Reciprocal Symmetry in State/Signal Systems

Damir Z. Arov
Division of Mathematical Analysis
Institute of Physics and Mathematics
South-Ukrainian Pedagogical University

Olof Staffans
Department of Mathematics
Åbo Akademi University
http://web.abo.fi/~staffans/

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Outline

- $J$-Conservative Discrete Time Input/State/Output Systems
- External and Internal I/S/O Reciprocity
- Conservative State/Signal Systems
- External and Internal S/S Reciprocity
Discrete Time-Invariant I/S/O Systems

A linear discrete-time-invariant i/s/o (input/state/output) system is of the form

\[ \sum_{i/s/o} : \begin{cases} x(n + 1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases} \tag{1} \]

\( A, B, C, D, \) are bounded linear operators and \( \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \).

the input \( u(n) \in \mathcal{U} = \) the input space,

the state \( x(n) \in \mathcal{X} = \) the state space,

the output \( y(n) \in \mathcal{Y} = \) the output space (all Hilbert spaces).

A trajectory = a triple of sequences \( (u, x, y) \) satisfying \( (1) \).
Forward $J$-Conservative I/S/O System

$\Sigma_{i/s/o}$ is forward $J$-conservative if all trajectories satisfy

$$\|x(n+1)\|^2_X = \|x(n)\|^2_X + \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{Y \oplus U}, \quad n \in \mathbb{Z}^+.$$  

Here

$$j(u, y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{Y \oplus U}.$$  

is the supply rate induced by the signature operator $J = J^* = J^{-1}$. 

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Adjoint I/S/O System

The (causal) adjoint system is given by

\[ \Sigma_{i/s/o}^*: \begin{cases} 
  x_*(n+1) = A^*x_*(n) + C^*y_*(n), & n \in \mathbb{Z}^+, \quad x_*(0) = x_{*0}, \\
  u_*(n) = B^*x_*(n) + D^*y_*(n), & n \in \mathbb{Z}^+. 
\end{cases} \tag{2} \]

The adjoint system is forward \( J_* \)-conservative if all the trajectories satisfy

\[ \|x_*(n + 1)\|_X^2 = \|x_*(n)\|_X^2 + \left\langle \begin{bmatrix} u_*(n) \\ y_*(n) \end{bmatrix}, J_* \begin{bmatrix} u_*(n) \\ y_*(n) \end{bmatrix} \right\rangle_{U \oplus Y}, \quad n \in \mathbb{Z}^+. \]

Here \( J_* = \begin{bmatrix} 0 & -1_Y \\ 1_Y & 0 \end{bmatrix} J \begin{bmatrix} 0 & -1_U \\ 1_U & 0 \end{bmatrix} \) defines the adjoint supply rate.
Simple $J$-conservative System

$\Sigma_{i/s/o}$ is $J$-conservative if $\Sigma_{i/s/o}$ is forward $J$-conservative and $\Sigma^*_{i/s/o}$ is forward $J^*$-conservative.
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The reachable subspace $R$ of $\Sigma_{i/s/o}$ is the closed linear span of all the values $x(n)$, $n \geq 0$, as $(u, x, y)$ varies over all trajectories of $\Sigma_{i/s/0}$ with $x_0 = 0$.

The unobservable subspace $U$ of $\Sigma_{i/s/o}$ is the set of all initial states $x(0)$ of all “unobservable” trajectories $(0, x, 0)$ of $\Sigma_{i/s/o}$ (i.e., both $u$ and $y$ are identically zero).

$\Sigma_{i/s/o}$ is simple if the closed linear span of $R$ and $U^\perp$ is all of $X$. 

Simple $J$-conservative System

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The reachable subspace $\mathcal{R}$ of $\Sigma_{i/s/o}$ is the closed linear span of all the values $x(n), n \geq 0$, as $(u, x, y)$ varies over all trajectories of $\Sigma_{i/s/o}$ with $x_0 = 0$.

The unobservable subspace $\mathcal{U}$ of $\Sigma_{i/s/o}$ is the set of all initial states $x(0)$ of all “unobservable” trajectories $(0, x, 0)$ of $\Sigma_{i/s/o}$ (i.e., both $u$ and $y$ are identically zero).

