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Henk de Snoo Seminar, Dec 17, 2010

Based on joint work with
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Grandpa, where do they come from?

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A **boundary control input/state/output system** can be written in the form

\[
\Sigma_{i/s/o} : \begin{cases} 
\dot{x}(t) = Lx(t), \\
u(t) = \Gamma_0 x(t), & t \geq 0 \\
y(t) = \Gamma_1 x(t), \\
x(0) = x_0.
\end{cases}
\]

(1)

\(\mathcal{X}\) is the *state space*, \(x(t) \in \mathcal{X}\), \(x_0 \in \mathcal{X}\),

\(\mathcal{U}\) is the *input space*, \(u(t) \in \mathcal{U}\),

\(\mathcal{Y}\) is the *output space*, \(y(t) \in \mathcal{Y}\) (these are Hilbert spaces),

\(L\) is the *main operator* (always unbounded),

\(\Gamma_0\) is the *boundary control operator* (surjective and unbounded),

\(\Gamma_1\) is the *observation operator* (can be bounded or unbounded).
A boundary control state/signal system is similar to a boundary control i/s/o system, but we no longer specify which part of the “boundary signal” \( w(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \) is the input, and which part is the output. After replacing \( \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix} \) by \( \Gamma \) we get an equation of the type

\[
\Sigma : \begin{cases} 
\dot{x}(t) = Lx(t), \\
w(t) = \Gamma x(t),
\end{cases} \quad t \geq 0; \quad x(0) = x_0. \tag{2}
\]

\( \mathcal{X} \) is the state space, \( x(t) \in \mathcal{X}, x_0 \in \mathcal{X} \), \( \mathcal{X} \) is a Hilbert space, \( \mathcal{W} \) is the signal space, \( w(t) \in \mathcal{W}, \mathcal{W} \) is a Kreĭn space, \( L \) is the main operator (always unbounded), \( \Gamma \) is the boundary operator (also unbounded), \( L \) and \( \Gamma \) have the same domain

\[
\text{Dom}(L) = \text{Dom}(\Gamma) = \text{Dom} \left( \begin{bmatrix} \Gamma \\ \Gamma \end{bmatrix} \right) \subset \mathcal{X}.
\]
There is an almost one-to-one correspondence between conservative boundary control s/s systems ↔ (conservative) boundary triplets
However, today I want to talk about the dynamics of boundary relations and not the dynamics of boundary triplets. To do this I have to go beyond the class of boundary s/s systems.
Given a boundary control s/s system

\[ \begin{array}{l}
\dot{x}(t) = Lx(t), \\
w(t) = \Gamma x(t), \\
t \geq 0; \\
x(0) = x_0.
\end{array} \tag{2} \]

we can rewrite it in the graph form

\[ \begin{array}{l}
\left[ \begin{array}{c}
\dot{x}(t) \\
x(t) \\
w(t)
\end{array} \right] \in V, \\
t \in \mathbb{R}^+, \\
x(0) = x_0,
\end{array} \tag{3} \]

where

\[ V := \left\{ \left[ \begin{array}{c}
Lx \\
x \\
\Gamma x
\end{array} \right] \in \mathcal{R} \mid x \in \text{Dom} \left( \left[ \begin{array}{c}
L \\
\Gamma
\end{array} \right] \right) \right\}. \tag{4} \]

Here \( V \) is the generating subspace, which is a subspace of the node space \( \left[ \begin{array}{c}
x \\
\dot{x} \\
w
\end{array} \right] \).
A general state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is of the form

$$\Sigma : \begin{cases} 
\begin{bmatrix} \dot{x}(t) \\
 x(t) \\
w(t) 
\end{bmatrix} \in V, & t \in \mathbb{R}^+, \ x(0) = x_0, 
\end{cases}$$

where $\mathcal{X}$ is the state space (a Hilbert space), and $\mathcal{W}$ is the signal space (a Kreĭn space).

The generating subspace $V$ is a closed subspace of the node space $\mathcal{K} := \left[ \begin{array}{c} \mathcal{X} \\
\mathcal{X} \\
\mathcal{W} \end{array} \right]$.

