quadratic cost minimization problem and its dual. The advantage with this approach is that we get rid of the finite cost condition. This is current joint work with Mark Opmeer.
References


[Yak75] ______, The frequency theorem for the case in which the state space and the control space are Hilbert spaces, and its application in certain