$\Sigma_{i/s/o}$ is simple if the closed linear span of $\mathcal{R}$ and $\mathcal{U}^\perp$ is all of $\mathcal{X}$.

**Theorem 1.** An simple $J$-conservative i/s/o system $\Sigma_{i/s/o}$ is uniquely determined, up to a unitary similarity transformation in its state space, by its transfer function (defined in some neighborhood of the origin)$^{1}$

$$\mathcal{D}(z) := zC(1 - zA)^{-1}B + D.$$  

$^{1}$The same statement is true true for the balanced minimal realization.
Outline

• $J$-Conservative Discrete Time Input/State/Output Systems

• External and Internal I/S/O Reciprocity

• Conservative State/Signal Systems

• External and Internal S/S Reciprocity
Externally Reciprocal Impedance Systems

By an impedance supply rate we mean the following: There is a unitary operator \( \Psi : \mathcal{U} \to \mathcal{V} \) (= a "unit resistance") such that the supply rate (power product) is given by \( j_{\text{imp}}(u, y) = 2 \Re(y, \Psi u) \). The signature operator is \( J_{\text{imp}} = \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix} \), and the dual signature operator is \( J_* = \begin{bmatrix} 0 & \Psi^* \\ \Psi & 0 \end{bmatrix} \).

The impedance (= transfer) function \( \mathcal{D} \) is always a \( \Psi \)-Nevanlinna (= positive real) function in the unit disk, i.e., \( \Psi^* \mathcal{D}(z) + \mathcal{D}(z)^* \Psi \geq 0 \) for all \( z \in \mathbb{D} \).
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If \( \varphi \) satisfies, in addition,

\[
\varphi(z) = \Psi \varphi^*(\overline{z}) \Psi, \quad z \in \mathbb{D},
\]

where \( \Psi : \mathcal{U} \rightarrow \mathcal{Y} \) is unitary, then we call \( \varphi \) is \( \Psi \)-reciprocal, and say that \( \Sigma_{i/s/o} \) is externally reciprocal.
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\( 2\Sigma_{i/s/o} \) is impedance conservative \( \Leftrightarrow \begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} = \begin{bmatrix} 1^* & C^*\Psi \\ \Psi^*C & \Psi^*D + D^*\Psi \end{bmatrix} \cdots \)
Internally Reciprocal Impedance Systems

(External) reciprocity is a very common property:

- If $\dim \mathcal{U} = \dim \mathcal{Y} = 1$, and $\varphi$ is real on the real axis, then $\varphi$ is reciprocal.
- The impedance function (transfer function from current to voltage) of every passive electrical circuit which does not contain any gyrators is reciprocal.
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- The impedance function (transfer function from current to voltage) of every passive electrical circuit which does not contain any gyrators is reciprocal.

**Theorem 2.** A pure (= strictly positive real) Nevanlinna function \( \mathfrak{D} \) is \( \Psi \)-reciprocal if and only if the (essentially unique) simple conservative realization \( \Sigma_{i/s/o} \) of \( \mathfrak{D} \) is internally reciprocal (= signature similar to its adjoint) in the sense that there exists a signature operator \( \mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1} \) such that

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
\mathcal{I} & 0 \\
0 & \Psi
\end{bmatrix}
\begin{bmatrix}
A^* & C^* \\
B^* & D^*
\end{bmatrix}
\begin{bmatrix}
\mathcal{I} & 0 \\
0 & \Psi
\end{bmatrix} \quad (\Rightarrow A = \mathcal{I}^* A^* \mathcal{I}).
\]
Internally Reciprocal Impedance Systems

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- If \( \dim \mathcal{U} = \dim \mathcal{V} = 1 \), and \( \varphi \) is real on the real axis, then \( \varphi \) is reciprocal.