$x(t) \in \mathcal{X}$ is the state at time $t \in \mathbb{R}^+$,

$x_0 \in \mathcal{X}$ is the initial state at time zero,

$w(t) \in \mathcal{W}$ is the signal at time $t \in \mathbb{R}^+$.
A **system node** is a construction used in the theory of well-posed (and non-wellposed) linear systems. It has a state space $\mathcal{X}$ (a Hilbert space), input space $\mathcal{U}$ (a Hilbert space), output space $\mathcal{Y}$ (a Hilbert space). It is a closed operator $S : [\mathcal{X} \ U] \rightarrow [\mathcal{X} \ Y]$. The dynamics of a system node is described by

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (5)$$

We can rewrite this as a state/signal system by taking $\mathcal{W} = [\mathcal{Y} \ U]$ and defining

$$\mathcal{V} := \left\{ \begin{bmatrix} z \\ \dot{x} \\ \dot{y} \\ \dot{u} \end{bmatrix} \subset [\mathcal{X} \ \mathcal{Y}] \ \mid \begin{bmatrix} z \\
\dot{x} \\
\dot{y} \\
\dot{u} \end{bmatrix} = S \begin{bmatrix} x \\
u \end{bmatrix} \right\}. \quad (6)$$
Example: Classical I/S/O System

Consider the classical input/state/output system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \quad (7)$$

Here $A$, $B$, $C$, and $D$ are bounded linear operators.

We can rewrite this as a state/signal system by taking $\mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} (= \mathcal{Y} \times \mathcal{U})$ and defining

$$\mathcal{V} := \left\{ \begin{bmatrix} z \\ \dot{x} \\ \dot{y} \\ u \end{bmatrix} \subset \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \mid z = Ax + Bu, \quad y = Cx + Du \right\}. \quad (8)$$
Thus, state/signal systems need not have anything to do with boundary control!
However, there is an almost one-to-one correspondence between conservative state/signal systems ↔ (conservative) boundary relations!
Thus, boundary relations do not necessarily have anything to do with boundary control!
We recall the equation describing the dynamics:

\[
\Sigma : \left\{ \begin{array}{l}
\dot{x}(t) \\
x(t) \\
w(t)
\end{array} \right\} \in V, \quad t \in \mathbb{R}^+, \quad x(0) = x_0.
\]

(3)

- \([\dot{x}, w]\) is a **classical trajectory** of \(\Sigma\) if \([\dot{x}, w]\) \in \left[ C^1(\mathbb{R}^+; \mathcal{X}) \right] \times \left[ C(\mathbb{R}^+; \mathcal{X}) \right] \) and (3) holds for all \(t \in \mathbb{R}^+\).

- \([\dot{x}, w]\) is a **generalized trajectory** of \(\Sigma\) if \([\dot{x}, w]\) \in \left[ C(\mathbb{R}^+; \mathcal{X}) \right] \times \left[ L^2_{loc}(\mathbb{R}^+; \mathcal{W}) \right] \) and there exists a sequence of classical trajectories \([\dot{x}_n, w_n]\) such that \(x_n \to x\) uniformly on bounded intervals and \(w_n \to w\) in \(L^2_{loc}(\mathbb{R}^+; \mathcal{W})\).
In this talk I focus on state/signal systems which are conservative, as studied in (Kur10). They are well-posed in the sense of (KS09). 

**Simplifying Assumption:** In the equation describing the dynamics

\[
\Sigma : \begin{cases} 
  \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} & \in V, \\
  t & \in \mathbb{R}^+, \\
  x(0) & = x_0.
\end{cases}
\] (3)

I throughout make the simplifying assumption that the present state \(x(t)\) and the present signal \(w(t)\) determine the value of \(\dot{x}(t)\) uniquely. To guarantee this I assume (for simplicity) that

\[
\begin{bmatrix} z_0 \\ 0 \end{bmatrix} \in V \Rightarrow z = 0.
\] (9)

The assumption can always be made “without loss of generality” (by factoring out an unreachable and unobservable part of the state space).
A conservative s/s system

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \\ t \in \mathbb{R}^+, \\ x(0) = x_0, \end{cases} \quad (3)$$

preserves energy, and so does the dual system. Preservation of energy means that

$$\frac{d}{dt} \| x(t) \|_X^2 = [w(t), w(t)]_W. \quad (10)$$

Here $\frac{1}{2} \| x(t) \|_X^2$ is the internal energy stored state at time $t$ (= the Hamiltonian), and $\frac{1}{2} [w(t), w(t)]_W$ represents the power entering into the system from the outside world. Thus, if we want to allow the energy to flow in both directions, then we must allow the right-hand side to take both positive and negative values, and we cannot replace the indefinite inner product $[\cdot, \cdot]_W$ in $W$ by a positive definite Hilbert space inner product $(\cdot, \cdot)_W$ in $W$. 
By carrying out the differentiation in the power balance equation

\[
\frac{d}{dt} \|x(t)\|^2 = [w(t), w(t)]_\mathcal{W}
\]  

(10)

we get the *Lagrangian identity*

\[
-(\dot{x}(t), x(t))_\mathcal{X} - (x(t), \dot{x}(t))_\mathcal{X} + [w(t), w(t)]_\mathcal{W} = 0.
\]  

(11)

At \( t = 0 \) the vector \( \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \) can be an arbitrary vector in \( \mathcal{V} \), and hence (11) with \( t = 0 \) implies

\[
-(z, x)_\mathcal{X} - (x, z)_\mathcal{X} + [w, w]_\mathcal{W} = 0, \quad \begin{bmatrix} z \\ \dot{x} \\ w \end{bmatrix} \in \mathcal{V}.
\]  

(12)

This inequality says that \( \mathcal{V} \) is a neutral subspace of the node space \( \mathcal{K} \) with respect to a suitable indefinite inner product!
Define

\[
\begin{bmatrix}
  z_1 \\ x_1 \\ w_1
\end{bmatrix},
\begin{bmatrix}
  z_2 \\ x_2 \\ w_2
\end{bmatrix}
\begin{bmatrix}
  0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  z_1 \\ x_1 \\ w_1
\end{bmatrix},
\begin{bmatrix}
  z_2 \\ x_2 \\ w_2
\end{bmatrix}
\begin{bmatrix}
  0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1
\end{bmatrix}.
\]

Then

\[-(z, x) x - (x, z) x + [w, w] \in V = 0,
\begin{bmatrix}
  z \\ x \\ w
\end{bmatrix} \in V \quad (12)
\]
says that

\[
\begin{bmatrix}
  z \\ x \\ w
\end{bmatrix},
\begin{bmatrix}
  z \\ x \\ w
\end{bmatrix}
\begin{bmatrix}
  0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  z \\ x \\ w
\end{bmatrix} = 0,
\begin{bmatrix}
  z \\ x \\ w
\end{bmatrix} \in V \quad (14)
\]

In other words, \( V \) is a neutral subspace of the node space \( \mathcal{R} \) with respect to the inner product (13). Equivalently, \( V \subset V[\perp] \).
We get the dual system by replacing $V$ by $V^{[\perp]}$. The duals system preserves energy if $V^{[\perp]}$ is neutral, i.e., if $V^{[\perp]} \subset V$.

**Definition**

The state/signal system

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \\ t \in \mathbb{R}^+, \\ x(0) = x_0, \end{cases}$$

is conservative if $V$ is Lagrangian, i.e., if $V = V^{[\perp]}$. 

\[ (3) \]
By a Lagrangian decomposition of the Kreĭn signal space \( \mathcal{W} \) we mean a direct sum decomposition \( \mathcal{W} = \mathcal{U} \oplus \mathcal{Y} \) where both \( \mathcal{U} \) and \( \mathcal{Y} \) are Lagrangian subspaces of \( \mathcal{W} \), i.e., \( \mathcal{U} = \mathcal{U}^\perp \) and \( \mathcal{Y} = \mathcal{Y}^\perp \). With suitable choices of norms in \( \mathcal{U} \) and \( \mathcal{Y} \) we can write the inner product in \( \mathcal{W} \) in the form

\[
[y_1 + u_1, y_2 + u_2]_{\mathcal{W}} = (\Psi y_1, u_2)_{\mathcal{U}} + (u_1, \Psi y_2)_{\mathcal{U}},
\]

for all \( u_1, u_2 \in \mathcal{U} \), and \( y_1, y_2 \in \mathcal{Y} \), and for some unitary operator \( \Psi : \mathcal{U} \rightarrow \mathcal{Y} \). We then write \( \mathcal{W} = \mathcal{U} \oplus \mathcal{Y} \).
Boundary Relation = Generating Subspace

Answer to question “Where do they come from”?: A boundary relation \( \simeq \) the generating subspace \( V \) of a conservative s/s system which has been reinterpreted as a relation.