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\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
\mathcal{I} & 0 \\
0 & \Psi
\end{bmatrix} \begin{bmatrix}
A^* & C^* \\
B^* & D^*
\end{bmatrix} \begin{bmatrix}
\mathcal{I} & 0 \\
0 & \Psi
\end{bmatrix} \quad \Rightarrow \quad A = \mathcal{I}^* A^* \mathcal{I}.
\]

Thus, external reciprocity of a pure impedance function \( \iff \) internal reciprocity of the simple conservative realization.\(^3\)

\(^3\)The same statement is true true for the balanced minimal realization.
Other Supply Rates

Analogous results are true for other supply rates as well (such as scattering and transmission).

Reciprocal i/s/o systems setting are discussed, e.g., in a finite-dimensional setting in

[Wil72], [OJ85], [ABGR90], and [LR95],

and in an infinite-dimensional setting in

[Fuh75], [Obe96], [GO99], and [AADR02].
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Claim: The simplest way to treat a general supply rate is to replace the input/state/output system $\Sigma_{i/s/o}$ by a state/signal signal (s/s) system.
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Signal Space and Energy Balance

We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one signal space $\mathcal{W} = \left[ \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \right]$. This signal space has a natural Kreĭn space inner product obtained from the signature operator $J$ in the supply rate $j$, namely

$$\left[ \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \right]_{\mathcal{W}} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}}.$$

The forward $J$-energy balance equation becomes (with $w(n) = \left[ \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right]$)

$$\|x(n + 1)\|_{\mathcal{X}}^2 = \|x(n)\|_{\mathcal{X}}^2 + [w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+,$$

or equivalently,

$$-(x(n + 1), x(n + 1))_{\mathcal{X}} + (x(n), x(n))_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+. $$
Graph Representation of I/S/O System

The basic i/s/o relation

\[
\Sigma_{i/s/o} : \begin{cases} 
    x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\
    y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+.
\end{cases}
\]  

(1)

can be written in graph form

\[
\Sigma_{s/s} : \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0.
\]  

(3)
Graph Representation of I/S/O System

The basic i/s/o relation

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\Sigma_{i/s/o} : \begin{cases} 
  x(n + 1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\
  y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+.
\end{cases}
\] (1)

can be written in graph form\(^4\)

\[
\Sigma_{s/s} : \begin{bmatrix} 
  x(n + 1) \\
  x(n) \\
  w(n)
\end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0.
\] (4)

\[\text{CDPS 2009}\]
The dynamics of a discrete time state/signal system $\Sigma$ is defined by

$$\Sigma : \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0,$$

where $V$ is the generating subspace of the node space $\mathcal{K} := \begin{bmatrix} X \\ X \end{bmatrix}$.

By a trajectory of $\Sigma$ we mean a pair of sequences $(x, w)$ satisfying (4).

We call $x$ the state component and $w$ the signal component of the trajectory.

$\Sigma$ is well-posed if (4) defines a “reasonable dynamics”.5

---

5For every $x_0 \in X$ there is a trajectory with $x(0) = x_0$, and this trajectory depends continuously on $x_0$ and the signal part $w(\cdot)$. 
Forward Conservativity of State/Signal Node

The forward energy balance

\[-(x(n+1), x(n+1)) + (x(n), x(n)) + [w(n), w(n)] = 0, \quad n \in \mathbb{Z}^+, \quad (5)\]

tells us to use the following natural (indefinite) Kreĭn space inner product in \( \mathcal{K} \):

\[
\left[ \begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \right]_{\mathcal{K}} = -(z_1, z_2)x + (x_1, x_2)x + [w_1, w_2]w. \quad (6)
\]

It is easy to see that \((5)\) holds for all trajectories of \( \Sigma \) if and only if

\[
\left[ \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix}, \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} \right]_{\mathcal{K}} = 0 \quad \forall \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} \in V.
\]

In other words, \((5)\) holds if and only if \( V \) is a neutral subspace of \( \mathcal{K} \) with the inner product \((6)\).
Conservativity of State/Signal Node

\[ V^{[\perp]} = \left\{ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in \mathcal{R} \left| \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} \tilde{z} \\ \tilde{x} \\ \tilde{w} \end{bmatrix} \right. = \forall \begin{bmatrix} \tilde{z} \\ \tilde{x} \\ \tilde{w} \end{bmatrix} \in V \right\}. \]

\( \Sigma_{s/s} \) is forward conservative \( \Leftrightarrow V \subset V^{[\perp]} \).