**Theorem**

Let \((V; \mathcal{X}, \mathcal{W})\) be a conservative s/s node and assume that there exists a Lagrangian decomposition \( \mathcal{W} = \mathcal{U} + \mathcal{Y} \). Interpret \( V \) as the (slightly modified) graph of a relation \( \Gamma: \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{U} \end{bmatrix} : \)

\[
V = \left\{ \begin{bmatrix} iz \\ xu \\ i\psi^*y \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \mid \{[x], [u]\} \in \Gamma \right\}, \tag{16}
\]

and set \( R := \text{Ker} (\Gamma) \). Then \( R \) is a closed symmetric operator in \( \mathcal{X} \), \( R^* \) is the closure of \( \text{dom} (\Gamma) \) in \( \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix} \), and \( \Gamma \) is a conservative boundary relation for \( R^* \).

Boundary control s/s system \( \Rightarrow \) \( \Gamma \) is an operator.
Taking Laplace transforms in the formula \[
\begin{bmatrix}
\dot{x}(t) \\
x(t) \\
w(t)
\end{bmatrix} \in V \text{ for all } t > 0,
\]
we get
\[
\begin{bmatrix}
\lambda \hat{x}(\lambda) - x(0) \\
\hat{x}(\lambda) \\
\hat{w}(\lambda)
\end{bmatrix} \in V, \quad \lambda \in \mathbb{C}^+.
\] (17)

**Definition**

The **characteristic manifold** of the s/s system \(\Sigma = (V; X, W)\) is the family of subspaces \(\hat{\mathcal{V}}(\lambda)\) defined by

\[
\hat{\mathcal{V}}(\lambda) = \left\{ \begin{bmatrix} x \\ x_0 \\ w \end{bmatrix} \in \begin{bmatrix} X \\ X \\ W \end{bmatrix} \mid \begin{bmatrix} \lambda x - x_0 \\ x \\ w \end{bmatrix} \in V \right\}.
\] (18)

The domain of \(\hat{\mathcal{V}}(\lambda)\) consists of all those points \(\lambda \in \mathbb{C}\) where this manifold is analytic.

Here \(\hat{\mathcal{V}}\) is **analytic** at a point \(\lambda_0\) if \(\hat{\mathcal{V}}(\lambda)\) has a graph representation in some neighborhood of \(\lambda_0\) with an analytic angle operator.
The characteristic manifold $\hat{\mathcal{V}}$ is defined and analytic (at least) in the open right-half plane.

The Weyl family and the Gamma field can be obtained from the characteristic manifold by first intersecting $\hat{\mathcal{V}}(\lambda)$ with $\begin{bmatrix} \chi \\ 0 \\ \mathcal{W} \end{bmatrix}$, then projecting it onto either $\begin{bmatrix} 0 \\ \chi \\ \mathcal{W} \end{bmatrix}$ or $\begin{bmatrix} \chi \\ 0 \\ \mathcal{U} \end{bmatrix}$, and finally interpreting the result as a relation.

Here $\mathcal{U}$ is one of the two components in the Lagrangian decomposition $\mathcal{W} = \mathcal{U} \oplus \mathcal{Y}$. 
Above I only discussed conservative state/signal systems. **Question:** What happens when the state/signal system is well-posed but not conservative?

**Answer:**

- We will then have to deal with two different generating subspaces $\mathcal{V}$ and $\mathcal{V}^{\perp} \neq \mathcal{V}$, and two different s/s systems $\Sigma = (\mathcal{V}; \mathcal{X}, \mathcal{W})$ and $\Sigma^{\perp} = (\mathcal{V}^{\perp}; \mathcal{X}, \mathcal{W})$.

- To each of these s/s systems corresponds a “non-conservative boundary relation”.

- Thus, we end up with pairs of boundary relations instead of just one boundary relation.

- In this case the “Lagrangian identity” simply says that the two systems are dual to each other.

- Details will be worked out later.
Boundary relations = generating subspaces of conservative state/signal systems, reinterpreted as relations.

The Weyl family and the Gamma fields are obtained from the characteristic manifold of the state/signal system by intersections and projections.

Pairs of boundary relations are related to non-conservative state/signal systems.

Boundary relations do not in reality have much to do with boundary control, only historically.


Mikael Kurula and Olof J. Staffans, *Connections between smooth and generalized trajectories of a state/signal system*, accepted for publication, 2010.