The “adjoint system” is forward conservative \( \Leftrightarrow V^{[\perp]} \subset V \).

Define: \( \Sigma_{s/s} \) is conservative if \( V = V^{[\perp]} \).

If \( \Sigma_{s/s} \) is conservative, then it is automatically well-posed.
Conservativity of State/Signal Node

\[ V^{[\perp]} = \left\{ \begin{bmatrix} z^* \\ x^* \\ w^* \end{bmatrix} \in \mathcal{R} \left| \begin{bmatrix} z^* \\ x^* \\ w^* \end{bmatrix}, \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right| \mathcal{R} = \forall \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}. \]

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The “adjoint system” is forward conservative \( \iff \) \( V^{[\perp]} \subset V \).

Define: \( \Sigma_{s/s} \) is conservative if \( V = V^{[\perp]} \).

If \( \Sigma_{s/s} \) is conservative, then it is automatically well-posed.

The reachable subspace \( \mathcal{R} \) and the unobservable subspace \( \mathcal{U} \) are defined in the same way as for i/s/o systems.

\( \Sigma_{i/s/o} \) is simple if the closed linear span of \( \mathcal{R} \) and \( \mathcal{U}^{\perp} \) is all of \( \mathcal{X} \).
The Behavior of a State/Signal Systems

In s/s theory the transfer function of an i/s/o system is replaced by the (frequency domain) behavior of the s/s system.

behavior of s/s system $\simeq$ graph of the transfer function of a i/s/o system.

More precisely, the behavior is the subspace of all $H^2$-functions $\hat{w}(\cdot)$ on $\mathbb{D}$ which satisfy

$$
\begin{bmatrix}
\frac{1}{z}\hat{x}(z) \\
\hat{x}(z) \\
\hat{w}(z)
\end{bmatrix}
\in V, \quad z \in \mathbb{D},
$$

(7)

for some analytic function $\hat{x}(z)$.

Interpretation: $\hat{x}(z)$ is the $Z$-transform of the state part and $\hat{w}(z)$ is the $Z$-transform of the signal part of a trajectory $(x, w)$ of $\Sigma_{s/s}$ with $x(0) = 0$ and $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$. 
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**behavior of s/s system** \(\simeq\) **graph of the transfer function** of a i/s/o system.

More precisely, the behavior is the subspace of all \(H^2\)-functions \(\hat{w}(\cdot)\) on \(\mathbb{D}\) which satisfy

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**Denote:** \(\mathcal{W} = \text{behavior of } \Sigma_{s/s} \text{ and } \mathcal{W}(z) := \{\hat{w}(z) | \hat{w}(\cdot) \in \mathcal{W}\}, z \in \mathbb{D}.$$
Passive Behaviors

It turns out that

- the behavior $\mathcal{W}$ of a conservative s/s system is a maximal nonnegative shift-invariant subspace of $H^2(\mathbb{D}; \mathcal{W})$

with respect to the indefinite inner product inherited from the Kreĭn space $\mathcal{W}$ (shift-invariance means that it is invariant under multiplication with $z$).
Passive Behaviors

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- the behavior $\mathcal{W}$ of a conservative s/s system is a maximal nonnegative shift-invariant subspace of $H^2(\mathbb{D}; \mathcal{W})$ with respect to the indefinite inner product inherited from the Kreĭn space $\mathcal{W}$ (shift-invariance means that it is invariant under multiplication with $z$).

- Passive behavior $= a$ maximal nonnegative shift-invariant subspace of $H^2(\mathbb{D}; \mathcal{W})$.

- Strictly passive behavior $= a$ maximal strictly positive shift-invariant subspace of $H^2(\mathbb{D}; \mathcal{W})$. 

References

More details about state/signal systems can be found in [AS05, AS07a, AS07b, AS07c, AS09a, AS09b] and [Sta06].

Continuous time state/signal systems have been studied in [KS09, Kur09].
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External and Internal Reciprocity

The state/signal analogue of external reciprocity, i.e., reciprocity of the transfer function, is the following:

A passive behavior $\mathcal{W}$ is $J$-reciprocal if $J = -J^{*} = J^{-1}$ is a skew-adjoint involution in the signal space $\mathcal{W}$ and $\mathcal{W}(z) = J\mathcal{W}(\bar{z})^{[\perp]}$, $z \in \mathbb{D}$. (In the impedance i/s/o case we may take $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.)

A conservative s/s system $\Sigma_{s/s}$ is externally reciprocal if the behavior $\mathcal{W}$ of $\Sigma_{s/s}$ is $J$-reciprocal for some skew-adjoint involution $J$. 
External and Internal Reciprocity

The state/signal analogue of external reciprocity, i.e., reciprocity of the transfer function, is the following:

A passive behavior $\mathcal{W}$ is \textit{J-reciprocal} if $\mathcal{J} = -\mathcal{J}^* = \mathcal{J}^{-1}$ is a skew-adjoint involution in the signal space $\mathcal{W}$ and $\mathcal{W}(z) = \mathcal{J}\mathcal{W}(\bar{z})^{\perp}$, $z \in \mathbb{D}$. (In the impedance i/s/o case we may take $\mathcal{J} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.)

A conservative s/s system $\Sigma_{s/s}$ is \textit{externally reciprocal} if the behavior $\mathcal{W}$ of $\Sigma_{s/s}$ is $\mathcal{J}$-reciprocal for some skew-adjoint involution $\mathcal{J}$.

A conservative s/s system $\Sigma_{s/s}$ is \textit{internally reciprocal} if it is internally signature similar to its adjoint, i.e., there exists a signature operator $\mathcal{I} = \mathcal{I}^* = \mathcal{I}^{-1}$ in the state space $\mathcal{X}$ and a boundedly invertible operator $\mathcal{J} \in \mathcal{B}(\mathcal{W})$ such that

$$V = \begin{bmatrix} 0 & \mathcal{I} & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{J} \end{bmatrix} V^{\perp}.$$
Connection Between External and Internal Reciprocity

**Theorem 3.** Let $\mathcal{W}$ be a passive behavior on the signal space $\mathcal{W}$.

(i) If $\mathcal{W}$ is $\mathcal{J}$-reciprocal for some skew-adjoint involution $\mathcal{J}$ in $\mathcal{W}$, then the (essentially unique) simple conservative realization $\Sigma = (V; \mathcal{X}; \mathcal{W})$ of $\mathcal{W}$ satisfies

$$ V = \begin{bmatrix} 0 & \mathcal{I} & 0 \\ \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{J} \end{bmatrix} V^{\perp} $$

for some signature operator $\mathcal{I}$. (Here $V^{\perp} = V$ since $\Sigma$ is conservative.)

(ii) If $\Sigma = (V; \mathcal{X}; \mathcal{W})$ is a conservative realization of $\mathcal{W}$ satisfying (8) for some signature operator $\mathcal{I}$ and some skew-adjoint involution $\mathcal{J}$, then $\mathcal{W}$ is $\mathcal{J}$-reciprocal.

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6 The same claim is true for minimal passive balanced systems (in which case $V^{\perp} \neq V$).
Further Questions
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• $\mathcal{J}$ defines a continuous non-degenerate \textit{anti-symmetric bilinear form} in $\mathcal{W}$ (which corresponds to the \textit{reactive power} in classical circuit theory).
Further Questions

• Both $I$ and $J$ are determined uniquely by $V$. Exactly to what extent is $J$ determined uniquely by $W$?

• $J$ defines a continuous non-degenerate anti-symmetric bilinear form in $W$ (which corresponds to the reactive power in classical circuit theory).

• There is a one-to-one correspondence between the set of all skew-adjoint involutions $J$ in $W$ and all Lagrangian decompositions $W = \mathcal{F} \perp \mathcal{E}$ of $W$. In particular, a necessary condition for reciprocity is that $\dim_+ W = \dim_- W$ (the input and output dimensions must be the same).
Further Questions

- Both $I$ and $J$ are determined uniquely by $V$. Exactly to what extent is $J$ determined uniquely by $W$?

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- This leads to a connection to the theory of port-Hamiltonian systems!
References


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[Wil72] Jan C. Willems, Dissipative dynamical systems Part II: Linear systems